BULL. AUSTRAL. MATH. SOC. 49D37
VOL. 31 (1985), 369-375.

A NOTE ON ARCWISE CONNECTED SETS AND FUNCTIONS

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We introduce a new class of generalized arcwise connected functions and discuss their basic properties. Our generalization is illustrated by an example and an application is given for a mathematical programming problem involving this new class of functions.

1. Introduction

Following Ortega and Rheinboldt ([2]), Avriel and Zang ([1]) expanded the classes of generalized convex functions. They extended the concepts of convexity, quasiconvexity and pseudoconvexity for functions to corresponding forms of arcwise connectedness and characterized their local-global minimum properties. Singh [3] discussed some basic properties for arcwise connected sets and functions. In the present note we introduce another type of generalized class of arcwise connected (GCN) functions and discuss some of their basic properties. We give an example to support our generalization of arcwise connected functions. Also we include an application of the GCN class of functions in the field of nonlinear programming.

Received 5 November 1984.

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DEFINITION 2.1. The set $X \subset R^n$ is said to be arcwise connected (AC) if, for every pair of points $x_0^1, x^2 \in X$, there exists a continuous real valued function H_{x^1,x^2} called an arc, defined on the unit interval $[0, 1] \subset R$ with values in X such that

$$H_{x^1,x^2}(0) = x^1 \quad \text{and} \quad H_{x^1,x^2}(1) = x^2 .$$

DEFINITION 2.2. Let $X \subset R^n$ be an arcwise connected set, and let f be a real valued function defined on X. We say that f is arcwise connected (CN) if, for every $x^1, x^2 \in X$ there exists an arc H_{x^1,x^2} in X such that

$$f\left(H_{x^1,x^2}(\theta)\right) \le (1-\theta)f\left(x^1\right) + \theta f\left(x^2\right) , \text{ for all } \theta , \quad 0 \le \theta \le 1 .$$

We give the following definition for a generalized arcwise connected function.

DEFINITION 2.3. Let $X \subset R^n$ be an arcwise connected set, and let f be a real valued function defined on X. We say that f is generalized arcwise connected (GCN) if, for every $x^1, x^2 \in X$, there exists an arc H_{x^1,x^2} in X such that

$$f\left(H_{x^1,x^2}(\theta)\right) \le \max\left\{f\left(x^1\right), f\left(x^2\right)\right\} \text{ for all } \theta , \quad 0 \le \theta \le 1 .$$

The following results (of the type in [3]) are also valid for GCN.

PROPOSITION 2.1. *If a real valued function f is* GCN *on an* AC *set* $X \subset R^n$, *then*

(i) kf *is also* GCN *for* $k \ge 0$,

(ii) $f \pm k$ *is* CN *for any real number* k.

The following result gives a necessary and sufficient conditions for f to be GCN.

THEOREM 2.1. *Let* X *be an arcwise connected subset of* R^n *, let* $f : X \to R$ *. The function* f *is* GCN *if and only if* $S_\alpha = \{x \in X : f(x) \le \alpha\}$ *is* AC *for all* $\alpha \in R$ *.*

Proof. Assume that f if GCN on X . Let $x^1, x^2 \in S_\alpha$. Then $f(x^1) \le \alpha$ and $f(x^2) \le \alpha$, hence $\max\{f(x^1), f(x^2)\} \le \alpha$. Since f is GCN, there is $H_{x^1,x^2}(\theta)$ such that

$$f\left(H_{x^1,x^2}(\theta)\right) \le \max\{f(x^1), f(x^2)\} \le \alpha ,$$

that is, $H_{x^1,x^2}(\theta) \in S_\alpha$. Since α was arbitrary, S_α is AC for each $\alpha \in R$. Now assume that S_α is AC for each $\alpha \in R$. Let $x^1, x^2 \in X$ and let $\theta \in (0, 1)$. Let $\max\{f(x^1), f(x^2)\} = \beta$. Then $x^1, x^2 \in S_\beta$. Since S_β is AC, $H_{x^1,x^2}(\theta) \in S_\beta$, hence $f\left(H_{x^1,x^2}(\theta)\right) \le \beta = \max\{f(x^1), f(x^2)\}$, which shows that f is GCN.

The following example, given in the form of a theorem, illustrates our generalization.

THEOREM 2.2. *If* $f : X \to R$ *and* $g : X \to R$ *are functions satisfying (*X *being* AC*):*

(a) f *is* CN *on* X *and* $g(x) > 0$ *for all* $x \in X$ *;*

(b) g *is* CN *on* X *and* $f(x) \le 0$ *for all* $x \in X$ *, then* f/g *is* GCN *on* X *;*

(c) f *and* g *are* CN *as stated in (a) and (b) with respect to the same arc for each pair of points.*

Proof. By Theorem 2.1 we need to show that the set

$$S_\alpha = \left\{x \in X : \frac{f(x)}{g(x)} \le \alpha\right\}$$

is AC for each $\alpha \in R$. Now

$$S_\alpha = \left\{x \in X : \frac{f(x)}{g(x)} \le \alpha\right\} = \{x \in X : f(x)-\alpha g(x) \le 0\} .$$

Suppose $x^1, x^2 \in S_\alpha$, we need to show that $H_{x^1,x^2}(\theta) \in S_\alpha$. Now for $\alpha > 0$ using condition *(a)*,

$$f\left(H_{x^1,x^2}(\theta)\right) - \alpha g\left(H_{x^1,x^2}(\theta)\right) \le (1-\theta)f\left(x^1\right) + \theta f\left(x^2\right) - \alpha g\left(H_{x^1,x^2}(\theta)\right) \le 0 ;$$

hence

$$H_{x^1,x^2}(\theta) \in S_\alpha .$$

For $\alpha < 0$,

$$\begin{aligned} f\left(H_{x^1,x^2}(\theta)\right) - \alpha g\left(H_{x^1,x^2}(\theta)\right) &\le 0 - \alpha(1-\theta)g\left(x^1\right) - \alpha\theta g\left(x^2\right) \\ &\le 0 + 0 + 0 ; \end{aligned}$$

hence $H_{x^1,x^2}(\theta) \in S_\alpha$ in this case also. Here we also use the property *(c)* in our proof.

We have the following lemma on boundedness of a function in GCN.

LEMMA 2.2. *Suppose*

(i) f *is a real valued* GCN *function defined on an* AC *set* $X \in R^n$,

(ii) f *is bounded above by* M *on a neighbourhood* $N_d\left(x^0\right)$ *of* $x^0 \in X$,

(iii) f *is bounded below by* m *on a subneighbourhood* $N_\varepsilon\left(x^0\right)$ *of* x^0.

Then f *is bounded on* $N_\varepsilon\left(x^0\right)$.

Proof. In view of Proposition 2.2 (iii) of [3] and Proposition 2.1 *(ii)* of this note we may assume that $x^0 = 0$ and $f(0) = 0$. If $\varepsilon = d$, there is nothing to prove. So we assume that $0 < \varepsilon < d$. Let $x \in N_d(0)$. Since f is GCN on X, there exists an arc $H_{\bar{0},x}$ in X such that $x^0 \in X$, $f\left(H_{\bar{0},x}(\theta)\right) \le \max\{f(\bar{0}), f(x)\} = f(x)$. Since $H_{0,x}$ is continuous, for $0 < \varepsilon < d$, there exists $\delta > 0$ such that $\theta \in \{\alpha : 0 \le \alpha < \delta\}$, $H_{\bar{0},x}(\theta) \in N_\varepsilon(\bar{0})$.

For some positive integer n, $\varepsilon/n < \delta$ implies

$$H_{\bar{0},x}(\varepsilon/n) \in N_\varepsilon(0) .$$

Letting

$$\hat{x} = H_{\bar{0},x}(\varepsilon/n) ,$$

we have

$$f(\hat{x}) \le \max\{f(\bar{0}), f(x)\} = f(x) .$$

Therefore

$$f(x) \ge f(\hat{x}) \ge m ,$$

since $m \le f(y)$ for all y in $N_\varepsilon(\bar{0})$. Hence, for $x \in N_d(\bar{0})$,

$$m \le f(x) \le M ;$$

that is, f is bounded on $N_\varepsilon(\bar{0})$.

DEFINITION 2.4. A function $F : X \to R$ on a convex set $X \in R^n$ is called quasi-convex if $F\left((1-\lambda)x^1+\lambda x^2\right) \le \max\left\{F\left(x^1\right), F\left(x^2\right)\right\}$ for any pair $x^1, x^2 \in X$.

PROPOSITION 2.3. *Suppose*

(i) *f is a real valued function defined on an arcwise connected set $X \subset R^n$,*

(ii) *for $x^1, x^2 \in X$, let H_{x^1,x^2} be an arc in X, and let*

$$F(\theta) = f\left(H_{x^1,x^2}(\theta)\right) , \quad 0 \le \theta \le 1 ;$$

then f is GCN with respect to the arc H_{x^1,x^2} if F is quasi-convex.

Proof. Suppose F is quasi-convex. Then

$$f\left(H_{x^1,x^2}(\theta)\right) = F(\theta) = F\left(\theta.1+(1-\theta).\bar{0}\right)$$
$$\le \max\{F(1),\ F(\bar{0})\}$$
$$= \max\left\{f\left(x^2\right),\ f\left(x^1\right)\right\}\ .$$

DEFINITION 2.5. Let $S \subset R^n$ be a nonempty AC set. Let $f : S \to R$. The function f is said to be strictly GCN if, for any two points $x_1, x_2 \in S$ such that $f\left(x_1\right) \ne f\left(x_2\right)$ and for all $\theta \in (0, 1)$, we have

$$f\left(H_{x_1,x_2}(\theta)\right) < \max\left\{f\left(x_1\right),\ f\left(x_2\right)\right\}\ .$$

THEOREM 2.4. *Let the programming problem be* $\min f(x)$, $x \in S$, *where* $S \subset R^n$ *is a nonempty* AC *set and let* $f : R^n \to R$ *be strictly* GCN *on* S. *If* x^* *is a local minimum of* $f(x)$, *then it is the global* min.

Proof. Assume to the contrary that x^* is not the global minimum. Then there exists a $x_0 \in S$ such that $f\left(x_0\right) < f(x^*)$. Since S is GCN, $H_{x_0,x^*}(\theta) \in S$. Since x^* is a local minimum there exists a $\delta \in (0, 1)$ such that

$$f(x^*) \le f\left(H_{x_0,x^*}(\lambda)\right) \quad \text{for all} \quad \lambda \in (0, \delta)\ .$$

Since f is strictly GCN, for each $\lambda \in (0, 1)$, we have

$$f\left(H_{x_0,x^*}(\lambda)\right) < \max\left\{f\left(x_0\right),\ f(x^*)\right\} = f(x^*)\ ,$$

which is a contradiction. Hence our proof is complete.

3.

Consider the following programming problem:

$$\text{(P)} \qquad \min_{x \in R^n} f(x) \quad \text{subject to} \quad g_i(x) \le 0\ , \quad i = 1, 2, \ldots, m\ ,$$

where f is strictly GCN and the g_i's $(i = 1, 2, \ldots, m)$ are GCN. Then we have the following result.

THEOREM 3.1. *Let* x^* *be a local minimum of the problem* (P), *then it is also a global minimum.*

Proof. In view of Theorem 2.4 we need to show that the set

$$S = \left\{x \in R^n : g_i(x) \leq 0, \; i = 1, \ldots, m\right\}$$

is an AC set in R^n. Now since, for fixed i, $g_i(x)$ is GCN, therefore, if $x_1, x_2 \in S$, there is a function $H_{x_1,x_2}(\theta)$ such that

$$g_i\left(H_{x_1,x_2}(\theta)\right) \leq (1-\theta)g_i\left(x_1\right) + \theta g_i\left(x_2\right) = 0 ,$$

which is also true for $i = 1, 2, \ldots, m$. Hence $H_{x_1,x_2}(\theta) \in S$ and hence S is an AC set in R^n. But $f(x)$ is strictly GCN and hence invoking Theorem 2.4 we have the result.

References

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