JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **191**, 450–459 (1995)

On the Pseudo-Differential Operator $(-x^{-1}D)^{\nu}$

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Submitted by Bruce C. Berndt

Received September 8, 1993

For a certain Frechet space F consisting of complex-valued C^{∞} even functions defined on R and rapidly decreasing as $|x| \to \infty$, we show that if ν is any complex number,

(i) The pseudo-differential operator $(-x^{-1}D)^{\nu}$ is an automorphism on F.

(ii) $e^{-\alpha x}$, Re $\alpha > 0$, is an eigenfunction of the pseudo-differential operator $(-x^{-1}D)^{\nu}$.

(iii) For f in X, a linear subsapce of the Hilbert space $L^2(\mathbf{R})$ generated by the even-order Hermite functions $(H_{2n}(x)e^{-x^2/2})n = 0, 1, 2, \ldots,$

$$(-x^{-1}D)^{\nu}f(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} C_{2k}a_{n-j}(x^2 - 2\nu^*)^j e^{-x^2/2},$$

where C_{2k} and a_{n-j} are constants and

$$\nu^{*j} = \begin{cases} \nu(\nu-1)\cdots(\nu-j+1) & \text{for } j \geq 1, \nu \neq 1 \\ 0 & \text{for } j \geq 2, \nu = 1. \end{cases}$$

1. INTRODUCTION

For any complex μ, F_{μ} is the space of all C^{∞} complex-valued functions $f(x)$ defined on $I = (0, \infty)$ such that

$$f(x) = x^{\mu+1/2} \sum_{j=0}^{k} a_j x^{2j} + o(x^{2k}) \tag{1.1}$$

near the origin and is rapidly decreasing as $x \to \infty$.

0022-247X/95 $6.00

We equip the space F_{μ} with the topology generated by the countable family of separating seminorms $(\gamma^{\mu}_{m,k})^{\infty}_{m,k=0}$ defined by

$$\gamma^{\mu}_{m,k}(f) = \sup_{x \in I} |x^m (x^{-1}D)^k x^{-\mu-1/2} f(x)|, \tag{1.2}$$

where $D = d/dx$ [2–4; 7; 8, p. 8].

It follows from Lemma 2 of [7] that F_{μ} is a Frechet space.

This is a sequel to the paper [4] where the authors obtained a Fourier-Basel series representation for the pseudo-differential operator $(-x^{-1}D)^{\nu}$ for each real ν on a certain subspace F_b of F. The space F consists of all C^{∞} complex-valued functions defined on I and satisfying the same boundary conditions as the functions in $F_{-1/2}$. For $f \in F$ and any real $\mu \neq -\frac{1}{2}$,

$$\bar{\gamma}^{\mu}_{m,k}(f) = \sup_{x \in I} |x^m \Delta^k_{\mu,x} f(x)| < \infty, \tag{1.3}$$

for each m, $k = 0, 1, 2, \ldots$, where $\Delta_{\mu,x} = D^2 + x^{-1}(2^{\mu} + 1)D$ [4]. F is a Frechet space. Its topology is generated by the countable family of separating seminorms $(\bar{\gamma}^{\mu}_{m,k})_{m,k=0,1,2,\ldots}$ [4; 8, p. 8]. Note that the topology assigned to F through the seminorms $\bar{\gamma}^{\mu}_{m,k}$ is independent of μ [4, Theorem 2.1(iii)].

In this paper we show that for every complex ν,

(i) The pseudo-differential operator $(-x^{-1}D)^{\nu}$ is a topological automorphism on the space $F_{-1/2}$.

(ii) For Re $\alpha > 0$, $(-x^{-1}D)^{\nu} e^{-\alpha x^2} = (2\alpha)^{\nu} e^{-\alpha x^2}$, $x \varepsilon R$.

(iii) Let $P_{2n}(x) = \sum_{j=0}^{n} a_{n-j} x^{2j}$ be a polynomial of degree $2n$, then

$$(-x^{-1}D)^{\nu}[P_{2n}(x)e^{-x^2/2}] = \sum_{j=0}^{n} a_{n-j}(x^2 - 2^{\nu*})^j e^{-x^2/2}.$$

(iv) Let X be the linear subspace of the Hilbert space $L^2(R)$ generated by the even-order Hermite functions $H_{2n}(x)e^{-x^2/2}$, $n = 0, 1, 2, \ldots$. We obtain a Fourier-Hermite series representation of the pseudo-differential operator $(-x^{-1}D)^{\nu}$ on the linear space X.

2. Preliminaries

Zemanian [7, 8] showed that the Hankel transform h_{μ}, defined by

$$h_{\mu}(f(x))(y) = \int_0^{\infty} f(x)(xy)^{1/2} J_{\mu}(xy)\, dx \tag{2.1}$$

where $J_\mu(x)$ is a Bessel function of order μ, is a self-reciprocal automorphism on F_μ for $\mu \geq -\frac{1}{2}$.

LEMMA 2.1 *For* Re $\mu \geq \frac{1}{2}$, h_μ *as defined by* (2.1) *is an automorphism on* F.

Proof. Let $f \in F_\mu$. The integral (2.1) is well defined since $f \sim 0(x^{\mu+1/2})$ near the origin and Re $\mu \geq -\frac{1}{2}$ ensure its convergence at the origin, and the rapid descent of f near infinity along with Weber's formula

$$\int_0^\infty t^{-\mu-1+m} J_\mu(t)\, dt = [\Gamma(\tfrac{1}{2} m)/[2^{\mu-m+1}\Gamma(\mu - \tfrac{1}{2} m + 1))],$$
$$0 < \operatorname{Re} m < \operatorname{Re} \mu + \tfrac{1}{2}$$

[6, p. 391] ensure its convergence at infinity. The rest of the proof is similar to that of Zemanian [7, Lemma 8].

Note that $f \to x^{\nu-\mu}$ is a homeomorphism from F_μ onto F_ν since $\gamma^{\nu}_{m,k}(x^{\nu-\mu}f) = \gamma^{\mu}_{m,k}(f)$. Hence the map h defined by

$$h(f)(x) = [h_\nu^{-1}(y^{\nu-\mu}h_\mu(f)](x) = [h_\nu(y^{\nu-\mu}h_\mu(f)](x) \quad \text{(since } h_\nu = h_\mu^{-1}) \tag{2.2}$$

is also a homeomorphism from F_μ onto F_ν. So we have the commutative diagram

$$\begin{array}{ccc} & h_\mu & \\ F_\mu & \longleftrightarrow & F_\mu \\ x^{\nu-\mu} \downarrow & & \downarrow h \\ F_\nu & \longleftrightarrow & F_\nu \\ & h_\nu & \end{array}.$$

In view of the above commutative diagram, we have the following.

DEFINITION. For $\mu \in C$, define the Hankel transform h_μ by

$$[h_\mu f(x)](y) = y^{\mu-\nu}[h_\nu 0 h(f(x))](y), \qquad f \in H_\mu,$$

where ν is chosen so that Re $\nu \geq -\frac{1}{2}$.

Remark 1. Note that $f \to x^{\mu+1/2} f$ is a homeomorphism from F onto F_μ [4, Theorem 2.1].

THEOREM 2.1. *If* $\nu - \mu \in N$ *in Eq.* (2.2), *then*

$$h = x^{\nu+1/2}(-x^{-1}D)^{\nu-\mu}x^{-\mu-1/2}. \tag{2.3}$$

Proof. Let the right-hand side of Eq. (2.3) be denoted by $\bar{h}$. Then for $f \in F_{\mu}$,

$$\gamma^{\nu}_{m,k}(\bar{h}f) = \gamma^{\mu}_{m,k+n}(f),$$

from (1.2), where $n = \nu - \mu$. Hence $\bar{h}$ is a continuous map. Let f_1, f_2 in F be such that $\bar{h}f_1 = \bar{h}f_2$. Then $\gamma^{\mu}_{m,k+n}(f_1 - f_2) = 0$, for each $m, k = 0, 1, 2, \ldots$. Hence, in particular, $\gamma^{\mu}_{m,n}(f_1 - f_2) = 0$.

Therefore, $(d/dx)[(x^{-1}D)^{n-1}(f_1 - f_2)] = 0$. Since $f_1 - f_2$ in F_{μ} is a rapidly decreasing function as $x \to \infty$, we get $(x^{-1}D)^{n-1}(f_1 - f_2) = 0$. Repeating the process $(n - 1)$ times, we conclude that $f_1 = f_2$ in F_{μ}. Hence $\bar{h}$ is injective. For $n = 1$ and g in F_{ν}, let

$$f(x) = -x^{\mu+1/2} \int_0^x t^{-\nu+1/2} g(t)\, dt,$$

then $f \in F_{\mu}$ and $\bar{h}f = g$. Using induction on n, it can be shown that $\bar{h}$ is surjective.

That $\bar{h}$ is a homeomorphism follows from the open mapping theorem [5, p. 172].

For $f \in F_{\mu}$,

$$\bar{h}[h_{\mu}(f(x))(y)] = y^{\nu+1/2}(-y^{-1}(d/dy))^n y^{-\mu-1/2} \int_0^{\infty} f(x)(xy)^{1/2} J_{\mu}(xy)\, dx.$$

Differentiating under the integral sign, we get

$$= y^{\nu+1/2} \int_0^{\infty} f(x) x^{\nu+n+1/2} (xy)^{-\nu} J_{\nu}(xy)\, dx$$

(since $(xy)^{-\nu}J_{\nu}(xy)$ is a C^{∞} bounded function on $0 < xy < \infty$, differentiation under the integral sign is valid) and

$$= \int_0^{\infty} x^n f(x)(xy)^{1/2} J_{\nu}(xy)\, dx = h_{\nu}(x^n f(x))(y)$$

$$= h[h_{\mu}(f(x))(y)],$$

thus proving the theorem.

From Eqs. (2.2) and (2.3), we get

$$(-x^{-1}D)^n = x^{-\nu-1/2} h_{\nu} y^n h_{\mu} x^{\mu+1/2}, \qquad n = \nu - \mu. \tag{2.4}$$

It is easily seen that the operator $(-x^{-1}D)^n$ is an automorphism on the Frechet space $F_{-1/2}$. Note that the spaces F and $F_{-1/2}$ are homeomorphic under the identify map. Henceforth we will drop the suffix $_{-1/2}$ from $F_{-1/2}$ and write it as F. Writing $\mu = 0$ in (2.4), we obtain the following useful representation for the operator $(-x^{-1}D)^n$:

$$(-x^{-1}D)^n = x^{-n-1/2} h_n y^n h_0 x^{1/2}. \tag{2.5}$$

3. The Main Result

The equation (2.5) motivates us to propose the following.

Definition. *For $\nu \in C$, define the pseudo-differential operator $(-x^{-1}D)^\nu$ by*

$$(-x^{-1}D)^\nu f(x) = x^{-\nu-1/2} h_\nu y^\nu h_0 x^{1/2} f(x), \qquad f \in F. \tag{3.1}$$

Then $(-x^{-1}D)^\nu$ is clearly an automorphism on F for each complex ν.

From (3.1), we get

$$(-x^{-1}D)^\nu f(x) = x^{-\nu-1/2} \int_0^\infty dy (xy)^{1/2} J_\nu(xy)\, y^\nu \int_0^\infty dx\, x^{1/2} f(x) (xy)^{1/2} J_0(xy). \tag{3.2}$$

Remark 2. In view of the above definition (3.1) of $(-x^{-1}D)^\nu$, Theorem 2.1 is valid with $\nu - \mu \in C$.

For distributions ψ in F', define $(-x^{-1}D)^\nu$ by

$$\langle (-x^{-1}D)^\nu \psi, f \rangle = \langle \psi, (-x^{-1}D)^\nu f \rangle, \qquad f \in F.$$

Theorem 3.1. *The pseudo-differential operator $(-x^{-1}D)^\nu$ is an automorphism on the space F and hence on its dual F' for each complex ν.*

Remark 3. Since f in F satisfies the boundary condition (1.1) near the origin and is rapidly decreasing as $x \to \infty$, we can extend f from $(0, \infty)$ to R by defining $f(x) = f(-x)$ for $-\infty < x < 0$ and $f(0) = a_0$, where a_0 is the constant term in (1.1). With this extension, the space F (now containing C^∞ complex-valued functions defined on $\mathbf{R}$ and satisfying (1.1) near the origin and rapidly decreasing as $|x| \to \infty$) becomes a linear subspace of the Hilbert space $L^2(\mathbf{R})$. Also, for $f \in F$,

$$(-x^{-1}D)^\nu f(x) = (x^{-\nu} h_{\nu-1/2} y^\nu h_{-1/2}) f(x)$$

(obtained by taking $\mu = -\frac{1}{2}$ and replacing $\nu + \frac{1}{2}$ by ν in (2.4))

$$= \tfrac{1}{2}\int_0^{\infty} dy\, y^{\nu+1/2}(xy)^{-\nu+1/2}J_{\nu-1/2}(xy)\int_{\mathbf{R}} dx\, f(x)(xy)^{1/2}J_{-1/2}(xy), \quad (3.3)$$

since $z^{\nu}J_{-\nu}(z)$ is an even entire function for $\nu \in C$. Thus we see that $(-x^{-1}D)^{\nu}f(x)$ is valid for $x \in \mathbf{R}$ and that the pseudo-differential operator $(-x^{-1}D)^{\nu}$ is an automorphism on extended F.

EXAMPLE 1. The function $e^{-x^2} \in F$ $(x \in \mathbf{R})$. Hence $(-x^{-1}D)^{\nu}e^{-x^2} = x^{-\nu-1/2}h_{\nu}y^{\nu}h_0(x^{1/2}e^{-x^2})$, from (3.1).

Using the well-known result

$$h_{\nu}(x^{\nu+1/2}e^{-\alpha x^2}) = [y^{\nu+1/2}/(2\alpha)^{\nu+1}]\exp(-y^2/4\alpha), \qquad \operatorname{Re}\alpha > 0, \operatorname{Re}\nu > -1 \quad (3.4)$$

[1, Eq. (10) on p. 29], we obtain $(-x^{-1}D)^{\nu}e^{-x^2} = 2^{\nu}e^{-x^2}$, for Re $\nu > -1$.

Similarly, it can be shown that for Re $\alpha > 0$,

$$(-x^{-1}D)^{\nu}e^{-\alpha x^2} = (2\alpha)^{\nu}e^{-\alpha x^2} \qquad \text{for Re } \nu > -1.$$

For Re $\nu \leq -1$, without loss of generality we may write $\nu = \nu_1 + \nu_2$ such that Re ν_1, Re $\nu_2 > -1$. Then, using the commutativity of the diagram

$$\begin{array}{ccccc} & F_o & \overset{h_o}{\longleftrightarrow} & F_o & \\ x^{\nu_1} & \downarrow & & \downarrow & x^{\nu_1+1/2}(-x^{-1}D)^{\nu_1}x^{-1/2} \qquad (=h) \\ & F_{\nu_1} & \overset{h_{\nu_1}}{\longleftrightarrow} & F_{\nu_1} & \\ x^{\nu_2} & \downarrow & & \downarrow & x^{\nu+1/2}(-x^{-1}D)^{\nu_2}x^{-\nu_1-1/2} \qquad (=h), \\ & F_{\nu} & \overset{h_{\nu}}{\longleftrightarrow} & F_{\nu} & \end{array}$$

it can be proved that $(-x^{-1}D)^{\nu} = (-x^{-1}D)^{\nu_1}o(-x^{-1}D)^{\nu_2}$. Hence,

$$(-x^{-1}D)^{\nu}e^{-\alpha x^2} = (2\alpha)^{\nu}e^{-\alpha x^2}, \operatorname{Re}\alpha > 0, \nu \in C. \quad (3.5)$$

Thus we have proved the following.

THEOREM 3.2. *For* Re $\alpha > 0$, *the functions* $e^{-\alpha x^2}$ *are eigenfunctions of the pseudo-differential operator* $(-x^{-1}D)^{\nu}$, $\nu \in C$. *Note that* $e^{-x^2/2}$ *is a fixed point of* $(-x^{-1}D)^{\nu}$.

EXAMPLE 2. We now show that

$$(-x^{-1}D)^{\nu}(x^2e^{-x^2/2}) = (x^2 - 2\nu)e^{-x^2/2}, \qquad \nu \in C. \quad (3.6)$$

The following two formulae along with (3.4) are needed to prove (3.6):

$$h_o(x^{2\mu-3/2}e^{-x^2/2}) = 2^{\mu-1}\Gamma(\mu)\,y^{1/2}\,{}_1F_1(\mu; 1; -\tfrac{1}{2}y^2), \qquad \operatorname{Re}\mu > 0, \quad (3.7)$$

[1, Eq. (21) on p. 9] and

$$h_\nu(x^{\mu-1/2}e^{-\alpha x^2}) = \frac{y^{\nu+1/2}\Gamma((1/2)(\nu+\mu+1))}{2^{\nu+1}\alpha^{(1/2)(\nu+\mu+1)}\Gamma(\nu+1)}\,{}_1F_1\left(\frac{\nu+\mu+1}{2}; \nu+1; -y^2/4\alpha\right),$$
$$\operatorname{Re}\alpha > 0, \operatorname{Re}(\nu+\mu) > -1 \qquad (3.8)$$

[1, Eq. (14) on p. 30].

Taking $\mu = 2$ in (3.7), we get

$$h_o(x^{5/2}e^{-x^2/2}) = 2y^{1/2}\,{}_1F_1(2; 1; -\tfrac{1}{2}y^2) = 2y^{1/2}(1 - \tfrac{1}{2}y^2)e^{-y^2/2}$$

because ${}_1F_1(2; 1; -\frac{1}{2}y^2) = \sum_0^\infty [(1 + r)/(r!)](-\frac{1}{2}y^2)r$ [1, p. 429].

Also,

$$h_\nu(y^{\nu+1/2}e^{-y^2/2}) = x^{\nu+1/2}e^{-x^2/2}, \qquad \operatorname{Re}\nu > -1,$$

and

$$h_\nu(y^{\nu+5/2}e^{-y^2/2}) = 2x^{\nu+1/2}(\nu+1)\,{}_1F_1(\nu+2; \nu+1; -x^2/2), \qquad \operatorname{Re}\nu > -3/2$$

(follows from (3.4) and (3.8)).

Since ${}_1F_1(\nu + 2; \nu + 1; -x^2/2) = [1 - (x^2/(1 + \nu)]e^{-x^2/2}$, we see that

$$(-x^{-1}D)^\nu(x^2e^{-x^2/2}) = (x^2 - 2\nu)e^{-x^2/2}, \qquad \operatorname{Re}\nu > -1,$$

by putting various terms together. Again, using the fact that $(-x^{-1}D)^\nu = (-x^{-1}D)^{\nu_1}o(x^{-1}D)^{\nu_2}$ for $\nu = \nu_1 + \nu_2$, we extend the above result for each $\nu \in C$.

Similarly, taking $\mu = 3$ in (3.7) and $\mu = \nu + 5$, $\alpha = \frac{1}{2}$ in (3.8), it follows that for $\nu \in C$,

$$(-x^{-1}D)^\nu(x^4e^{-x^2/2}) = [x^4 - 4\nu x^2 + 4\nu(\nu-1)]e^{-x^2/2} = (x^2 - 2^{\nu *})^2e^{-x^2/2},$$

where ν^* is defined by (3.11).

Taking appropriate values of μ in (3.7), (3.8) and using them along with (3.4), $(-x^{-1}D)^\nu(x^{2n}e^{-x^2/2})$ can be evaluated easily for each $n = 0, 1, 2, \ldots$.

We record this as

LEMMA 3.1. *For $\nu \in C$ and $n = 0, 1, 2, \ldots,$*

(i) $(-x^{-1}D)^{\nu}(x^{2n}e^{-x^2/2}) = (x^2 - 2^{\nu *})^n e^{-x^2/2}$. (3.9)

(ii) *Let $P_{2n}(x) = \sum_{j=0}^{n} a_{n-j}x^{2j}$ be a polynomial; then*

$$(-x^{-1}D)^{\nu}(P_{2n}(x)e^{-x^2/2}) = \sum_{j=0}^{n} a_{n-j}(x^2 - 2^{\nu *})^j e^{-x^2/2} \tag{3.10}$$

where

$$\nu^{*j} = \nu(\nu-1)\cdots(\nu-j+1) \qquad \text{for } j \geq 1, \nu \neq 1;$$

and when $\nu = 1$,

$$\nu^{*j} = 0 \quad \textit{for } j \geq 2. \tag{3.11}$$

4. THE FOURIER HERMITE SERIES

DEFINITION. The polynomials defined by

$$H_o(x) = 1, H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2} \qquad \text{for } n > 0$$

are called Hermite polynomials.

By using Lemma 3.1 and rearranging, we obtain for $\nu \in C$

$$\begin{aligned}
(-x^{-1}D)^{\nu}[(X^1/x)e^{-x^2/2}] &= (X^1/x)e^{-x^2/2},\\
(-x^{-1}D)^{\nu}(X^2e^{-x^2/2}) &= (X^2 - 2^3\nu)e^{-x^2/2},\\
(-x^{-1}D)^{\nu}((X^3/x)e^{-x^2/2}) &= (X^1/x)(X^2 - 2^3\nu)e^{-x^2/2},\\
(-x^{-1}D)^{\nu}(X^4e^{-x^2/2}) &= (X^2 - 2^3\nu)^2e^{-x^2/2},\\
(-x^{-1}D)^{\nu}((X^5/x)e^{-x^2/2}) &= (X^1/x)(X^2 - 2^3\nu)^2e^{-x^2/2},\\
(-x^{-1}D)^{\nu}(X^6e^{-x^2/2}) &= [(X^2 - 2^3\nu)^3 - 2^8\nu]e^{-x^2/2},\\
(-x^{-1}D)^{\nu}(X^7/x)e^{-x^2/2}) &= (X^1/x)[(x^2 - 2^3\nu)^3 - 2^8\nu]e^{-x^2/2},
\end{aligned} \tag{4.1}$$

where we have used the notations

$$X^i = H_i(x) \text{ and } X^i \cdot X^j = H_{i+j}(x), \qquad i, j = 1, 2, 3, \ldots,$$

giving the lower triangular matrix representation for $(-x^{-1}D)^{\nu}H_{2n}(x)$.

$e^{-x^2/2}$. The Hermite functions $H_n(x)e^{-x^2/2}$ are the orthogonal basis for the Hilbert space $L^2(\mathbf{R})$. Let X be the linear subspace of $L^2(\mathbf{R})$, spanned by the even Hermite functions (as a Hamel basis). Then for $f \in X$,

$$f = \sum_{k=0}^{\infty} C_{2k} H_{2k}(x) e^{-x^2/2},$$

where

$$C_{2k} = [1/(2^{2k}(2k)!\pi^{1/2}] \int_{\mathbf{R}} f(x) H_{2k}(x) e^{-x^2/2}\, dx$$

is the Fourier-Hermite series for f in $L^2(\mathbf{R})$. So, for any $f \in X$ and $\nu \in C$,

$$(-x^{-1}D)^{\nu} f(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} C_{2k} a_{n-j} (x^2 - 2^{\nu *})^j e^{-x^2/2}, \qquad \text{in } L^2(\mathbf{R}), \quad (4.2)$$

where a_{n-j} is given by $H_{2k}(x) = \sum_{j=0}^{k} a_{n-j} x^{2j}$.

Thus (4.2) gives the Fourier-Hermite series representation for the pseudo-differential operator $(-x^{-1}D)^{\nu}$ on the linear space X.

Though some pattern seems to emerge from the formulae (4.1), it is fairly hard to predict the general formula for the lower triangular matrix representation of $(-x^{-1}D)^{\nu} H_{2n}(x)e^{-x^2/2}$. But if we define the Hermite polynomials through the generating function

$$H_n(x) = (-1)^n e^{x^2/2} D^n e^{-x^2/2} \qquad \text{for } n > 0 \text{ and } H_o(x) = 1, \qquad (4.3)$$

we have

THEOREM 4.1. *Let $H_{2n}(x)$ be the Hermite polynomial, generated by* (4.3), *of order* $2n$; *then*

$$(-x^{-1}D)^{\nu}(H_{2n}(x)e^{-x^2/2}) = (X^2 - 2^{\nu \cdot})^n e^{-x^2/2} \qquad (4.4)$$

where as before $X^i = H_i(x)$, $X^i \cdot X^j = H_{i+j}(x)$, and $\nu_*^j = \nu(\nu + 1)(\nu + 2) \cdots (\nu + j - 1)$ for each $j = 1, 2, 3, \ldots, \nu \neq -1$, and when $\nu = -1$, $\nu_*^j = 0$ for $j \geq 2$.

Thus each of $(-x^{-1}D)^{\nu} H_{2n}(x)e^{-x^2/2}$ has a lower triangular matrix representation. From (4.4) *we see that the Hermite function $H_{2n}(x)e^{-x^2/2}$ satisfies the recurrence relation*

$$[(-x^{-1}D)^{\nu} - 1]\, y_{2n}(x) = \sum_{m=1}^{n} ((-2)^m/m) \prod_{j=0}^{m-1} (n - j)(\nu + j)\, y_{2(n-m)}(x). \qquad (4.5)$$

Acknowledgment

The author thanks Professor V. V. Menon of the Department of Applied Mathematics, IT-BHU, for his suggestions.

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