Journal of Mathematical Analysis and Applications **235,** 567–577 (1999)
Article ID jmaa.1999.6409, available online at http://www.idealibrary.com on IDEAL®

Composite Nonsmooth Multiobjective Programs with V-ρ-Invexity

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Submitted by E. S. Lee

Received November 18, 1998

In this paper, some problems consisting of nonsmooth composite multiobjective programs have been treated with V-ρ-invexity type of conditions. In particular, we prove the generalized Karush–Kuhn–Tucker sufficient optimality theorem and duality theorems for nonsmooth composite multiobjective programs. Also, weak vector saddle point theorems are obtained for the composite programs under V-ρ-invexity conditions. The results obtained generalize the results of Hun Kuk *et al*. to some extent.

Key Words: composite nonsmooth multiobjective programs; V-ρ-invexity; duality; weak vector saddle point.

1. INTRODUCTION

Several classes of functions have been defined for the purpose of weakening the limitations of convexity in mathematical programming. Hanson [3] introduced the concept of invexity and proved that the Kuhn–Tucker conditions are sufficient for optimality of a nonlinear programming problem under invexity conditions. Recently, several new concepts concerning a generalized invex functions have been proposed. Among these, Jeyakumar and Mond [5] defined generalized V-invexity of differentiable multiobjective programming problems which preserve the sufficient optimality conditions and duality results as in the scalar case, and avoid the major difficulty of verifying that the inequality holds for the same function $\eta(\cdot,\cdot)$ for invex functions. Later, Mishra and Mukherjee [10] and Liu [9] further extended the results of Jeyakumar and Mond [5] to nonsmooth multiobjective programming problems. Consequently, Jeyakumar [6] introduced ρ-invexity for differentiable scalar-valued functions and

0022-247X/99 $30.00

investigated the sufficiency of the Karush–Kuhn–Tucker conditions, and he obtained some duality theorems for the scalar nonlinear programming problem. Later, Jeyakumar [7] defined ρ-invexity for nonsmooth scalar-valued functions, studied duality theorems for nonsmooth optimization problems, and gave relationships between saddle points and optima very recently, Hun Kuk *et al*. [4] defined V-ρ-invexity for vector-valued functions, established sufficient optimality conditions, derived duality results for nonsmooth multiobjective programs under the V-ρ-invexity assumptions, and obtained weak vector-saddle-point theorems in their paper.

On the other hand, Jeyakumar and Yang [8] considered nonsmooth convex composite multiobjective problems which are not necessarily convex programming problems. Also, Lagrangian necessary conditions, new sufficient optimality conditions, and duality results for efficient and properly efficient solutions were obtained by them. In a subsequent work, Mishra and Mukherjee [10] introduced generalized convex composite multiobjective nonsmooth programming under the context of proper and conditional proper efficiency and they obtained optimality and duality results in a similar context. Further, Mishra [11] studied Lagrange multipliers, saddle point properties, and scalarizations aspect of composite multiobjective nonsmooth programs.

Motivated by the above ideas, in this paper, we examine nonsmooth multiobjective problems where the objective functions and the constraints are compositions of V-ρ-invex functions. We prove generalized Karush–Kuhn–Tucker sufficient optimality theorems and duality theorems for composite nonsmooth multiobjective programs involving locally Lipschitz functions. Finally, weak vector saddle-point theorems are also obtained under V-ρ-invexity conditions.

The paper is organized as follows. Section 2 gives preliminary notations and definitions, while in Section 3, we will show that the generalized Karush–Kuhn–Tucker conditions are sufficient for a weak minimum of (CP). In Section 4, we will introduce Mond–Weir types of dual problems and obtain weak and strong duality theorems. Finally in Section 5, we prove weak vector saddle point theorems for the nonsmooth composite multiobjective program (CP) in which the functions are locally Lipschitz.

2. PRELIMINARIES AND DEFINITIONS

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. Throughout our discussion, the following will be needed in the sequel

$$x > y \quad \text{and only if } x_i > y_i, \qquad i = 1, 2, \ldots, n$$

$$x \geq y \quad \text{and only if } x_i \geq y_i, \qquad i = 1, 2, \ldots, n$$

and similarly for $x < y$ and $x \leq y$.

A real-valued function f: $R^n \to R$ is said to be locally Lipschitz if for any $z \in R^n$ there exists a positive constant K and a neighborhood N of z such that for each $x, y \in N$,

$$|f(x) - f(y)| \leq K\|x - y\|.$$

The generalized (Clarke [1]) directional derivative of a locally Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ is

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1}(f(y + td) - f(y)).$$

The Clarke generalized subgradient of f at x is denoted by

$$\partial^0 f(x) = \{\xi : f^0(x; d) \geq \xi^T d, \qquad \forall d \in R^n\}$$

In this paper, we consider the composite multiobjective programming problem (CP):

$$\text{(CP)} \qquad \text{V-Minimize} \quad (f_1(F_1)(x)), f_2(F_2(x)), \ldots, f_p(F_p(x)))$$

$$\text{subject to} \qquad g_j(G_j(x)) \leq 0, \qquad j = 1, 2, \ldots,$$

$$x \in C$$

where C is a convex subset of a Banach space X, f_i, g_j $(i = 1, 2, \ldots, p;$ $j = 1, 2, \ldots, m)$ are real-valued locally Lipschitz functions on R^n and F_i and G_j are locally Lipschitz and Gateaux differentiable functions from X into R^n, respectively. Suppose the feasible set is defined as follows:

$$E = \{x \in R^n : g_j(G_j(x)) \leq 0, j = 1, 2, \ldots, m\}.$$

Egudo and Hanson [2] defined invexity of locally Lipschitz functions as follows:

DEFINITION 2.1. A locally Lipschitz $f(x)$ is invex on $X_0 \subset R^n$ if for x, $u \in X_0$ there exists a function $\eta(x, u)$: $X_0 \times X_0 \to R^n$ such that

$$f(x) - f(u) \geq \xi^T \eta(x, u) \qquad \forall \xi \in \partial^0 f(u).$$

Egudo and Hanson [2] generalized the V-invexity of Jeyakumar and Mond [5] to the nonsmooth case as follows:

DEFINITION 2.2. A vector function $f\colon X_0 \to R^n$ is said to be V-invex if there exist functions $\eta\colon X_0 \to R^p$ and $\alpha_i\colon X_0 \times X_0 \to R_+ \backslash \{0\}$ such that for each $x, u \in X_0$:

$$f_i(x) - f_i(u) - \alpha_i(x,u)\xi_i^T \eta(x,u) \geq 0 \qquad \forall \xi_i \in \partial^0 f_i(u).$$

Hun Kuk *et al.* [4] introduced V-ρ-invexity to the nonsmooth case as follows:

DEFINITION 2.3. Let $f_i\colon R^n \to R$ and $g_j\colon R^n \to R$ be locally Lipschitz functions for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, m$ respectively.

(a) $f = (f_1, \ldots, f_p)$ is V-ρ-invex with respect to functions η and $\theta\colon R^n \times R^n \to R^n$ if there exists $\alpha_i\colon R^n \times R^n \to R_+ \backslash \{0\}$ and $\rho_i \in R$, $i = 1, 2, \ldots, p$ such that for any $x, u \in R^n$ and any $\xi_i \in \partial f_i(u)$,

$$\alpha_i(x,u)\left[f_i(x) - f_i(u)\right] \geq \xi_i \eta(x,u)^t + \rho_i \|\theta(x,u)\|^2$$

(b) $g = (g_1, \ldots, g_m)$ is V-ρ-invex with respect to functions η and $\theta\colon R^n \times R^n \to R^n$ if there exists $\beta_j\colon R^n \times R^n \to R_+ \backslash \{0\}$ and $\sigma_i \in R$, $i = 1, 2, \ldots, m$ such that for any $x, u \in R^n$ and any $\zeta_j \in \partial g_j(x)$,

$$\beta_j(x,u)\left[g_j(x) - g_j(u)\right] \geq \zeta_j \eta(x,u) + \sigma_j \|\theta(x,u)\|^2.$$

DEFINITION 2.4. Let $f_i\colon R^n \to R$ and $g_j\colon R^n \to R$ be locally Lipschitz functions for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, m$, and F_i and G_j are Gateaux differentiable functions, respectively.

(a) $f(F) = (f_1(F_1), f_2(F_2), \ldots, f_p(F_p))$ is V-ρ-invex with respect to functions η and $\theta\colon R^n \times R^n \to R^n$ is there exists $\alpha_i\colon R^n \times R^n \to R_+ \backslash \{0\}$ and $\rho_i \in R$, $i = 1, 2, \ldots, p$ such that for any $x, u \in R^n$ and any $\xi_i \in \partial f_i(F_i(u))$,

$$\alpha_i(x,u)\left[f_i(F_i(x)) - f_i(F_i(u))\right] \geq \xi_i \eta(x,u)^t + \rho_i \|\theta(x,u)\|^2$$

(b) $g(G) = (g_1(G_1), g_2(G_2), \ldots, g_m(G_m))$ is V-ρ-invex with respect to functions η and $\theta\colon R^n \times R^n \to R^n$ if there exists $\beta_j\colon R^n \times R^n \to R_+ \backslash \{0\}$ and $\rho_j \in R$, $j = 1, 2, \ldots, m$ such that for any $x, u \in R^n$ and any $\zeta_j \in \partial g_j(G_j(u))$,

$$\beta_j(x,u)\left[g_j\left(G_j(x)\right) - g_j\left(G_j(u)\right)\right] \geq \zeta_j \eta(x,u) + \rho_j \|\theta(x,u)\|^2.$$

DEFINITION 2.5. A point $u \in X$ is said to be weak minimum of (CP) if there exists no $x \in X$ such that

$$f_i\left(F_i(x)\right) < f_i\left(F_i(u)\right), \qquad i = 1, 2, \ldots, p.$$

3. SUFFICIENT OPTIMALITY CONDITIONS FOR V-ρ-INVEX COMPOSITE PROGRAMS

In this section, we present that the generalized Karush–Kuhn–Tucker conditions are sufficient for a weak minimum of (CP).

The following null space condition is as in Jeyakumar and Yang [8].

Let $x,\ a \in X$. Define K: $X \to R^{n(p+m)} := \eta R^p$ by $K(x) = (F_1(x), \dots, F_p(x), G_1(x), \dots, G_m(x))$. For each $x,\ a \in X$, the linear mapping $A_{x,a}$: $X \to R^{n(p+m)}$ is given by

$$A_{x,a}(y) = \big(\delta_1(x,a) F_1(a) y, \dots, \delta_p(x,a) F_p(a) y,\ \theta_1(x,a) G_1(a) y, \dots, \theta_m(x,a) G_m(a) y\big)$$

where $\delta_i(x,a)$, $i = 1,2,\dots,p$ and $\theta_j(x,a)$, $j = 1,2,\dots,m$, are real positive constants. Also, recall from the generalized Farkas lemma of Craven [1] that $K(x) - K(a) \in A_{a,x}(X)$ iff $A^T_{x,a}(y) = 0 \ \Rightarrow\ y^T(K(x) - K(a)) = 0$. Let us denote the null space of a function H by $N[H]$. For each $x, a \in X$, there exist real constants $\delta_i(x,a) > 0$, $i = 1,\dots,p$ and $\theta_j(x,a) > 0$, $j = 1,\dots,m$, such that

$$\text{(NC)} \qquad N[X_{x,a}] \subset N[K(x) - K(a)].$$

Equivalently, the null space condition means that for each x, $a \in X$, there exist real constants $\delta_i(x,a) > 0$, $i = 1,\dots,p$, and $\theta_j(x,a) > 0$, $j = 1,\dots,m$ and $\mu(x,a) \in X$ such that $F_i(x) - F_i(a) = \delta_i(x,a) F_i'(a) \mu(x,a)$ and $G_j(x) - G_j(a) = \theta_j(x,a) G_j'(a) \mu(x,a)$.

THEOREM 3.1. *Let* $(x, \tau, \lambda) \in R^n \times R^p \times R^m$ *satisfy the generalized Karush–Kuhn–Tucker conditions as follows*:

$$0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i(F_i(u)) F_i'(u) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j\big(G_j(u)\big) G_j'(u) - (c-u)^+$$

$$g_j\big(G_j(u)\big) \le 0 \quad \textit{and} \quad \lambda_j g_j\big(G_j(u)\big) = 0, \qquad j = 1,2,\dots,m.$$

$$\tau_i \ge 0, \qquad 1 = 1,2,\dots,p, \qquad \tau^t e > 0,$$

$$\lambda_j \ge 0, \qquad j = 1,2,\dots,m.$$

If $f(F)$ *is* V-ρ-*invex and* $g(G)$ *is* V-σ-*invex with respect to the same* η *and* θ *and*

$$\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \sigma_j \ge 0,$$

then u *is a weak minimum of* (P).

Proof. Since

$$0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i\big(F_i(u)\big) F_i'(u) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j\big(G_j(u)\big) G_j'(u) - (c-u)^+$$

implies the existence of $\xi_i \in \partial^0 f_i(F_i(u))$ and $\zeta_j \in \partial^0 g_j(G_j(u))$ such that

$$\sum_{i=1}^{p} \tau_i \xi_i^T F_i'(u) + \sum_{j=1}^{m} \lambda_j \zeta_j^T G_j'(u) = 0, \tag{1}$$

suppose that u is not a weak minimum of (CP). Then there exists $x \in X$ such that

$$f_i\big(F_i(x)\big) < f_i\big(F_i(u)\big), \qquad i = 1, 2, \dots, p.$$

Since $\alpha_i(x, u) > 0$, we have

$$\alpha_i(x,u) f_i\big(F_i(x)\big) < \alpha_i(x,u) f_i\big(F_i(u)\big), \qquad i = 1, 2, \dots, p.$$

By the V-ρ-invexity of $f(F)$, for all i, we have $\xi_i \eta(x,u) + \rho_i \|\theta(x,u)\|^2 < 0$, for each $\xi_i \in \partial^0 f_i(F_i(u))$. Hence, we obtain

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x,u) + \sum_{i=1}^{p} \tau_i \rho_i \|\theta(x,u)\|^2 < 0.$$

Since $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \sigma_j \geq 0$, it follows from (1) that

$$\sum_{j=1}^{m} \lambda_j \zeta_j^T \eta(x,u) F_i'(u) + \sum_{j=1}^{m} \lambda_j G_j'(u) \sigma_j \|\theta(x,u)\|^2 > 0.$$

Then, by the V-σ-invexity of g, we obtain

$$\sum_{j=1}^{m} \beta_j(x,u) G_j'(u) \big[\lambda_j g_j\big(G_j(x)\big) - \lambda_j g_j\big(G_j(u)\big)\big] > 0.$$

Since $\lambda_j g_j(G_j(u)) = 0$, $j = 1, 2, \dots, m$, we have

$$\sum_{j=1}^{m} \beta_j(x,u) G_j'(u) \lambda_j g_j\big(G_j(u)\big) > 0$$

which is a contradiction to the conditions $\beta_j(x,u) > 0$, $\lambda_j \geq 0$ and $g_j(G_j(x) \leq 0$. Hence, x is a weak minimum for (CP), which completes the proof. ∎

4. DUALITY

We consider the Mond–Weir type of dual problem (D) for the problem (CP):

(D) Maximize $\left(f_1(F_1(u)), f_2(F_2(u)), \dots, f_p\left(F_p(u)\right)\right)$

subject to

$$0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i\left(F_i(u)\right) F_i'(u) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j\left(G_j(u)\right) G_j'(u) - (c-u)^+$$

$$\lambda_j g_j\left(G_j(u)\right) \geq 0, \qquad j = 1, 2, \dots, m.$$

$$u \in X, \qquad \tau \in R^p, \qquad \tau_i \geq 0, \qquad i = 1, \dots, p,$$

$$\lambda_j \geq 0, \qquad j = 1, 2, \dots, m.$$

THEOREM 4.1 (weak duality). *Let x be feasible for* (CP) *and* (u, τ, λ) *be feasible for* (D). *Assume that* $\sum_{j=1}^{m} \lambda_j \rho_j \geq 0$. *If $f(F)$ is V-ρ-invex and $g(G)$ is V-σ-invex with respect to the same functions η and θ, then $f(F(x)) < f(F(u))$.*

Proof. Since (u, τ, λ) is feasible for (D) and $\beta_j(x, u) > 0$, we have

$$\beta_j(x,u) \lambda_j g_j\left(G_j(x)\right) \leq \beta_j(x,u) \lambda_j g_j\left(G_j(u)\right).$$

Then, by the V-σ-invexity of $g(G)$, we have

$$\lambda_j \zeta_j \eta(x,u) + \lambda_j \sigma_j \| \theta(x,u) \|^2 \leq 0, \quad \text{for each } \zeta_j \in \partial^0 g_j\left(G_j(u)\right).$$

Thus, we have

$$\sum_{j=1}^{m} \lambda_j \zeta_j \eta(x,u) + \sum_{j=1}^{m} \lambda_j \sigma_j \| \theta(x,u) \|^2 \leq 0,$$

$$\text{for each } \zeta_j \in \partial^0 g_j\left(G_j(u)\right)$$

Since $0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i(F_i(u)) F_i'(u) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j(G_j(u)) G_i'(u)$ implies the existence of $\xi_i \in \partial^0 f_i(F_i(u))$ and $\zeta_j \in \partial^0 g_j(G_j(u))$ such that $\sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \zeta_j = 0$,
which implies that

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x,u) + \sum_{j=1}^{m} \lambda_j \zeta_j \eta(x,u) = 0,$$

hence, from the assumption $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \rho_j \geq 0$, we obtain

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x, u) + \sum_{i=1}^{p} \tau_i \rho_i \| \theta(x, u) \|^2 \geq 0.$$

Since $f(F)$ is V-ρ-invex, we have

$$\sum_{i=1}^{p} \alpha_i(x, u) \big[\tau_i f_i(F_i(x)) - \tau_i f_i(F_i(u)) \big] \geq 0.$$

Again, since $\alpha_i(x, u) > 0$, $\tau_i \geq 0$, and $\tau^T e = 1$, we have $f(F(x)) < f(F(u))$, which completes the proof. ▮

THEOREM 4.2 (strong duality). *Let $\bar{x}$ be a weak minimum of* (CP) *and a constraint qualification is satisfied. Then there exist $\bar{\tau} \in R^p$ and $\bar{\lambda} \in R^m$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is feasible for* (D). *If $f(F)$ is V-ρ-invex and $g(G)$ is V-ρ-invex with respect to the same functions η and θ, then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a weak minimum of* (D).

Proof. $\bar{x}$ is a weak minimum of (CP) and a constraint qualification is satisfied at $\bar{x}$ and from the generalized Karush–Kuhn–Tucker theorem (Theorem 6.1.3 of Clarke [1]) there exist $\tau \in R^p$ and $\lambda \in R^m$ such that

$$0 \in \sum_{i=1}^{p} \tau_i \partial^0 f_i(F_i(\bar{x})) F_i'(u) + \sum_{j=1}^{m} \lambda_j \partial^0 g_j\big(G_j(\bar{x})\big) G_i'(u),$$

$$\lambda_j g_j\big(G_j(\bar{x})\big) = 0, \qquad j = 1, 2, \dots, m,$$

$$\tau_i \geq 0, \qquad i = 1, \dots, p, \qquad \tau^T e > 0,$$

$$\lambda_j \geq 0, \qquad j = 1, 2, \dots, m.$$

Since $\tau_i \geq 0$, $i = 1, 2, \dots, p$ and $\tau^T e > 0$ and setting $\bar{\tau}_i = \tau_i / \sum_{i=1}^{p} \tau_i$ and $\bar{\lambda}_j = \lambda_j / \sum_{j=1}^{m} \lambda_j$, then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is feasible for (D). Since $\bar{x}$ is feasible for (CP), it follows from the weak duality Theorem 4.1 that $f_i(F_i(x)) \leq f(F_i(u))$ for any feasible u for (D). Hence $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a weak minimum of (D), which completes the proof. ▮

5. WEAK VECTOR SADDLE POINT

In this section, we prove weak vector saddle point theorems for the nonsmooth composite multiobjective programs whose functions are locally Lipschitz and Gateaux differentiable.

For a problem (CP), a point (x, τ, λ) is called to be a critical point of (CP) if x is a feasible point for (CP) and

$$0 \in \partial^0 \left(\sum_{i=1}^{p} \tau_i f_i\big(F_i(x)\big) + \sum_{j=1}^{m} \lambda_j g_j\big(G_j(x)\big) \right),$$

$$\lambda_j g_j\big(G_j(x)\big) = 0, \qquad \lambda_j \geq 0, \qquad j = 1, 2, \dots, m,$$

$$\tau_i \geq 0, \qquad i = 1, \dots, p, \qquad \tau^T e > 1.$$

Note that

$$\partial^0 \left(\sum_{i=1}^{p} \tau_i f_i\big(F_i(x)\big) + \sum_{j=1}^{m} \lambda_j g_j\big(G_j(x)\big) \right)$$
$$= \sum_{i=1}^{p} \tau_i \partial^0 \left(f_i\big(F_i(x)\big) + \sum_{j=1}^{m} \lambda_j g_j\big(G_j(x)\big) \right).$$

Let $(x, \tau, \lambda) = f(F(x)) + \lambda^T g(G(x))\, e$, where $x \in R^m$ and $\lambda \in R^m_+$.

Whenever we introduce $L(x, \tau, \lambda)$, it means that $L(x, \tau, \lambda)$ has p components.

DEFINITION 5.1. A point $(\bar{x}, \bar{\lambda}) \in R^n \times R^m_+$ is said to be a weak vector saddle point if $L(\bar{x}, \tau, \lambda) > L(\bar{x}, \bar{\tau}, \bar{\lambda}) > L(x, \bar{\tau}, \bar{\lambda})$, for all $x \in R^n$, $\tau \in R^p$, and $\lambda \in R^m_+$.

THEOREM 5.1 (saddle-point conditions). *Let* $(\bar{x}, \bar{\tau}, \bar{\lambda})$ *be a critical point of* (CP). *Assume that* $f(F(\cdot)) + \lambda^T g(G(\cdot))\, e$ *is* V-ρ-*invex with respect to functions* η *and* θ *and* $\sum_{i=1}^{p} \bar{\tau} \rho_i \geq 0$. *Then* $(\bar{x}, \lambda)$ *is a weak vector saddle point of* (CP).

Proof. Since $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a critical point for (CP), there exists $\xi_i \in \partial^0(f_i(F_i(\bar{x})) + \sum_{j=1}^{m} \bar{\lambda}_j g_j(G_j(\bar{x})))$ such that $\sum_{i=1}^{p} \bar{\tau}_i \xi_i = 0$. Again, since $\sum_{i=1}^{p} \bar{\tau}_i \rho_i \geq 0$, $\sum_{i=1}^{p} \tau_i \xi_i \eta(\bar{x}, u) + \sum_{i=1}^{p} \bar{\tau}_i \rho_i \|\theta(x, \bar{x})\|^2 \geq 0$. From V-ρ-invexity of $f(F(\cdot)) + \bar{\lambda}^T g(G(\cdot))\, e$, we obtain

$$\sum_{i=1}^{p} \alpha_i(x, \bar{x}) \bar{\tau}_i \Big[f_i\big(F_i(x)\big) - f_i\big(F_i(\bar{x})\big) + \bar{\lambda}^T g\big(G(x)\big) - \bar{\lambda}^T g\big(G(\bar{x})\big) \Big] \geq 0$$

for any $x \in R^n$.

Since $\alpha_i(x, x) > 0$, $\tau_i \geq 0$ and $\tau^T e = 1$, $i = 1, \dots, p$,

$$f\big(F(\bar{x}) + \bar{\lambda}^T g(G(x))\, e > f(F(x)) + \bar{\lambda}^T g(G(x))\, e,$$

for any $x \in R^n$,

i.e.,

$$L(\bar{x}, \overline{\lambda}) > L(x, \overline{\lambda}), \quad \text{for any } x \in R^n.$$

Again, since

$$\lambda^T g(G(x)) \leq 0, \quad \text{for any } \lambda \in R^m_+,$$

$$\overline{\lambda}^T g(G(x)) - \overline{\lambda}^T g(G(\bar{x})) \geq 0, \quad \text{for any } \lambda \in R^m_+.$$

Thus, $f(F(\bar{x})) + \overline{\lambda}^T g(G(\bar{x}))\, e - (f(F(x)) + \lambda^T g(G(\bar{x}))\, e) \in R^p_+$ and hence

$$L(\bar{x}, \lambda) > L(\bar{x}, \overline{\lambda}), \quad \text{for any } \lambda \in R^m_+.$$

Thus (x, λ) is a weak vector saddle point of (CP) which completes the proof (D).

THEOREM 5.2. *If there exists* $\bar{\tau} \in R^p$, $\overline{\lambda} \in R^m_+$ *such that* $(\bar{x}, \bar{\tau}, \overline{\lambda})$ *is a weak vector saddle point, then* $\bar{x}$ *is a weak minimum of* (CP).

Proof. Suppose that $(\bar{x}, \bar{\tau}, \overline{\lambda})$ is a weak vector saddle point. It follows from the left side inequality of saddle-point conditions that

$$f(F(\bar{x})) + \overline{\lambda}^T g(G(\bar{x}))\, e > f(F(x)) + \overline{\lambda}^T g(G(\bar{x}))\, e, \quad \text{for any } \lambda \in R^m_+$$

Thus, $\overline{\lambda}^T g(G(\bar{x}))\, e > \overline{\lambda}^T g(G(\bar{x}))\, e$, for any $\lambda \in R^m_+$ and hence we have

$$\lambda^T g(G(\bar{x})) \leq \overline{\lambda}^T g(G(\bar{x})), \quad \text{for any } \lambda \in R^m_+.$$

Since λ can be arbitrarily large, $g(G(x)) \leq 0$. Hence $\overline{\lambda}^T g(G(\bar{x})) \leq 0$. Finally, letting $\lambda = 0$ in (2) we obtain $\overline{\lambda}^T g(G(\bar{x})) \geq 0$. Thus, $\overline{\lambda}^T g(G(\bar{x})) = 0$.

Similarly, from the right side inequality of saddle-point conditions and $\overline{\lambda}^T g(G(\bar{x})) = 0$, we have for any feasible x for (CP),

$$f(F(\bar{x})) > f(F(x)).$$

Then, $\bar{x}$ is a weak minimum of (CP), which completes the proof. ∎

6. CONCLUSION

As has been observed in [4] one can easily check the feature that multiobjective fractional programs with V-ρ-invexity cannot be regarded as multiobjective programs with V-ρ-invexity. So, one can extend the results of the present paper for composite programs for the case of composite fractional multiobjective programs with a slightly different approach.

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