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Multiobjective Control Problem with V-Invexity

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A multiobjective control problem is considered. Duality results are obtained for Mond–Weir-type duals under V-invexity assumptions and their generalizations.

1. INTRODUCTION

A number of duality theorems for the single-objective control problem have appeared in the literature; see [4–6, 9, 10]. In general, these references give conditions under which an extremal solution of the control problem yields a solution of the corresponding dual. Mond and Hanson [7] established the converse duality theorem which gives conditions under which a solution of the dual problem yields a solution of the control problem. Mond and Smart [8] extended the results of Mond and Hanson [7] for duality in control problems to invex functions. It is also shown in Mond and Smart [8] that, for invex functions, the necessary conditions for optimality in the control problem are also sufficient.

Recently, Bhatia and Kumar [1] extended the work of Mond and Hanson [7] to the content of multiobjective control problems and established duality results for Wolfe as well as Mond–Weir-type duals under ρ-invexity assumptions and their generalizations.

In this section we will obtain duality results for multiobjective control problems under V-invexity assumptions and their generalizations. The results of the present section extend the work of Bhatia and Kumar [1] to a wider class of functions.

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2. NOTATION AND PRELIMINARIES

The control problem is to choose, under given conditions, a control vector $u(t)$, such that the state vector $x(t)$ is brought from some specified initial state $x(a) = \alpha$ to some specified final state $x(b) = \beta$ in such a way as to minimize a given functional. A more precise mathematical formulation is given in the following problem:

$$\text{(VCP)} \quad \text{Minimize} \left(\int_a^b f_1(t, x, u)\, dt \cdots \int_a^b f_p(t, x, u)\, dt \right)$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta, \tag{1}$$

$$g(t, x, u) \leq 0, \qquad t \in I, \tag{2}$$

$$h(t, x, u) = x^0, \qquad t \in I. \tag{3}$$

Here R^n denotes an n-dimensional euclidean space and $I = [a, b]$ is a real interval. Each f_i: $1 \times R^n \times R^m \to R$ for $i = 1, 2, \ldots, p$, g: $I \times R^n \times R^m \to R^k$, and h: $I \times R^n \times R^m \to R^q$ is a continuously differentiable function.

Let x: $I \to R^n$ be differentiable with its derivative x^0, and let y: $I \to R^m$ be a smooth function. Denote the partial derivatives of f by f_t, f_x, and f_y, where

$$f_t = \frac{\partial f}{\partial t}, \quad f_x = \left[\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots, \frac{\partial f}{\partial x^n} \right], \quad f_u = \left[\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial f^2}, \ldots, \frac{\partial f}{\partial u^n} \right],$$

where the superscripts denote the vector components.

Similarly, we have g_t, g_x, g_u and h_t, h_x, h_u. X is the space of continuously differentiable state functions x: $I \to R^n$ such that $x(a) = \alpha$ and $x(b) = \beta$ and is equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$; and Y is the space of piecewise continuous control functions u: $I \to R^m$, and has the uniform norm $\|\cdot\|_\infty$. The differential equation (3) with initial conditions expressed as $x(t) = x(a) + \int_a^b h(s, x(s), u(s))\, ds$, $t \in I$ may be written as $H_x = H(x, y)$, where H: $X \times Y \to C(I, R^n)$, $C(I, R^n)$ being the space of continuous functions from I to R^n defined as $H(x, y) = h(t, x(t), u(t))$. A Mond–Weir-type dual for (VCP) is proposed and duality relationships are established under generalized V-invexity assumptions:

The Mond–Weir-type vector control dual:

(MVCD)

$$\text{Maximize} \left(\int_a^b f_1(t, y, v)\, dt \cdots \int_a^b f_p(t, y, v)\, dt \right)$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta, \tag{4}$$

$$\sum_{f=1}^{p} \tau_i f_{iy}(t, y, v) + \sum_{j=1}^{k} \lambda_j(t) g_{jy}(t, y, v) + \sum_{r=1}^{q} \mu_r(t) h_{ry}(t, y, v) + u^0(t) = 0, \qquad t \in I, \tag{5}$$

$$\sum_{i=1}^{p} \tau_i f_{iv}(t, y, v) + \sum_{j=1}^{k} \lambda_j(t) g_{jv}(t, y, v) + \sum_{r=1}^{q} \mu_r(t) h_{rv}(t, y, v) = 0, \qquad t \in I, \tag{6}$$

$$\int_a^b \sum_{r=1}^{q} \mu_r(t) \left[h(t, y, v) - x^0(t) \right] dt \geq 0, \qquad t \in I, \tag{7}$$

$$\int_a^b \sum_{j=1}^{k} \lambda_j(t) g_j(t, y, v)\, dt \geq 0, \qquad t \in I, \tag{8}$$

$$\lambda(t) \geq 0, \qquad t \in I, \tag{9}$$

$$\tau_i \geq 0, \qquad i = 1, 2, \dots, p, \qquad \sum_{i=1}^{p} \tau_i = 1. \tag{10}$$

Optimization in (VCP) and (MVCD) means obtaining efficient solutions for the corresponding programs.

Let $F_i = \int_a^b f_i(t, x, u)\, dt$ be Frechet differentiable. Let there exist functions $\eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \in R^p$ with $\eta = 0$ at t if $x(t) = \bar{x}(t)$, and $\xi(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \in R^m$.

DEFINITION 1. A vector function $F = (F_1, \dots, F_p)$ is said to be V-invex in X, X', and u on $[a, b]$ with respect to η, ξ, and α_i if there exist differentiable vector functions $\eta \in R^p$ and ξ in R^m and $\alpha_i \in R_+ \backslash \{0\}$ such that, for each $x, \bar{x} \in X_0$ and $u, \bar{u} \in Y$ and for $i = 1, 2, \ldots, p$,

$$F_i(x) - F_i(\bar{x}) \supseteq \int_a^b \Big\{ \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) \times \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) + \frac{d}{dt} \eta_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \times f_x^i(t, \bar{x}, \bar{x}', \bar{u}) + \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) h_u(t, \bar{x}, \bar{x}^i, \bar{u}) \times \xi(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \Big\} \, dt.$$

Definition 2. The vector function $F = (F_1, \ldots, F_p)$ is said to be V-pseudo-invex in x, x', and u on $[a, b]$ with respect to η, ξ, and β if there exist η, ξ as above and $\beta_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \in R_+ \backslash \{0\}$ such that, for each $x, \bar{x} \in X$ and $u, u \in Y$ and for $i = 1, 2, \ldots, p$,

$$\int_a^b \sum_{i=1}^p \Big\{ \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) + \frac{d}{dt} \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) + f_u^i(t, \bar{x}, \bar{x}', \bar{u}) \times \xi(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \Big\} \, dt \geq 0$$

$$\Rightarrow \quad \int_a^b \sum_{i=1}^p \beta_i(t, x, \bar{x}, \bar{u}) f_i(t, x, x', u) \, dt \geq \int_a^b \sum_{i=1}^p \beta_i(t, x, \bar{x}, \bar{u}) f_i(t, \bar{x}, \bar{x}', \bar{u}) \, dt$$

or, equivalently,

$$\int_a^b \sum_{i=1}^p \beta_i(t, x, \bar{x}, \bar{u}) f_i(t, x, x', u) \, dt < \int_a^b \sum_{i=1}^p \beta_i(t, x, \bar{x}, \bar{u}) f_i(t, \bar{x}, \bar{x}', \bar{u}) \, dt$$

$$\Rightarrow \quad \int_a^b \sum_{i=1}^{p} \Big\{ \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) + \frac{d}{dt} \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) + f_u^i(t, \bar{x}, \bar{x}', \bar{u}) \xi(t, x, \bar{x}, x, \bar{x}', u, \bar{u}) \Big\} \, dt < 0.$$

Definition 3. The vector function $F = (F_1, \dots, F_p)$ is said to be V-quasi-invex in x, x', and u on $[a, b]$ with respect to η, ξ, and γ if there exist η, ξ as above and the vector $\gamma_i \in R_+ \backslash \{0\}$ such that, for each $x, \bar{x} \in X$, $u, \bar{u} \in Y$,

$$\int_a^b \sum_{i=1}^{p} \gamma_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_i(t, x, x', u) \, dt \leq \int_a^b \sum_{i=1}^{p} \gamma_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_i(t, \bar{x}, \bar{x}', \bar{u}) \, dt$$
$$\Rightarrow \quad \int_a^b \sum_{i=1}^{p} \Big\{ \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) \, dt + \frac{d}{dt} \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) + f_u^i(t, \bar{x}, \bar{x}', \bar{u}) \xi(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \Big\} \, dt \leq 0,$$

or, equivalently,

$$\int_a^b \sum_{i=1}^{p} \Big\{ \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) \, dt + \frac{d}{dt} \eta(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) + f_u^i(t, \bar{x}, \bar{x}', \bar{u}) \xi(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \Big\} \, dt > 0$$
$$\Rightarrow \quad \int_a^b \sum_{i=1}^{p} \gamma_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_i(t, x, x', u) \, dt > \int_a^b \sum_{i=1}^{p} \gamma_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_i(t, \bar{x}, \bar{x}', \bar{u}) \, dt.$$

Remark 1. V-invexity is defined here for functionals instead of functions, unlike the definition given in Section 1 of Chapter 1 in Mishra [6] as well as in Mukherjee and Mishra [9]. This has been done so that the V-invexity of a functional F is necessary and sufficient for its critical points to be global minima, which coincides with the original concept of a V-invex function being one for which critical points are also global minima (Craven and Glover [3]). We thus have the following characterization result.

LEMMA 1. *$F(x) = \int_a^b f(t, x, x', u)\, dt$ is V-invex iff every critical point of F is a global minimum.*

Note 1. $(\bar{x}, \bar{u})$ is a critical point of F if $f_x^i(t, x, x', u) = (d/dt)$, $f_{x'}^i(t, \bar{x}, \bar{x}', \bar{u})$ and $f_u^i(t, \bar{x}, \bar{x}', \bar{u}) = 0$ almost everywhere in $[a, b]$. If $x(a)$ and $x(b)$ are free, the transversality conditions $h_{x'}(t, \bar{x}, \bar{x}', \bar{u}) = 0$ at a and b are included.

Proof of Lemma 1. $(\Rightarrow)$ Assume that there exist functions η, ξ, and α such that F is V-invex with respect to η, ξ, and α on $[a, b]$.

Let $(\bar{x}, \bar{u})$ be a critical point of F. Then

$$\begin{aligned}
F_i(x) - F_i(\bar{x}) \geq \int_a^b \Big\{ & \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) \\
& + \frac{d}{dt} \eta_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u})\, \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \\
& \times f_x^i(t, \bar{x}, \bar{x}', \bar{u}) \\
& + \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_u^i(t, \bar{x}, \bar{x}', \bar{u}) \\
& \times \xi_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \Big\}\, dt, \\
= \int_a^b \Big\{ & \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_x^i(t, \bar{x}, \bar{x}', \bar{u}) \\
& + \eta_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \\
& \times \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) \frac{d}{dt} f_{x'}^i(t, \bar{x}, \bar{x}', \bar{u}) \\
& + \alpha_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_u^i(t, \bar{x}, \bar{x}', \bar{u}) \\
& \times \xi_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u})\, dt \\
& + \eta_i(t, x, \bar{x}, x', \bar{x}', u, \bar{u}) f_{x'}^i(t, \bar{x}, \bar{x}', \bar{u}) u) \Big\} \\
= 0 \qquad & \text{(by integration by parts)} \qquad \forall\ i = 1, 2, \ldots, p
\end{aligned}$$

as $(\bar{x}, \bar{u})$ is a critical point of either fixed boundary conditions, which imply that $\eta = 0$ at a and b, or free boundary conditions, which imply conditions, which imply that $f_x^i = 0$ at a and b. Therefore, $(\bar{x}, \bar{u})$ is a global minimum of f.

Assume that every critical point is a global minimum. If $(\bar{x}, \bar{u})$ is a critical point, put $\eta = \xi = 0$. If $(\bar{x}, \bar{u})$ is not a critical point, then, if $f_x^i \neq (d/dt) f_{x'}^i$ at $(\bar{x}, \bar{u})$, put

$$\eta_i = \frac{f^i(t, x, x', u) - f^i(t, \bar{x}, \bar{x}', \bar{u})}{2\left[f_x^i - (d/dt) f_x^i\right]^T \left[f_x^i - (d/dt) f_{x'}^i\right]} \left[f_x^i - \left(\frac{d}{dt}\right) f_{x'}^i\right],$$

$\alpha = 1$, or, if $f_x^i = (d/dt) f_{x'}^i$, put $\eta = 0$; and, if $h_u \neq 0$, put

$$\xi_i = \frac{f^i(t, x, x', u) - f^i(t, \bar{x}, \bar{x}', \bar{u})}{2 f_u^{iT} f_u^i} f_u^i$$

and $\alpha_i = 1$, or, if $f_u^i = 0$, put $\xi = 0$. Then F is V-invex on a, b with respect to η, ξ, and α.

Chandra, Craven, and Husain [2] gave the Fritz–John necessary optimality conditions for the existence of an extremal solution for the single objective control problem (CP):

$$\text{(CP)} \qquad \int_a^b f(t, x, u)\, dt$$

subject to

$$x = h(t, x, u), \qquad g(t, x, u) \leq 0.$$

Mond and Hanson [7] pointed out that if the primal solution for (VCP) is normal, then Fritz–John conditions reduce to Kuhn–Tucker conditions.

LEMMA 2 (Kuhn–Tucker Necessary Optimality Conditions). *If* $(\bar{x}, \bar{u}) \in X \times Y$ *solves (VCP), if the Frechet derivative* $[D - F_x^i(x^0, u^0)]$ *is surjective, and if the optimal solutions* (x^0, y^0) *is normal, then there exist piecewise smooth* $\tau^0\colon I \to R^p$, $\lambda^0\colon I \to R$, *and* $\mu\colon I \to R^k$, *satisfying the following, for all* $t \in [a, b]$:

$$\sum_{r=1}^{m} \mu_r^0 h_x^r(t, x^0, u^0) + \mu_r^0(t) = 0, \tag{11}$$

$$\sum_{i=1}^{p} \tau_i^0 f_u^i(t, x^0, u^0) + \sum_{j=1}^{k} \lambda_j^0 g_u^j(t, x^0, u^0) + \sum_{r=1}^{m} \mu_r^0 h_u^r(t, x^0, u^0) = 0, \tag{12}$$

$$\sum_{j=1}^{k} \lambda_j^0 g(t, x^0, u^0) = 0, \tag{13}$$

$$\tau^0 > 0, \qquad \lambda^0 \geq 0, \qquad \sum_{i=1}^{p} \tau_i^0 = 1. \tag{14}$$

We shall now prove that (VCP) and (MVCD) are a dual pair subject to generalized V-invexity conditions on the objective and constraint functions.

3. DUALITY THEOREMS

THEOREM 1 (Weak Duality). *Assume that, for all feasible* (x, u) *for* (VCP) *and all feasible* $(y, v, \tau, \lambda, \mu)$ *for* $(MVCD)$, *if*

$$\text{(i)} \qquad \left(\int_a^b \tau_1 f_1(\cdot, \cdot, \cdot)\, dt, \dots, \int_a^b \tau_p f_p(\cdot, \cdot, \cdot)\, dt\right)$$

and

$$\text{(ii)} \qquad \left(\int_a^b \lambda_1 g_1(\cdot, \cdot, \cdot)\, dt, \dots, \int_a^b \lambda_m g_m(\cdot, \cdot, \cdot)\, dt\right)$$

are V-quasi-invex and

(iii)

$$\left(\int_a^b \mu_1(h_1(\cdot, \cdot, \cdot) - x)\, dt, \dots, \int_a^b \mu_k(h_k(\cdot, \cdot, \cdot) - x)\, dt\right)$$

are strictly V-quasi-invex with respect to the same η, ξ, *then the following cannot hold*:

$$\int_a^b f_i(t, x, u)\, dt \le \int_a^b f_i(t, y, v)\, dt, \qquad \forall\ i \in \{1, \dots, p\} \tag{15}$$

and

$$\int_a^b f_{i_0}(t, x, u)\, dt < \int_a^b f_{i_0}(t, y, v)\, dt \qquad \textit{for some } i_0 \in \{1, 2, \dots, p\}. \tag{16}$$

Proof. Suppose contrary to the result that (15) and (16) hold. Then (i) yields

$$\int_a^b \Big\{ \sum \eta_i(t, x, x', \bar{x}, \bar{x}', u, \bar{u}) \tau_i f_y(t, y, y, v) + f_v^i(t, y, y', v)\, \xi_i(t, x, x', \bar{x}, \bar{x}', u, \bar{u}) \Big\}\, dt < 0. \tag{17}$$

From the feasibility conditions,

$$\int_a^b \lambda_j g_j(t, x, x', u)\, dt \le \int_a^b \lambda_i g_j(t, x, x', u)\, dt$$

for each $j = 1, 2, \ldots, m$. Since $\beta_j(t, x, x', \bar{x}, \bar{x}', u, \bar{u}) \geq 0$, $\forall\, j = 1, 2, \ldots, m$, we have

$$\int_a^b \sum \beta_j(t, x, x', \bar{x}, \bar{x}', u, \bar{u})\, \lambda_j g_j(t, x, x', u)\, dt$$
$$\leq \int_a^b \sum \beta_j(t, x, x', \bar{x}, \bar{x}', u, \bar{u})\, \lambda_j g_j(t, x, x', u)\, dt.$$

Then (ii) yields

$$\int_a^b \sum \Big\{ \eta_j(t, x, x', \bar{x}, \bar{x}', u, \bar{u})\, \lambda_j g_x^j(t, x, x', u)$$
$$+ f_u^j(t, x, x', u)\, \xi_j(t, x, x', \bar{x}, \bar{x}', u, \bar{u}) \Big\}\, dt$$
$$\leq 0. \tag{18}$$

Similarly, we have

$$\int_a^b \sum_{r=1}^{k} \mu_r [h_r(t, x, u) - x']\, dt \leq \int_a^b \sum_{r=1}^{k} \mu_r [h_r(t, y, v) - x]\, dt.$$

From (iii) it follows that

$$\int_a^b \sum \Big\{ \eta(t, x, x', \bar{x}, \bar{x}', u, \bar{u})\, \mu_r h_y(t, y, y', u) - \frac{d}{dt} \eta(t, \ldots)\, \mu_r$$
$$+ \sum \mu_r h_v(t, y, y', v)\, \xi(t, x, x', \bar{x}, \bar{x}', u, \bar{u}) \Big\}\, dt < 0. \tag{19}$$

By integrating $(d/dt)\eta(t, x, x', \bar{x}, \bar{x}', u, \bar{u})\mu$ from a to b by parts and applying the boundary conditions (1), we have

$$\int_a^b \frac{d}{dt} \eta(t, x, x', \bar{x}, \bar{x}', u, \bar{u})\, \mu\, dt$$
$$= -\int_a^b \eta(t, x, x', \bar{x}, \bar{x}', u, \bar{u})\, \mu^0(t)\, dt. \tag{20}$$

Using (20) in (19), we have

$$\int_a^b \Big\{ \sum \eta(t, x, x', \bar{x}, \bar{x}', u, \bar{u})\, \mu_r h_y(t, y, y', v) + \mu^0(t)$$
$$+ \sum \mu_r h_v(t, y, y', v)\, \xi(t, x, x', \bar{x}, \bar{x}', u, \bar{u}) \Big\}\, dt < 0. \tag{21}$$

Adding (17), (18), and (21), we have

$$\int_a^b \Bigg\{ \eta \Bigg[\sum \tau_i \big(f_y^i(t, y, y', v) + \sum \lambda_j g_y^j(t, y, y', v) + \sum_{r=1}^{k} \mu_r h_y^r(t, y, y', v) \Bigg]$$

$$+ \xi \Bigg[\sum \tau_i f_v^i(t, y, y', v) + \sum \lambda_j g_v^j(t, y, y', v)$$

$$+ \sum_{r=1}^{k} \mu_r h_v^T(t, y, y', v) \Bigg] \Bigg\} \, dt < 0,$$

which is a contradiction to (5) and (6). ∎

COROLLARY 1. *Assume that weak duality* (*Theorem* 1) *holds between* (*VCP*) *and* (*MVCD*). *If* (y, v) *is feasible for* (*VCP*) *and* $(y, v, \tau, \lambda, \mu)$ *is feasible for* (*MVCD*), *than* (y, v) *is efficient for* (*VCP*) *and* $(y, v, \tau, \lambda, \mu)$ *is efficient for* (*MVCD*).

Proof. Suppose (y, v) is not efficient for (VCP). Then there exists some feasible (x, u) for (VCP) such that

$$\int_a^b f_i(t, x, x', u) \, dt \leq \int_a^b f_i(t, y, y', v) \, dt, \qquad \forall \ i \in \{1, 2, \dots, p\}$$

and

$$\int_a^b f_{i_0}(t, x, x', u) \, dt < \int_a^b f_{i_0}(t, y, y', v) \, dt \qquad \text{for some } i_0 \in \{1, 2, \dots, p\}.$$

This contradicts weak duality. Hence (y, v) is efficient for (VCP). Now suppose $(y, v, \tau, \lambda, \mu)$ is not efficient for (MVCD). Then there exist some $(x, u, \tau, \lambda, \mu)$ feasible for (MVCD) such that

$$\int_a^b f_i(t, x, x', u) \, dt \geq \int_a^b f_i(t, y, y', v) \, dt, \qquad \forall \ i \in \{1, 2, \dots, p\}$$

and

$$\int_a^b f_{i_0}(t, x, x', u) \, dt > \int_a^b f_{i_0}(t, y, y', v) \, dt \qquad \text{for some } i_0 \in \{1, 2, \dots, p\}.$$

This contradicts weak duality. Hence $(y, v, \tau, \lambda, \mu)$ is efficient for (MVCD). ∎

THEOREM 2 (Strong Duality). *Let* $(\bar{x}, \bar{u})$ *be efficient for* (*VCP*) *and assume that* $(\bar{x}, \bar{u})$ *satisfy the constraint qualification of Lemma* 2 *for at least one* $i \in \{1, 2, \dots, p\}$. *Then there exist* $\bar{\tau} \in R^p$ *and piecewise smooth* $\bar{\lambda}$:

$I \to R^m$ and $\overline{\mu}$: $I \to R^k$ such that $(\bar{x}, \overline{u}, \bar{\tau}, \overline{\lambda}, \overline{\mu})$ is feasible for (MVCD). If also weak duality (Theorem 1) holds between (VCP) and (MVCD), then $(\bar{x}, \overline{u}, \bar{\tau}, \overline{\lambda}, \overline{\mu})$ is efficient for (MVCD).

Proof. As $(\bar{x}, \overline{u})$ satisfy the constraint qualifications of Lemma 2, it follows that there exist piecewise smooth $\bar{\tau}$: $I \to R^p$, $\overline{\lambda}$: $I \to R^m$, $\overline{\mu}$: $I \to R^k$ satisfying for all $t \in I$ the following relations:

$$\sum_{i=1}^{p} \bar{\tau}_i f_x^i(t, \bar{x}, \bar{x}', \overline{u}) + \sum_{j=1}^{m} \overline{\lambda}_j g_x^j(t, \bar{x}, \bar{x}', \overline{u}) + \sum_{r=1}^{k} \overline{\mu}_r h_x^r(t, \bar{x}, \bar{x}', \overline{u}) + \overline{\mu}(t) = 0,$$

$$\sum_{i=1}^{p} \bar{\tau}_i f_u^i(t, \bar{x}, \bar{x}', \overline{u}) + \sum_{j=1}^{m} \overline{\lambda}_j g_u^j(t, \bar{x}, \bar{x}', \overline{u}) + \sum_{r=1}^{k} \overline{\mu}_r h_u^r(t, \bar{x}, \bar{x}', \overline{u}) = 0,$$

$$\sum_{j=1}^{m} \overline{\lambda}_j(t) g_j(t, \bar{x}, \bar{x}', \overline{u}) = 0,$$

$$\overline{\lambda}(t) \geq 0,$$

$$\bar{\tau}_1 \geq 0, \qquad \sum_{i=1}^{p} \bar{\tau}_i = 1.$$

The relations

$$\int_a^b \sum_{j=1}^{m} \lambda_j g_j(t, \bar{x}, \bar{x}', \overline{u})\, dt = 0,$$

$$\int_a^b \sum_{r=1}^{k} \overline{\mu}_r \left[h_r(t, \bar{x}, \bar{x}', \overline{u}) - x \right] dt \geq 0$$

are obvious.

The preceding relations imply that $(\bar{x}, \overline{u}, \bar{\tau}, \overline{\lambda}, \overline{\mu})$ is feasible for (MVCD). The result now follows from Corollary 1. ∎

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