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On Multiple-Objective Optimization with Generalized Univexity

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A multiple-objective optimization problem involving generalized univex functions is considered. Kuhn–Tucker type sufficient optimality conditions are obtained for a feasible point to be an efficient or properly efficient solution. Mond–Weir type duality results are obtained. Further, a vector-valued Lagrangian is introduced and certain vector saddlepoint results are presented.

Key Words: univexity; type I function; pseudo-type I function; quasi-type I function; optimality; duality; efficient solutions; properly efficient solutions.

1. INTRODUCTION

Hanson and Mond [5] introduced two new classes of functions called type I and type II functions, which are not only sufficient but also are necessary for optimality in primal and dual problems, respectively. Consider the following nonlinear programming problem,

$$\text{Min } f(x),$$
$$\text{subject to } g(x) \leq 0;$$

$f(x)$ and $g(x)$ are type I objective and constraint functions, respectively, with respect to η at x_0 [5], if there exists a vector function $\eta(x)$, defined for all x, $x_0 \in P = \{x; g(x) \leq 0\}$, such that

$$f(x) - f(x_0) \geqq [\nabla f(x_0)]^t \eta(x, x_0),$$
$$-g(x_0) \geqq [\nabla g(x_0)]^t \eta(x, x_0).$$

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Reuda, Hanson [7] further extended type I functions to pseudo-type I and quasi-type I functions and have obtained sufficient optimality criteria for a nonlinear programming problem involving these functions.

Bector and Singh [1] introduced a new class of functions, called b-vex functions. Optimality and duality results for these functions were proved by Bector, Suneja, and Lalitha [3]. A further generalization was defined by Bector, Suneja, and Gupta [2], called univex functions.

Let X be a nonempty open set in $\mathbb{R}^n$, f: $X \to \mathbb{R}$, η: $X.X \to \mathbb{R}^n$, ϕ: $\mathbb{R} \to \mathbb{R}$, and b: $X \times [0,1] \to \mathbb{R}_+$, $b = b(x, u, \lambda)$. If the function f is differentiable then b does not depend on λ [1, 2].

DEFINITION 1.1. A differentiable function f is said to be univex at $x_0 \in X$ with respect to η, ϕ, and b if $\forall x \in X$ we have

$$b(x, x_0)\phi[f(x) - f(x_0)] \geqq [\nabla f(x_0)]^t \eta(x, x_0).$$

DEFINITION 1.2. A functional f: $X \to \mathbb{R}$ is sublinear if $F(x+y) \leq F(x) + F(y)$ $\forall x, y \in X$ and $F(\alpha x) = \alpha F(x)$ $\forall x \in X$ and every nonnegative real number α.

Recently, Rueda, Hanson, and Singh [8] obtained optimality and duality results for several mathematical programs by combining the concepts of type I and univex functions.

In this article, we consider a multiple objective nonlinear programming problem and we obtain optimality and duality results by combining the concepts of type I, type II, pseudo-type I, quasi-type I, quasi-pseudo-type I, pseudo-quasi-type I, strictly pseudo-quasi-type I, and univex functions.

2. OPTIMALITY CRITERIA

Throughout this article we consider the following multiple-objective primal problem,

$$\text{(VP)} \quad \text{Min } f(x) = \big(f_1(x), f_2(x), \dots, f_p(x)\big), \qquad x \in X \subseteq \mathbb{R}^n,$$

$$\text{subject to } g(x) \leqq 0,$$

when f: $X \to \mathbb{R}^p$ and g: $X \to \mathbb{R}^m$ are differentiable functions on a set $X \subseteq \mathbb{R}^n$ and minimization means obtaining efficient solution of (VP).

Let $P := \{x \colon x \in X,\ g(x) \leq 0\}$. For a feasible point $x^* \in P$, we denote by $I(x^*)$ the set,

$$I(x^*) = \{i \colon g_i(x^*) = 0\}.$$

A feasible solution x^* for (VP) is efficient for (VP) if and only if there is no other feasible x for (VP) such that, for some $i \in \{1, 2, \dots, p\}$,

$$f_i(x) < f_i(x^*),$$
$$f_j(x) \leqq f_j(x^*), \qquad \forall j \neq i.$$

An efficient solution x^* for (VP) is properly efficient for (VP) if there exists a scalar $M > 0$ such that, for each i,

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leqq M,$$

for some j such that $f_j(x) > f_j(x^*)$ whenever x is feasible for (VP) and $f_i(x) < f_i(x^*)$.

Let $x, y \in \mathbb{R}^n$. By $x \leq y$, we mean $x_i \leqq y_i$ $\forall i$; by $x \leq y$, we mean $x_i \leqq y_i$ $\forall i$ and $x_j < y_j$ for at least one j, $1 \leqq j \leqq n$; by $x < y$, we mean $x_i < y_i$ $\forall i$.

In this section, we obtain sufficient optimality conditions for a feasible solution x^* to be efficient or properly efficient for (VP) in the form of the following theorems.

THEOREM 2.1. *Let x^* be (VP)-feasible. Suppose that there exist η, ϕ_0, b_0, and ϕ_1, b_1, $\lambda_0^* \geqq 0$, $i = 1, 2, \dots, p$, $\sum_{i=1}^{p} \lambda_i^* = 1$, $\mu_i^* \geqq 0$, $i \in I$ such that*

$$b_0(x, x^*)\phi_0\left[\sum_{i=1}^{p} \lambda_i^* f_i(x) - \sum_{i=1}^{p} \lambda_i^* f_i(x^*)\right] > \sum_{i=1}^{p} \lambda_i^* [\nabla f_i(x^*)]^T \eta(x, x^*), \tag{2.1}$$

$$-b_1(x, x^*)\phi_1\left[\sum_{i \in I(x^*)} \mu_i^* g_i(x^*)\right] \geqq \sum_{i \in I(x^*)} \mu_i^* [\nabla g_i(x^*)] \eta(x, x^*), \quad \forall\ (VP)\text{-feasible } x \tag{2.2}$$

and

$$\sum_{i=1}^{p} \lambda_i^* \nabla f_i(x^*) + \sum_{i \in I(x^*)} \mu_i^* \nabla g_i(x^*) = 0. \tag{2.3}$$

Further suppose

$$a \leqq 0 \quad \Rightarrow \quad \phi_0(a) \leqq 0, \tag{2.4}$$

$$\phi_1(a) \leqq 0 \quad \Rightarrow \quad a > 0, \tag{2.5}$$

$$b_0(x, x^*) > 0, \qquad b_1(x, x^*) \geqq 0, \tag{2.6}$$

for all feasible x. Then x^* is an efficient solution for (VP).

Proof. Suppose that x^* is not an efficient solution for (VP). Then, there exists a feasible x for (VP) and an index j such that

$$f_j(x) < f_j(x^*),$$
$$f_i(x) \leqq f_i(x^*), \qquad \forall i \neq j.$$

These two inequalities lead to

$$0 \geqq \sum_{i=1}^{p} \lambda_i^* f_i(x) - \sum_{i=1}^{p} \lambda_i^* f_i(x^*).$$

From (2.4) and (2.5) it follows that

$$b_0(x, x^*) \phi_0 \left[\sum_{i=1}^{p} \lambda_i^* f_i(x) - \sum_{i=1}^{p} \lambda_i^* f_i(x^*) \right] \leqq 0.$$

Therefore, by (2.1), we have

$$\sum_{i=1}^{p} \lambda_i^* [\nabla f_i(x^*)]^T \eta(x, x^*) < 0. \tag{2.7}$$

Then, by (2.3), we have

$$\sum_{i \in I(x^*)} \mu_i^* [\nabla g_i(x^*)]^T \eta(x, x^*) \geqq 0. \tag{2.8}$$

From (2.2) and (2.8), we obtain

$$b_1(x, x^*) \phi_1 \left[\sum_{i \in I(x^*)} \mu_i^* g_i(x^*) \right] \leqq 0. \tag{2.9}$$

By (2.5), (2.6), and (2.9) it follows that

$$\sum_{i \in I(x^*)} \mu_i^* g_i(x^*) > 0,$$

which is a contradiction to the (VP) feasibility of x^*, because $\mu_i^* \geqq 0$, $i \in I$. Therefore, x^* is an efficient solution for (VP). ∎

THEOREM 2.2. *Let x^* be (VP)-feasible. Suppose that there exist $\lambda_i^* > 0$, $i = 1, 2, \dots, p$, $\mu_i^* \geqq 0$, $i \in I(x^*)$, η, b_0, b_1, ϕ_0, and ϕ_1 such that*

$$\sum_{i=1}^{p} \lambda_i^* [\nabla f_i(x^*)]^T \eta(x, x^*) \geqq 0$$

$$\Rightarrow \quad b_0(x, x^*) \phi_0 \left[\sum_{i=1}^{p} \lambda_i^* f_i(x) - \sum_{i=1}^{p} \lambda_i^* f_i(x^*) \right] \geqq 0, \tag{2.10}$$

and

$$-b_1(x, x^*)\,\phi_1\left[\sum_{i \in I(x^*)} \mu_i^* g_i(x^*)\right] \leqq 0$$

$$\Rightarrow \quad \sum_{i \in I(x^*)} \mu_i^* [\nabla g_i(x^*)]^T \eta(x, x^*) \leqq 0, \tag{2.11}$$

for all (*VP*)*-feasible* x, *and* (2.3) *of Theorem* 2.1 *hold. Further, suppose*

$$a \geqq 0 \quad \Rightarrow \quad \phi_1(a) \geqq 0, \tag{2.12}$$

$$\phi_0(a) \geqq 0 \quad \Rightarrow \quad a \geqq 0, \tag{2.13}$$

$$b_1(x, x^*) > 0, \qquad b_0(x, x^*) \geqq 0, \tag{2.14}$$

$\forall$ *feasible* x. *Then* x^* *is a properly efficient solution for* (*VP*).

Proof. Because $g_I(x^*) = 0$, $\mu_i^* \geqq 0$, $i \in I(x^*)$, $\sum_{i \in I(x^*)} \mu_i^* g_i(x^*) \geqq 0$, and $b_1(x, x^*) \geqq 0$ and (2.12) and (2.11), we have $\sum_{i \in I(x^*)} \mu_i^* [\nabla g_i(x^*)]^T \eta(x, x^*) \leqq 0$, which on using (2.3) and (2.10) yields $b_0(x, x^*)\phi_0[\sum_{i=1}^p \lambda_i^* f_i(x) - \sum_{i=1}^p \lambda_i^* f_i(x)] \geqq 0$. By (2.13) and (2.14), we get $\sum_{i=1}^p \lambda_i^* f_i(x) \geqq \sum_{i=1}^p \lambda_i^* f_i(x^*)$. Therefore, by Theorem 1 of Geoffrion [4], x^* is a properly efficient solution for (VP). ∎

THEOREM 2.3. *Let* x^* *be* (*VP*)*-feasible. Suppose that there exist* $\lambda_i^* \geqq 0$, $i = 1, 2, \ldots, p$, $\sum_{i=1}^p \lambda_i^* = 1$, $\mu_i^* \geqq 0$, $i \in I(x^*)$, η, b_0, b_1, ϕ_0, *and* ϕ_1 *such that*

$$\sum_{i=1}^{p} \lambda_i^* [\nabla f_i(x^*)]^T \eta(x, x^*) \geqq 0$$

$$\Rightarrow \quad b_0(x, x^*)\,\phi_0\left[\sum_{i=1}^{p} \lambda_i^* f_i(x) - \sum_{i=1}^{p} \lambda_i^* f_i(x^*)\right] > 0, \tag{2.15}$$

or equivalently,

$$b_0(x, x^*)\,\phi_0\left[\sum_{i=1}^{p} \lambda_i^* f_i(x) - \sum_{i=1}^{p} \lambda_i^* f_i(x^*)\right] \leqq 0$$

$$\Rightarrow \quad \sum_{i=1}^{p} \lambda_i^* [\nabla f_i(x^*)]^T \eta(x, x^*) < 0. \tag{2.16}$$

and

$$-b_1(x, x^*)\,\phi_1\left[\sum_{i \in I(x^*)} \mu_i^* g_i(x^*)\right] \leqq 0$$

$$\Rightarrow \quad \sum_{i \in I(x^*)} \mu_i^* [\nabla g_i(x^*)]^T \eta(x, x^*) \leqq 0, \tag{2.17}$$

for all (VP)*-feasible* x, *and* (2.8) *of Theorem* 2.1 *hold. Further, suppose,*

$$a \leqq 0 \quad \Rightarrow \quad \phi_0(a) \leqq 0, \tag{2.18}$$

$$a \geqq 0 \quad \Rightarrow \quad \phi_1(a) \geqq 0, \tag{2.19}$$

$$b_0(x, x^*) > 0, \qquad b_1(x, x^*) \geqq 0. \tag{2.20}$$

Then, x^* *is an efficient solution for* (VP).

Proof. Suppose that x^* is not efficient for (VP). Then, there exist a feasible x for (VP) such that $\sum_{i=1}^{p} \lambda_i^* f_i(x) - \sum_{i=1}^{p} \lambda_i^* f_i(x^*) \leqq 0$, which on using (2.18), (2.20), and (2.16) yields,

$$\sum_{i=1}^{p} \lambda_i^* [\nabla f_i(x^*)]^T \eta(x, x^*) < 0. \tag{2.21}$$

Because $\sum_{i \in I(x^*)} \mu_i^* g_i(x^*) \geqq 0$, by (2.20), (2.19), and (2.17), we have

$$\sum_{i \in I(x^*)} \mu_i^* [\nabla g_i(x^*)]^T \eta(x, x^*) \leqq 0. \tag{2.22}$$

Now, on adding (2.21) and (2.22), we obtain a contradiction to (2.3). Hence, x^* is an efficient solution for (VP). ∎

Remark 2.1. Proceeding along similar lines as in Theorem 2.2, it can be easily seen that x^* becomes properly efficient for (VP) in the preceding theorem if $\lambda_i > 0$, $\forall i = 1, 2, \ldots, p$.

THEOREM 2.4. *Let* x^* *be* (VP)*-feasible. Suppose that there exist,* $\lambda_i^* \geqq 0$, $i = 1, 2, \ldots, p$, $\sum_{i=1}^{p} \lambda_i^* = 1$, $\mu_i^* \geqq 0$, $i \in I(x^*)$, b_0, b_1, ϕ_0, ϕ_1, *and* η *such that* (2.3) *of Theorem* 2.1 *holds, and if* $I(x^*) \neq \phi$ *and*

$$b_0(x, x^*) \phi_0 \left[\sum_{i=1}^{p} \lambda_i^* f_i(x) - \sum_{i=1}^{p} \lambda_i^* f_i(x^*) \right] \leqq 0$$
$$\Rightarrow \quad \sum_{i=1}^{p} \lambda_i^* [\nabla f_i(x^*)]^T \eta(x, x^*) \leqq 0, \tag{2.23}$$

and

$$\sum_{i \in I(x^*)} \mu_i^* [\nabla g_i(x^*)]^T \eta(x, x^*) \geqq 0$$
$$\Rightarrow \quad -b_1(x, x^*) \phi_1 \left[\sum_{i \in I(x^*)} \mu_i^* g_i(x^*) \right] > 0, \tag{2.24}$$

for all feasible x. Further suppose,

$$a \leqq 0 \quad \Rightarrow \quad \phi_0(a) \leqq 0, \tag{2.25}$$

$$\phi_1(a) < 0 \quad \Rightarrow \quad a > 0, \tag{2.26}$$

$$b_0(x, x^*) > 0, \qquad b_1(x, x^*) \geqq 0. \tag{2.27}$$

Then x^* is an efficient solution for (VP).

Proof. The proof is similar to the proof of Theorem 2.3 ∎

Remark 2.2. Proceeding along similar lines as in Theorem 2.2, it can be easily seen that x^* becomes properly efficient for (VP) in the previous theorem if $\lambda_i^* > 0, \forall i = 1, 2, \dots, p$.

3. DUALITY

In this section we consider the Mond–Weir type dual and generalize duality results of Rueda, Hanson, and Singh [8] as well as Kaul, Suneja, and Srivastavva [6] under weaker univexity assumptions.

Consider the following Mond–Weir type dual of (VP),

$$\text{(VD)} \quad \text{Max } f(u),$$

$$\text{subject to } \sum_{i=1}^{p} \lambda_i \nabla f_i(u) + \sum_{j=1}^{m} \mu_j \nabla g_j(u) = 0, \tag{3.1}$$

$$\sum_{j=1}^{m} \mu_j g_j(u) \geqq 0, \tag{3.2}$$

$$\mu_j \geqq 0, \qquad j = 1, 2, \dots, m, \tag{3.3}$$

$$\lambda_i \geqq 0, \qquad i = 1, 2, \dots, p, \qquad \sum_{i=1}^{p} \lambda_i = 1, \tag{3.4}$$

where $e = (1, 1, \dots, 1) \in \mathbb{R}^p$.

Assuming (f_i, g) to type I, semistrictly type I pseudo-quasi-type I etc. for the same η, Kaul, Suneja, and Srivastava [6] established various duality results for (VP) and (VD). We shall generalize various duality results for (VP) and (VD) by combining univex and type I and its generalizations, these generalizing results of Rueda, Hanson, and Singh [8] and as a byproduct the results of Kaul, Suneja, and Srivastava [6].

THEOREM 3.1 (Weak Duality). *Let x be feasible for* (*VP*) *and let the triplet* (u, λ, μ) *be feasible for* (*VD*). *Let for* $i = 1, 2, \ldots, p$, $\lambda_i > 0$, $\mu_j \geqq 0$, $j = 1, 2, \ldots, m$, η, b_0, b_1, ϕ_0, ϕ_1 *such that*

$$b_0(x,u)\phi_0\left[\sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \lambda_i f_i(u)\right] \geqq \sum_{i=1}^{p} \lambda_i [\nabla f_i(u)]^T \eta(x,u), \quad (3.5)$$

and

$$-b_1(x,u)\phi_1\left[\sum_{i=1}^{p} \mu_j g_j(u)\right] \geqq \sum_{j=1}^{m} \mu_j [\nabla g_i(u)]^T \eta(x,u), \quad (3.6)$$

at u over P;

Further suppose,

$$a \leqq 0 \quad \Rightarrow \quad \phi_0(a) \leqq 0, \quad (3.7)$$

$$\phi_1(a) \leqq 0 \quad \Rightarrow \quad a > 0, \quad (3.8)$$

$$b_0(x,u) > 0, \qquad b_1(x,u) \geqq 0. \quad (3.9)$$

Then $f(x) \nleq f(u)$.

Proof. The proof is similar to the proof of Theorem 2.1 ∎

THEOREM 3.2 (Weak Duality). *Let x be feasible for* (*VP*) *and let the triplet* (u, λ, μ) *be feasible for* (*VD*). *Let either* (*a*) *or* (*b*) *Hold*:

(a) *for* $i = 1, 2, \ldots, p$, $\lambda_i > 0$, *and* $j = 1, 2, \ldots, m$, $\mu_j \geqq 0$ *and there exist* η, b_0, b_1, ϕ_0, *and* ϕ_1 *such that*

$$\sum_{i=1}^{p} \lambda_i [\nabla f_i(u)]^T \eta(x,u) \geqq 0$$

$$\Rightarrow \quad b_0(x,u)\phi_0\left[\sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \lambda_i f_i(u)\right] \geqq 0, \quad (3.10)$$

(*or equivalently,*

$$b_0(x,u)\phi_0\left[\sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \lambda_i f_i(u)\right] < 0$$

$$\Rightarrow \quad \sum_{i=1}^{p} \lambda_i [\nabla f_i(u)]^T \eta(x,u) < 0),$$

and

$$-b_1(x,u)\phi_1\left[\sum_{j=1}^{m}\mu_j g_j(u)\right] \leqq 0$$
$$\Rightarrow \quad \sum_{j=1}^{m}\mu_j[\nabla g_j(u)]^T\eta(x,u) \leqq 0, \tag{3.11}$$

at u over p;

Further, suppose

$$a<0 \quad \Rightarrow \quad \phi_0(a)<0, \tag{3.12}$$
$$a \geqq 0 \quad \Rightarrow \quad \phi_1(a) \geqq 0, \tag{3.13}$$
$$b_0(x,u)>0, \qquad b_1(x,u) \geqq 0. \tag{3.14}$$

(b) *for* $i=1,2,\ldots,p$, $\lambda_i>0$, *and* $j=1,2,\ldots,m$, $\mu_j \geqq 0$, *and there exist* η, b_0, b_1, ϕ_0, *and* ϕ_1 *such that*

$$b_0(x,u)\phi_0\left[\sum_{i=1}^{p}\lambda_i f_i(x) - \sum_{i=1}^{p}\lambda_i f_i(u)\right] \leqq 0$$
$$\Rightarrow \quad \sum_{i=1}^{p}\lambda_i[\nabla f_i(u)]^T\eta(x,u) \leqq 0, \tag{3.15}$$

and

$$\sum_{j=1}^{m}\mu_j[\nabla g_i(u)]^T\eta(x,u) \geqq 0$$
$$\Rightarrow \quad -b_1(x,u)\phi_1\left[\sum_{j=1}^{m}\mu_j g_j(u)\right] > 0, \tag{3.16}$$

(*or equivalently,*

$$-b_1(x,u)\phi_1\left[\sum_{j=1}^{m}\mu_j g_j(u)\right] \leqq 0$$
$$\Rightarrow \quad \sum_{j=1}^{m}\mu_j[\nabla g_i(u)]^T\eta(x,u) < 0),$$

at u over P.

Further suppose,

$$a \leqq 0 \quad \Rightarrow \quad \phi_0(a) \leqq 0, \tag{3.17}$$

$$a \geqq 0 \quad \Rightarrow \quad \phi_1(a) \geqq 0, \tag{3.18}$$

$$b_0(x,u) \geqq 0, \qquad b_1(x,u) \geqq 0. \tag{3.19}$$

Then $f(x) \nleq f(u)$.

Proof. (a) If possible, let $f(x) \leq f(u)$. Then, there exists an index i_0 such that

$$f_{i_0}(x) < f_{i_0}(u),$$

$$f_i(x) \leqq f_i(u), \qquad \forall i \neq i_0.$$

Because $\lambda_i > 0$, $i = 1, 2, \dots, p$, the foregoing inequalities yield

$$\sum_{i=1}^{p} \lambda_i f_i(x) < \sum_{i=1}^{p} \lambda_i f_i(u).$$

On using (3.12) and (3.14), from the earlier strict inequality, we have

$$b_0(x,u)\phi_0\left[\sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{j=1}^{p} \lambda_i f_i(u)\right] < 0.$$

Using (3.10) from the previous strict inequality, we have

$$\sum_{i=1}^{p} \lambda_i \left[\nabla f_i(u)\right]^T \eta(x,u) < 0, \qquad \forall x \in P. \tag{3.20}$$

Using (3.2), (3.13), and (3.14), we obtain

$$-b_1(x,u)\phi_1\left[\sum_{j=1}^{m} \mu_j g_j(u)\right] \leqq 0.$$

Now, by using (3.11), the foregoing inequality yields

$$\sum_{j=1}^{m} \mu_j \left[\nabla g_j(u)\right]^T \eta(x,u) \leqq 0. \tag{3.21}$$

On adding (3.20) and (3.21), we obtain

$$\sum_{i=1}^{p} \lambda_i \left[\nabla f_i(u)\right]^T \eta(x,u) + \sum_{i=1}^{p} \mu_j \left[\nabla g_j(u)\right]^T \eta(x,u) < 0,$$

which contradicts (3.1). Hence,

$$f(x) \nleq f(u).$$

(b) If possible, let $f(x) \leq f(u)$. Then, there exists an index i_0 such that

$$f_{i_0}(x) < f_{i_0}(u),$$

$$f_i(x) \leqq f_i(u), \qquad \forall i \neq i_0.$$

Now $\lambda_i \geqq 0$, and $\sum_{i=1}^{p} \lambda_i = 1$, therefore, from the preceding inequalities, we get

$$\sum_{i=1}^{p} \lambda_i f_i(x) \leqq \sum_{i=1}^{p} \lambda_i f_i(u),$$

i.e.,

$$\sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \lambda_i f_i(u) \leqq 0.$$

Then by (3.17), (3.19), and the previous inequality, we have

$$b_0(x,u)\phi_0\left[\sum_{i=1}^{p} \lambda_i f_i(x) - \sum_{i=1}^{p} \lambda_i f_i(u)\right] \leqq 0.$$

Using (3.15), the foregoing inequality yields

$$\sum_{i=1}^{p} \lambda_i [\nabla f_i(u)]^T \eta(x,u) \leqq 0. \tag{3.22}$$

Again, from (3.2), (3.18), and (3.19), we obtain

$$-b_1(x,u)\phi_1\left[\sum_{j=1}^{m} \mu_j g_j(u)\right] \leqq 0.$$

Using (3.16), we get

$$\sum_{j=1}^{m} \mu_j [\nabla g_j(u)]^T \eta(x,u) < 0. \tag{3.23}$$

Now, on adding (3.22) and (3.23), we obtain,

$$\sum_{i=1}^{p} \lambda_i [\nabla f_i(u)]^T \eta(x,u) + \sum_{j=1}^{m} \mu_j [\nabla g_j(u)]^T \eta(x,u) < 0,$$

which contradicts (3.1). Hence,

$$f(x) \nleq f(u).$$

∎

THEOREM 3.3 (Strong Duality). *If x^* is a properly efficient solution for (VP) at which a constraint qualification is satisfied, then there exists $(\lambda^*, \mu^*) \in \mathbb{R}^p \times \mathbb{R}^m$ such that (x^*, λ^*, μ^*) is (VD)-feasible and the values of the objective functions for (VP) and (VD) are equal at x^* and (x^*, λ^*, μ^*), respectively. Furthermore, if for all (VP)-feasible x and (VD)-feasible (u, λ, μ) the hypotheses of Theorem 3.2(a) are satisfied, then (x^*, λ^*, μ^*) is properly efficient for (VD).*

Proof. Because a constraint qualification is satisfied at x^* then there exist scalars $\lambda_i^* \geqq 0$, $i = 1, 2, \dots, p$, $\sum_{i=1}^p \lambda_i^* = 1$, $\mu_j^* \geqq 0$, $j = 1, 2, \dots, m$ such that

$$\sum_{i=1}^{p} \lambda_i^* \nabla f_i(x^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0, \tag{3.24}$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0. \tag{3.25}$$

Therefore (x^*, λ^*, μ^*) is feasible for (VD).

Now, the proper efficiency of (x^*, λ^*, μ^*) follows as in Theorem 4 of Weir [9] by using part (a) of the weak duality Theorem 3.2. ∎

THEOREM 3.4. *Let x^* be an efficient solution for (VP) at which the Kuhn–Tucker constraint (u, λ, μ) of (VD), $(\sum_{i=1}^p \lambda_i f_i, \sum_{j=1}^m \mu_j g_j)$ satisfies strict inequalities (3.10) and (3.11) at u over P, then there exists $(\lambda^*, \mu^*) \in \mathbb{R}^p \times \mathbb{R}^m$ such that (x^*, λ^*, μ^*) is efficient for (VD) and the objective function values of (VP) and (VD) are equal.*

Proof. Because the Kuhn–Tucker constraint qualification is satisfied at x^* then there exist scalars $\lambda_i^* > 0$, $i = 1, 2, \dots, p$ and $\mu_i^* \geqq 0$, $i \in I(x^*)$, such that (3.24) and (3.25) hold. The scalars $\lambda_i^* > 0$ may be normalized according to $\sum_{i=1}^p \lambda_i^* = 1$. Setting $\mu_i^* = 0$, $i \notin I(x^*)$, then gives that the triplet (x^*, λ^*, μ^*) is not efficient, then there exist a feasible (u, λ, μ) for (VD) and an index i_0 such that

$$f_{i_0}(x^*) < f_{i_0}(u),$$

$$f_i(x) \leqq f_i(u), \qquad \forall i \neq i_0.$$

On using (3.25), we obtain a contradiction to part (b) of the weak duality Theorem 3.2 for feasible solutions x^* for (VP) and (u, λ, μ) for (VD). Hence, (x^*, λ^*, μ^*) is efficient for (VD). ∎

THEOREM 3.5. *Suppose that there exists a feasible x^* for (VP) and (x^*, λ^*, μ^*) for (VD) such that*

$$f_i(x^*) = f_i(u^*), \qquad \forall i = 1, 2, \dots, p. \tag{3.26}$$

If $\lambda_i^* > 0$, *for* $i = 1, 2, \ldots, p$ *and* $(\sum_{i=1}^{p} \lambda_i^* f_i, \sum_{j=1}^{m} \mu_j g_j)$ *satisfy* (3.10) *and* (3.11) *at* u^* *over* P, *then* x^* *is properly efficient for* (VP). *Also if for each feasible* (u, λ, μ) *of* (VD), $(\sum_{i=1}^{p} \lambda_i^* f_i, \sum_{j=1}^{m} \mu_j^* g_j)$ *satisfy* (3.10) *and* (3.11) *at* u *over* P *and suppose* (3.12)–(3.14) *hold, then* (x^*, λ^*, μ^*) *is properly efficient for* (VD).

Proof. The proof of this theorem is similar to that of Theorem 4.5 of Kaul, Suneja, and Srivastava [6]. ∎

4. VECTOR LAGRANGIAN AND SADDLE POINT ANALYSIS

In this section we give as a consequence of Theorem 2.1, a Lagrange multipliers theorem and consider saddle point of the vector Lagrangian function.

THEOREM 4.1. *If Theorem* 2.1 *holds, then equivalent multiobjective programming problem* (EVP) *for* (VP) *is given by*

$$(\text{EVP}) \quad V\text{-}Minimize \left(f_1(x) + \mu^T g(x), \ldots, f_p(x) + \mu^T g(x) \right)$$

$$\text{subject to } \mu_j g_j(x^*) = 0, \qquad j = 1, 2, \ldots, m,$$

$$\mu_j \geq 0, \qquad j = 1, 2, \ldots, m.$$

Proof. Let x^* be an efficient solution of (VP), from the (KT) optimality conditions, we have

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*) = 0, \tag{4.1}$$

and

$$\mu_j g_j(x^*) = 0, \qquad j = 1, 2, \ldots, m. \tag{4.2}$$

Using (4.2) in (4.1), we get

$$\sum_{i=1}^{p} \lambda_i \left[\nabla f_i(x^*) + \mu^T g(x^*) \right] + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*) = 0. \tag{4.3}$$

If (x^*, λ^*, μ^*) is not properly efficient for (VD), then there exists a feasible (u, λ, μ) of (VD) and an index i such that

$$f_i(u) - f_i(u^*) > M\left(f_j(u^*) - f_j(u) \right),$$

$\forall M > 0$ and $\forall j$ such that

$$f_j(u^*) > f_j(u),$$

whenever

$$f_i(u^*) < f_i(u).$$

On using (3.26), we get

$$f_i(u) - f_i(x^*) > M\big(f_j(x^*) - f_j(u)\big),$$

$\forall M > 0$ and $\forall j$ such that

$$f_j(x^*) > f_j(u),$$

whenever

$$f_i(x^*) < f_i(u).$$

Now

$$\sum_{i=1}^{p} \lambda_i^* \big[f_i(x^*) - f_i(u)\big] < 0,$$

which contradicts the weak duality for feasible solutions x^* of (VP) and feasible solution (u, λ, μ) of (VD) by Theorem 3.2(a). Thus, (x^*, λ^*, μ^*) is properly efficient for (VD). Now, applying the arguments of Theorem 2.1 by replacing f_i by $f_i(\cdot) + \mu^T G(\cdot)$ yields the result. ∎

We now introduce the vector valued Lagrangian function and we study its saddle point. Theorem 4.1 suggests the vector valued Lagrangian function $L(x, \mu)$ as $L\colon X \times \mathbb{R}_+^m \to \mathbb{R}^p$ given by

$$L(x, \mu) = \big(L_1(x, \mu), L_2(x, \mu), \dots, L_p(x, \mu)\big),$$

where

$$L_i(x, \mu) = f_i(x) + \mu^T g(x), \qquad i = 1, 2, \dots, p.$$

DEFINITION 4.1. A point $(x^*, \mu^*) \in X \times \mathbb{R}_+^m$ is said to be a vector saddle point of the vector valued Lagrangian function $L(x, \mu)$ if it satisfies the following conditions,

$$L(x^*, \mu) \ngeq L(x^*, \mu^*), \qquad \forall \mu \in \mathbb{R}_+^m, \tag{4.4}$$

$$L(x^*, \mu^*) \ngeq L(x, \mu^*), \qquad \forall x \in X. \tag{4.5}$$

THEOREM 4.2. *If (x^*, μ^*) is a vector saddle point of $L(x, \mu)$, then x^* is a properly efficient solution of (VP).*

Proof. Because (x^*, μ^*) is a vector saddle point of $L(x, \mu)$, therefore, (4.4) implies that $L_i(x^*, \mu) \leq L_i(x^*, \mu^*)$ for at least one $i = 1, 2, \ldots, p$, $\forall \mu \in \mathbb{R}^m_+$,

$$\Rightarrow \quad f_i(x^*) + \mu^T g(x^*) \leq f_i(x^*) + \mu^{*T} g(x^*),$$

$$\text{for at least one } i \text{ and } \forall \mu \in \mathbb{R}^m_+,$$

$$\Rightarrow \quad (\mu - \mu^*)^T g(x^*) \leqq 0, \qquad \forall \mu \in \mathbb{R}^m_+. \tag{4.6}$$

For any $j = 1, 2, \ldots, m$, set

$$\mu_k = \bar{\mu}_k, \quad \text{for } k = 1, 2, \ldots, j-1, j+1, \ldots, m,$$
$$\mu_j = \bar{\mu}_{j+1}.$$

From which it follows that

$$g_j(x^*) \leqq 0.$$

Repeating this process for $j = 1, 2, \ldots, m$, we have

$$g(x^*) \leqq 0.$$

Hence, x^* is feasible for (VP). Again, because $\mu^* \in \mathbb{R}^m_+$ and $g(x^*) \leqq 0$, we have $\mu^{*T} g(x^*) \leqq 0$. But from (4.6), we have by setting $\mu = 0$, that $\mu^{*T} g(x^*) \geqq 0$. Thus, $\mu^{*T} g(x^*) \leq 0$ and $\mu^{*T} g(x^*) \geqq 0$ yield

$$\mu^{*T} g(x^*) = 0. \tag{4.7}$$

Now, we assume that x^* is not an efficient solution of the problem (VP). Therefore, there exists feasible x with $g(x) \leq 0$, such that

$$f_i(x) \leqq f_i(x^*), \qquad \forall i = 1, 2, \ldots, p,$$

and

$$f_{i_0}(x) < f_{i_0}(x^*), \quad \text{for at least one } i_0 \in \{1, 2, \ldots, p\}.$$

These along with (4.2) and (4.7) yield

$$f_i(x) + \mu^{*T} g(x) \leq f_i(x^*) + \mu^{*T} g(x^*),$$

$$\forall i = 1, 2, \ldots, p \text{ and } \forall x \in X,$$

and

$$f_{i_0}(x) + \mu^{*T}g(x) < f_{i_0}(x^*) + \mu^{*T}g(x^*)$$
$$\text{for at least one } i_0 \in \{1, 2, \dots, p\} \text{ and } \forall x \in X.$$

That is

$$L_i(x, \mu^*) \leq L_i(x^*, \mu^*), \qquad \forall i = 1, 2, \dots, p \text{ and } \forall x \in X,$$
$$L_{i_0}(x, \mu^*) < L_{i_0}(x^*, \mu^*),$$
$$\text{for at least one } i_0 \in \{1, 2, \dots, p\} \text{ and } \forall x \in X.$$

which is a contradiction to (4.5). Hence, x^* is an efficient solution of (VP).

We now suppose that x^* is not a properly efficient solution of (VP). Therefore, there exists a feasible point x for (VP) and an index i_0 such that for every positive function $M > 0$, we have

$$f_i(x) - f_i(x^*) > M\big(f_j(x^*) - f_j(x)\big),$$

for all j satisfying

$$f_j(x) > f_j(x^*),$$

wherever

$$f_i(x) < f_i(x^*).$$

This along with (4.2) and (4.7) yields

$$f_i(x) + \mu^{*T}g(x) < f_i(x^*) + \mu^{*T}g(x^*),$$
$$\forall i = 1, 2, \dots, p \text{ and } x \in X,$$

i.e., $L_i(x, \mu^*) < L_i(x^*, \mu^*)$, which is a contradiction to (4.5). Hence, x^* is a properly efficient solution of (VP).

Similarly by assuming (4.5) we can get a contradiction to (4.4). ∎

THEOREM 4.3. *Let x^* be a properly efficient solution of (VP) and let an x^* slater type constraint qualification be satisfied. If* $(\sum_{i=1}^{p} \lambda_i f_i, \sum_{j=1}^{m} \mu_j g_j)$ *satisfy* (2.1) *and* (2.2) *and* (2.4)–(2.6) *hold, then there exists $\mu^* \in \mathbb{R}_+^m$ such that (x^*, μ^*) is a vector saddle point of $L(x, \mu)$.*

Proof. Because x^* is a properly efficient solution of (VP), therefore, x^* is also an efficient solution of (VP) and because at x^*, slater type constraint qualification is satisfied, therefore, by (KT) conditions, there exist $\lambda > 0$, $\lambda \in \mathbb{R}^p$, $\mu^* \in \mathbb{R}^m_+$, such that the following hold

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0, \tag{4.8}$$

$$\mu_j^* g_j(x^*) = 0, \qquad j = 1, 2, \ldots, m. \tag{4.9}$$

Now for all $i = 1, 2, \ldots, p$, $\forall x \in X$, we have

$$\begin{aligned} L_i(x, \mu^*) - L_i(x^*, \mu^*) &= f_i(x) - f_i(x^*) + \mu^{*T}[g(x) - g(x^*)] \\ &\geq -\eta(x, x^*)^T \left[\sum_{i=1}^{p} \mu^{*T} \nabla g_j(x^*) \right] \\ &\geq 0, \text{(by (2.2))}. \end{aligned}$$

Because $\lambda_i \in \mathbb{R}^p$, $\lambda > 0$, therefore,

$$L_i(x^*, \mu^*) \not\geq L_i(x, \mu^*), \qquad \forall x \in X.$$

The other part,

$$L_i(x^*, \mu) \not\geq L_i(x, \mu^*), \qquad \forall x \in \mathbb{R}^m_+,$$

of the vector saddle point inequality follows from

$$L(x^*, \mu) - L(x^*, \mu^*) = (\mu - \mu^*)^T g(x^*) \leq 0.$$

Hence (x^*, μ^*) is a vector saddle point of $L(x, \mu)$. ∎

Remark 4.1. Theorem 4.3 can be established under weaker assumptions used in previous sections.

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