Journal of Mathematical Physics

Further comments on the behavior of acceleration waves of arbitrary shape

Cite as: J. Math. Phys. **22**, 683 (1981); https://doi.org/10.1063/1.524977
Published Online: 04 June 1998

V. D. Sharma, and V. V. Menon

Further comments on the behavior of acceleration waves of arbitrary shape

V. D. Sharma[a] and V. V. Menon[b]
Department of Aerospace Engineering, University of Maryland, College Park, Maryland 20742

(Received 25 February 1980; accepted for publication 12 December 1980)

Converging waves with nonzero initial critical amplitude are completely characterized. It is shown that for a converging wave a necessary and sufficient condition for the initial critical amplitude to be zero is that the converging wave is spherical.

PACS numbers: 03.40.Kf

I. INTRODUCTION

Bowen and Chen[1] have discussed the growth and decay behavior of converging and diverging waves, and they have completely characterized diverging waves with nonzero initial critical amplitude; but they do not refer to the corresponding characterization for converging waves. The purpose of the present paper is to characterize these converging waves completely. It is shown that all spherically converging waves will grow into shock waves before the formation of the focus, no matter how small their initial amplitude.

II. BEHAVIOR OF CONVERGING WAVES

The differential equation governing the amplitude $a(t)$ of an acceleration wave propagating into a homogeneous material medium, assumed to be at rest initially, is of the form[1]

$$\frac{da}{dt} = -(\mu_0 - \tfrac{1}{2}u_n\bar{K})a + \beta_0 a^2, \tag{2.1}$$

where μ_0 is a constant depending on the type of material under study and the uniform conditions prevailing ahead of the wave, β_0 is a nonzero constant depending solely on the elastic response of the material, u_n is the constant normal speed (taken to be positive), $\bar{K}$ is the mean curvature of the wavefront at any time t expressed as

$$\bar{K} = (\bar{K}_0 - 2K_0u_nt)/(1 - \bar{K}_0u_nt + K_0u_n^2t^2), \tag{2.2}$$

where $\bar{K}_0 = k_1 + k_2$ is the initial mean curvature and $K_0 = k_1k_2$ is the initial total curvature with k_1 and k_2 being the initial principal curvatures. When k_1 and k_2 are both nonpositive, the wave is divergent; and when one or both the initial principal curvatures are positive, the wave is convergent. In the following discussion, we shall consider only converging waves.

The solution of (2.1) in view of (2.2) can be written as

$$a(t) = [I_1(t)\exp(-\mu_0t)]/[1/a(0) - \beta_0I_2(t)], \tag{2.3}$$

where $a(0)\neq 0$ is the initial amplitude, and the functions $I_1(t)$ and $I_2(t)$ are given by

$$I_1(t) = \{(1 - k_1u_nt)(1 - k_2u_nt)\}^{-1/2}, \tag{2.4}$$

$$I_2(t) = \int_0^t \{(1 - k_1u_n\tau)(1 - k_2u_n\tau)\}^{-1/2}\exp(-\mu_0\tau)d\tau. \tag{2.5}$$

In the analysis of converging waves, where *at least one of the initial principle curvatures is positive*, the integral $I_2(t^*)$ plays a crucial role, where t^* is the smallest positive root of the equation $(1 - k_1u_nt^*)(1 - k_2u_nt^*) = 0$. One can easily show that the integral $I_2(t^*)$ in (2.5) is infinite if and only if both k_1 and k_2 are positive and equal, i.e., $k_1 = k_2 > 0$. For, if $k_1 = k_2 > 0$, by substituting $z = t^* - t$, we find that the singularity $t\to t^*$ in the integrand of $I_2(t^*)$ is of the form $z^{-1}\phi(z)$ as $z\to 0$, where $\phi(z)$ is both bounded and bounded away from zero, and the function z^{-1} is not integrable on any interval $[0,T]$, $T>0$. If $k_1\neq k_2$ and at least one of k_1, k_2 is positive, then the integral $I_2(t^*)$ is finite; this follows from the argument that the singularity as $t\to t^*$ in the integrand of $I_2(t^*)$ is of the type $z^{-\frac{1}{2}}\psi(z)$ as $z\to 0$, where the function $\psi(z)$ is again bounded and bounded away from zero, and the function $z^{-\frac{1}{2}}$ is integrable over every interval $[0,T]$, $T>0$.

Thus for converging waves, irrespective of the sign of μ_0, we have the following two situations:

(i) $k_1\neq k_2$ *and at least one of them is positive*: in this case, when $\operatorname{sgn}a(0) = \operatorname{sgn}\beta_0$, it follows from (2.3) that *not* all converging waves will grow into shock waves, i.e., there exists a critical value of the initial wave amplitude, given by

$$\gamma = [|\beta_0|I_2(t^*)]^{-1}, \tag{2.6}$$

such that waves with initial amplitude less than γ form a focus (i.e., $|a(t)|\to\infty$ as $t\to t^*$), waves with initial amplitude greater than γ form a shock before the focus (i.e., there exists a positive $\hat{t}\,(<t^*)$ given by $I_2(\hat{t}) = [\beta_0a(0)]^{-1}$, such that $|a(t)|\to\infty$ as $t\to\hat{t}$), and waves with initial amplitude equal to γ form a shock and focus simultaneously (i.e., $\hat{t} = t^*$ and $|a(t)|\to\infty$ as $t\to t^*$).

(ii) $k_1 = k_2 > 0$: in this case, which corresponds to a spherically converging wave, the integral $I_2(t^*)$ is infinite and thus, the initial critical amplitude given by (2.6) vanishes; further, it follows from (2.3) that when $\operatorname{sgn}a(0) = \operatorname{sgn}\beta_0$, there exists a positive $\tilde{t} < t^*$, given by

$$\int_0^{\tilde{t}} (1 - k_1u_n\tau)^{-1}\exp(-\mu_0\tau)\,d\tau = 1/[\beta_0a(0)],$$

such that as t approaches $\tilde{t}$, the denominator of (2.3) vanishes, whereas the numerator remains finite, i.e., $|a(t)|\to\infty$ as $t\to\tilde{t}$. This means that all spherically converging waves will grow into shock waves before the formation of the focus, no matter how small be their initial amplitude. This result is, in

[a]On leave from Applied Mathematics Section, I. T., B. H. U., India.
[b]Applied Mathematics Section, I. T., B. H. U., India.

 0022-2488/81/040683-02$01.00

effect, that for a converging wave a necessary and sufficient condition for the initial critical amplitude to be zero is for the converging wave to be spherical.

It is interesting to note that when $\text{sgn}\,a(0) = -\text{sgn}\,\beta_0$, the denominator in (2.3), in both the situations mentioned above, is always bounded away from zero, and $|a(t)| \rightarrow \infty$ as $t \rightarrow t^*$, i.e., in this case all converging waves form a focus only.

ACKNOWLEDGMENTS

The authors are thankful to the referee for making certain points more explicit.

Research support to V.D.S. from the Ministry of Education, Govt. of India, is gratefully acknowledged.

[1]R. M. Bowen and P. J. Chen, J. Math. Phys. **13**, 948 (1972).