Journal of Mathematical Analysis and Applications **240**, 299–310 (1999)
Article ID jmaa.1999.6334, available online at http://www.idealibrary.com on IDEAL®

Some Results on Mathematical Programming with Generalized Ratio Invexity

L. Venkateswara Reddy and R. N. Mukherjee

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi, 221 005, India

Submitted by Koichi Mizukami

Received September 8, 1997

In this paper, a generalized ratio invexity concept has been applied for single objective fractional programming problems. A concept which has been invoked seems to be more general than the one used earlier by Khan and Hanson in such contexts. Further, duality results for fractional programs have also been obtained.

Key Words: mathematical programming; ratio invexity; duality.

1. INTRODUCTION

Recently Khan and Hanson (Ref. [11]) have used the ratio invexity concept to characterize optimality and duality results in fractional programming. This concept seems to be new and it introduces a modified kind of characterization in sufficient optimality conditions. Slightly away from this but introducing invexity conditions with indices ρ and θ, Suneja and Lalitha (Ref. [15]) have also characterized multiobjective fractional programming problem for duality results. In the ensuing paragraph we present an account of the fractional programming problem as depicted in Khan and Hanson (Ref. [11]).

Consider the nonlinear fractional programming problem as follows

$$\text{(FP)} \quad \text{minimize } \frac{f(x)}{g(x)}$$

$$\text{subject to } h(x) \leq 0, \; x \in X_0 \tag{1}$$

where X_0 is a subset of R^n; let f and g be real-valued functions defined on X_0; and let h be an m-dimensional vector valued function also defined

0022-247X/99 $30.00

on X_0. We let $\Delta = \{x \in X_0, h(x) \leq 0\}$ be the set of all feasible solutions. We assume that $f(x) \leq 0$ for all $x \in \Delta$ and $g(x) > 0$ for all $x \in \Delta$, and the functions f, g, and h satisfy

$$x, a \in \Delta \Rightarrow \begin{cases} f(x) - f(a) - \nabla f(a)\eta(x,a) \geq 0, \\ -g(x) + g(a) + \nabla g(a)\eta(x,a) \geq 0, \\ h(x) - h(a) - \nabla h(a)\eta(x,a) \geq 0 \end{cases}$$

with h: $X_0 \times X_0 \to R^n$.

These are called invex functions, first introduced by Hanson [7]. He presented sufficient optimality conditions and a weak duality theorem for the generalized convex programming problem

$$\text{(P)} \quad \text{minimize } f(x)$$
$$\text{subject to } h(x) \leq 0, \ x \in X.$$

The problem (P) is characterized as an invex problem, as was quoted in Craven (Ref. [2]). The problem (FP) as introduced above is said to be a convex–concave problem if f is convex, g is concave, and h is convex. It is then transformed into an invex problem if all the functions, f, g, and h, are taken as invex functions. Most of the references like Israel and Mond [1], Reiland [13], and Khan [10] have discussed invex problems and their generalizations for the multiobjective case. The paper of Khan and Hanson [11] in this respect could be thought of as a beginning of some investigation for invex fractional programming problems. The earlier papers of Craven [2], Weir [16], and Singh and Hanson [14] gave some partial references for the same subject. In [9], Jeyakumar and Mond have defined generalized invex functions called B-invex functions. The same concept has also been applied to treat fractional programming problems.

The paper is organized as follows. In Sect. 2 we will give some notations and preliminary definitions. In Sect. 3 we will state and prove necessary and sufficient optimality results. Finally, in Sect. 4 we will obtain duality results in the newer context.

2. PRELIMINARIES AND DEFINITIONS

The following definitions, which will be useful later in the sequel to our discussion, are given below.

DEFINITION 2.1 (Ref. [8]). Let f be a numerical function defined on an open set $X \subseteq R^n$ and let f be differentiable at $\bar{x} \in X$. Let ρ be a real number and $\eta, \theta: X \times X \to R^n$ be two functions. The function f is said to be

(a) ρ-invex at $\bar{x}$ with respect to η, θ if for each $x \in X$, such that

$$f(x) - f(\bar{x}) \geq \eta(x, \bar{x})^t \nabla f(\bar{x}) + \rho \|\theta(x, \bar{x})\|^2.$$

(b) ρ-pseudoinvex at $\bar{x}$ with respect η, θ if for each $x \in X$, such that

$$\eta(x, \bar{x})^t \nabla f(\bar{x}) + \rho \|\theta(x, \bar{x})\|^2 \geq 0 \Rightarrow f(x) \geq f(\bar{x}).$$

(c) ρ-strictly pseudoinvex at $\bar{x}$ with respect η, θ if for each $x \in X$, $x \neq \bar{x}$, such that

$$\eta(x, \bar{x})^t \nabla f(\bar{x}) + \rho \|\theta(x, \bar{x})\|^2 \geq 0 \Rightarrow f(x) > f(\bar{x}).$$

(d) ρ-quasiinvex at $\bar{x}$ with respect to η, θ if for each $x \in X$, such that

$$f(x) \leq f(\bar{x}) \Rightarrow \eta(x, \bar{x})^t \nabla f(\bar{x}) + \rho \|\theta(x, \bar{x})\|^2 \leq 0.$$

Remark 2.1. If f is ρ-invex at each $\bar{x} \in X$ with respect to η, θ then it is ρ-invex on X with respect to η, θ. The definition for other functions is similar.

A function f is said to be strongly invex if $\rho > 0$, invex if $\rho = 0$, and weakly invex if $\rho < 0$, which clearly shows that a strongly invex function is invex and hence an invex function is in turn weakly invex. Finally, every ρ-invex function is both ρ-pseudoinvex and ρ-quasiinvex.

We reconsider the nonlinear fractional programming problem (FP) as follows

$$\text{(FP)} \quad \text{minimize } \frac{f(x)}{g(x)}$$

$$\text{subject to } h(x) \leq 0, \ x \in X \subseteq R^n,$$

where $X \subseteq R^n$. Let f and g be real-valued functions defined on X and let h be an m-dimensional vector-valued function also defined on X.

The duality results for the above (FP) will be discussed in Sect. 4.

3. RESULTS

THEOREM 3.1. *Let f, g be two numerical functions defined on some open set $X \subseteq R^n$ such that $f(x) \leq 0$, $g(x) > 0$. If f is ρ-invex with η, θ then $\frac{f(x)}{g(x)}$ is ρ-invex with $\eta'(x, y)\frac{g(y)}{g(x)}$ and $\theta' = (\frac{1}{g(x)})^{1/2}(1 - \frac{f(y)}{g(y)})^{1/2}\theta$.*

Proof. If f is ρ-invex with η, θ and $-g$ is ρ-invex with the same η, θ, then we have

$$f(x) - f(\bar{x}) \geq \eta(x, \bar{x})^t \nabla f(\bar{x}) + \rho \|\theta(x, \bar{x})\|^2 \tag{2}$$

and

$$-g(x) + g(\bar{x}) \geq -\eta(x, \bar{x})^t \nabla g(\bar{x}) + \rho \|\theta(x, \bar{x})\|^2 \tag{3}$$

for every $x, \bar{x} \in X$.

Next, choose

$$\begin{aligned}
&\eta^T(x, \bar{x}) \nabla \left(\frac{f(\bar{x})}{g(\bar{x})} \right) \\
&\quad = n^t(x, \bar{x}) \left\{ \frac{1}{[g(\bar{x})]} \nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})^2} \nabla g(\bar{x}) \right\} \\
&\quad = \frac{1}{[g(\bar{x})]^2} \left[g(\bar{x}) \eta^t(x, \bar{x}) \nabla f(\bar{x}) - f(\bar{x}) \eta^t(x, \bar{x}) \nabla g(\bar{x}) \right]
\end{aligned}$$

and

$$\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} = \frac{f(x) - f(\bar{x})}{g(x)} - \frac{f(x)\{g(x) - g(\bar{x})\}}{g(x) g(\bar{x})}. \tag{4}$$

Since $f(x)$ and $-g(x)$ are ρ-invex functions with respect to $\eta(x, \bar{x})$ and $f(x) \leq 0$, $g(x) > 0$, Eq. (4) implies that

$$\begin{aligned}
&\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} \\
&\quad \geq \frac{1}{g(x)} \eta^t(x, \bar{x}) \nabla f(\bar{x}) + \frac{\rho}{g(x)} \|\theta(x, \bar{x})\|^2 \\
&\qquad - \frac{f(\bar{x})}{g(\bar{x}) g(x)} \eta^T(x, \bar{x}) \nabla g(\bar{x}) + \frac{f(\bar{x})}{g(\bar{x}) g(x)} \rho \|\theta(x, \bar{x})\|^2
\end{aligned}$$

$$= \frac{1}{g(x)} \eta^t(x, \bar{x}) \nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x}) g(x)} \eta^T(x, \bar{x}) \nabla g(\bar{x})$$

$$+ \frac{1}{g(x)} \left(1 - \frac{f(\bar{x})}{g(\bar{x})}\right) \rho \|\theta(x, \bar{x})\|^2$$

$$= \frac{1}{g(x)} \eta^t(x, \bar{x}) \nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x}) g(x)} \eta^t(x, \bar{x}) \nabla g(\bar{x})$$

$$+ \rho \left\| \left\{ \frac{1}{g(x)} \left(1 - \frac{f(\bar{x})}{g(\bar{x})}\right) \right\}^{1/2} \theta(x, \bar{x}) \right\|^2$$

$$\geq \frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla \left(\frac{f(\bar{x})}{g(\bar{x})} \right)$$

$$+ \rho \left\| \left\{ \frac{1}{g(x)} \left(1 - \frac{f(\bar{x})}{g(\bar{x})}\right) \right\}^{1/2} \theta(x, \bar{x}) \right\|^2 .$$

Hence the above inequality implies that $\frac{f(x)}{g(x)}$ is $(\frac{g(\bar{x})}{g(x)} \eta^T(x, \bar{x}), \rho)$—$\rho$-invex with $\theta' = (\frac{1}{g(x)})(1 - \frac{f(\bar{x})}{g(\bar{x})})^{1/2}\theta$.

This completes the proof of the theorem. ∎

Remark 3.1. If $\bar{x} \in X$ is a minimum for nonlinear, single-objective fractional programming problem (FP) and a constraint qualification (Ref. [12]) is satisfied, then the following Kuhn–Tucker conditions are necessary.

If f, $-g$ are ρ-invex with respect to η, θ then $\frac{f}{g}$ is ρ-invex with respect to $\eta^t(x, \bar{x})\frac{g(\bar{x})}{g(x)}$ and θ' such that

$$\nabla \left(\frac{f(\bar{x})}{g(\bar{x})} \right) + \nabla v_0 h(\bar{x}) = 0, \tag{5}$$

$$v_0 h(\bar{x}) = 0 \tag{6}$$

and

$$v_0 \geq 0. \tag{7}$$

THEOREM 3.2. *For a feasible solution $\bar{x}$ of (FP). Suppose the Kuhn–Tucker conditions* (5)–(7) *are satisfied at $\bar{x}$. Let $f(x) \leq 0$, $g(x) > 0$, where f and $-g$ are ρ-invex functions with respect to $\eta(x, \bar{x}) = \eta^t(x, \bar{x})\frac{g(\bar{x})}{g(x)}$*

and θ', and let h_i be ρ_i'-invex with respect to the same $\eta(x, \bar{x})^t \frac{g(\bar{x})}{g(x)}$ and $\theta' = (\frac{1}{g(x)})(1 - \frac{f(\bar{x})}{g(\bar{x})})^{1/2}\theta$ $(i = 1, 2, \ldots, m)$. Also $(\rho + v_0 \rho') \geq 0$ ($v_0 \rho'$ stands for the inner product of the vectors v_0 and ρ'). Then $\bar{x}$ is minimum for (FP).

Proof.

$$\begin{aligned}
&\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} \\
&\quad \geq \frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla \frac{f(\bar{x})}{g(\bar{x})} + \rho \left\| \frac{1}{g(x)} \left(1 - \frac{f(\bar{x})}{g(\bar{x})}\right)^{1/2} \theta(x, \bar{x}) \right\|^2 \\
&\quad = \frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla v_0 h(\bar{x}) + \rho \left\| \frac{1}{g(x)} \left(1 - \frac{f(\bar{x})}{g(\bar{x})}\right)^{1/2} \theta(x, \bar{x}) \right\|^2 \\
&\quad \geq -\frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla (v_0 h(\bar{x})) + \rho \left\| \frac{1}{g(x)} \left(1 - \frac{f(\bar{x})}{g(\bar{x})}\right)^{1/2} \theta(x, \bar{x}) \right\|^2 \\
&\qquad + v_0 h(x) - v_0 h(\bar{x}).
\end{aligned} \tag{8}$$

Since each h_i is ρ-invex with ρ_i' and θ'

$$\begin{aligned}
&\geq (\rho + v_0 \rho') \|\theta(x, \bar{x})\|^2 \\
&\geq 0,
\end{aligned} \tag{9}$$

we get

$$\left(\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} \right) \geq 0. \tag{10}$$

Therefore, $\bar{x}$ is a minimum. This completes the proof. ∎

In the following it is assumed that $f(\bar{x})$ and $-g(\bar{x})$ are ρ-invex with η, θ at u hence $\frac{f(x)}{g(\bar{x})}$ is ρ-invex with respect to $\eta(x, \bar{x}) = \eta(x, \bar{x})^t \cdot \frac{g(\bar{x})}{g(x)}$ and $\theta' = (\frac{1}{g(x)})(1 - \frac{f(\bar{x})}{g(\bar{x})})^{1/2}\theta$. Also, h_i is ρ_i'-invex at u with $\eta^t(\bar{x}, u)\frac{g(u)}{g(\bar{x})}$ and θ'. Also $(\rho + v_0 \rho') \geq 0$ ($v_0 \rho'$ stands for the inner product of the vectors v_0 and ρ'). In the following, we give a variant of Theorem 3.2.

THEOREM 3.3. *Suppose $\bar{x}$ is feasible for (FP) and that the Kuhn–Tucker conditions* (5)–(7) *are satisfied. Let $f \leq 0$ and $g > 0$, where f/g is ρ-pseudo-*

invex functions with respect to $\eta(x, \bar{x}) = \eta^T(x, \bar{x})\frac{g(\bar{x})}{g(x)}$ *and* θ' *and let* h_i *be* ρ_i*-quasiinvex with respect to the same* $\eta^t(x, x)\frac{g(\bar{x})}{g(x)}$ *and* θ', *as well as* $(\rho + v_0 \rho') \geq 0$ ($v_0 \rho'$ *stands for the inner product of the vectors* v_0 *and* ρ'). *Then* $\bar{x}$ *is a minimum for* (*FP*).

Proof. The proof is omitted as it can be given by the previous methods of Ref. [15]. ▮

4. DUALITY RESULTS

In this section, we consider the pair of invex fractional programming problems defined on $X \subseteq R^n$.

(FP) Primal problem:

$$\text{minimize } \frac{f(x)}{g(x)}$$

$$\text{subject to } h(x) \leq 0, \ x \in X \subseteq R^n. \tag{11}$$

The dual problem to the above primal problem is as follows:

(FD) Dual problem:

$$\text{maximize } \frac{f(\bar{x})}{g(\bar{x})}$$

$$\text{subject to } \nabla \frac{f(\bar{x})}{g(\bar{x})} + \nabla v_0 h(\bar{x}) = 0, \tag{12}$$

$$v_0 h(\bar{x}) = 0, \tag{13}$$

and

$$v_0 \geq 0. \tag{14}$$

THEOREM 4.1 (Weak duality). *If* x *is feasible for the primal problem* (*FP*) *and* $\bar{x}$ *is feasible for* (*FD*) *then*

$$\frac{f(x)}{g(x)} \geq \frac{g(\bar{x})}{g(\bar{x})}.$$

Proof. Since x is feasible for (FP) and $(\bar{x}, v_0)$ is feasible for (FD) and a constraint qualification [15] is satisfied at $(\bar{x}, v_0)$, therefore the following

Kuhn–Tucker conditions hold:

$$\nabla \frac{f(\bar{x})}{g(\bar{x})} + \nabla v_0 h(\bar{x}) = 0, \tag{15}$$

$$v_0 h(\bar{x}) = 0, \tag{16}$$

$$v_0 \geq 0. \tag{17}$$

For any $x \in X \subseteq R^n$ satisfying the constraint of (FP), we have

$$\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} \geq \frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla \frac{f(\bar{x})}{g(\bar{x})}$$
$$+ \rho \left\| \frac{1}{g(x)} \left\{ \left(1 - \frac{f(\bar{x})}{g(\bar{x})} \right) \right\}^{1/2} \theta(x, \bar{x}) \right\|^2 .$$

Since $\frac{f(x)}{g(x)}$ is ρ-invex w.r.t. $\frac{g(\bar{x})}{g(v)} \eta^t(x, \bar{x})$ at $\bar{x}$, using (15), we get

$$\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})}$$
$$= -\frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla v_0 h(\bar{x}) + \rho \left\| \frac{1}{g(x)} \left\{ \left(1 - \frac{f(\bar{x})}{g(\bar{x})} \right) \right\}^{1/2} \theta(x, \bar{x}) \right\|^2$$
$$\geq -\frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla v_0 h(\bar{x}) + \rho \left\| \frac{1}{g(x)} \left\{ \left(1 - \frac{f(\bar{x})}{g(\bar{x})} \right) \right\}^{1/2} \theta(x, \bar{x}) \right\|^2$$
$$+ v_0 h(\bar{x}) - v_0 h(\bar{x}).$$

Since $h_i(x)$ is ρ-invex w.r.t. $\frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x})$ at $\bar{x}$, we obtain

$$\geq (\rho + v_0 \rho')$$
$$\geq 0,$$
$$\frac{f(x)}{g(x)} \geq \frac{f(\bar{x})}{g(\bar{x})}.$$

This completes the proof. ∎

In the following theorem, it is assumed that $f(\bar{x})$ and $-g(\bar{x})$ are ρ-invex at u with respect to $\eta(x, \bar{x})$, hence $\frac{f(\bar{x})}{g(\bar{x})}$ (according to Theorem 4.1) and $v_0^T h_i(\bar{x})$ are ρ_i'-invex at u with respect to $(\frac{g(u)}{g(\bar{x})})\eta(\bar{x}, u)$.

THEOREM 4.2 (Strong duality). *Under the Kuhn–Tucker conditions if $\bar{x}$ is minimal for (FP) then there exists $0 \leq u \in R^n$ such that $(\bar{x}, \bar{u})$ is maximal for (FD) and the optimal values of (FP) and (FD) are equal.*

Proof. Let us suppose any (u, v) be vector, which also satisfies the constraints of (FD). For $(\bar{x}, v_0)$ to be maximal of (FD), we must show that

$$\frac{f(\bar{x})}{g(\bar{x})} - \frac{f(u)}{g(u)} \geq 0.$$

From the constraint (3) we obtain

$$\begin{aligned} \frac{f(\bar{x})}{g(\bar{x})} - \frac{f(u)}{g(u)} &\geq \frac{f(\bar{x})}{g(\bar{x})} - \frac{f(u)}{g(u)} - v^t h(u) \\ &\geq \frac{g(u)}{g(\bar{x})} \eta^t(\bar{x}, u) \nabla \frac{f(u)}{g(u)} - v^t h(u) \\ &\quad + \rho \left\| \left\{ \frac{1}{g(\bar{x})} \left(1 - \frac{f(\bar{x})}{g(\bar{x})} \right) \right\}^{1/2} \theta(\bar{x}, u) \right\|^2. \end{aligned}$$

Because h is ρ-invex with respect to $(\frac{g(\bar{x})}{g(u)}\eta^t(\bar{x}, u)\rho, \theta')$, we will get

$$\begin{aligned} &\geq -\frac{g(u)}{g(\bar{x})} \eta^t(\bar{x}, u) \nabla h(u) - v^t h(u) \\ &\quad + \rho \left\| \left\{ \frac{1}{g(\bar{x})} \left(1 - \frac{f(\bar{x})}{g(\bar{x})} \right) \right\}^{1/2} \theta(\bar{x}, u) \right\|^2. \end{aligned}$$

Since h_i is ρ'-invex with respect $\frac{g(u)}{g(\bar{x})}\eta^t(\bar{x}, u)$ at u with η and θ'

$$\begin{aligned} &\geq (\rho + v_0 \rho') \\ &\geq 0. \end{aligned}$$

Hence $\frac{f(\bar{x})}{g(\bar{x})} - \frac{f(u)}{g(u)} \geq 0$.

Hence $(\bar{x}, v_0)$ is maximal for (FD) and the objective values are equal in the two problems. This proves the theorem. ∎

THEOREM 4.3 (Converse duality). *Let $(\bar{x}, v_0)$ be a dual maximal for dual problem (FD) and a dual constraint qualification (Ref. [15]) holds at $(\bar{x}, v_0)$, then $(\bar{x}, v_0)$ satisfies the Kuhn–Tucker conditions*

$$\nabla \frac{f(\bar{x})}{g(\bar{x})} + \nabla v_0 h(\bar{x}) = 0, \tag{18}$$

$$v_0 h(\bar{x}) = 0, \tag{19}$$

$$v_0 \geq 0. \tag{20}$$

Proof. Since $(\bar{x}, v_0)$ is a maximal for the dual problem (FD) and a dual constraint qualification (Ref. [15]) holds at $(\bar{x}, v_0)$, then $(\bar{x}, v_0)$ satisfies the Kuhn–Tucker conditions

$$\nabla \frac{f(\bar{x})}{g(\bar{x})} + \nabla v_0 h(\bar{x}) = 0, \tag{21}$$

$$v_0 h(\bar{x}) = 0, \tag{22}$$

$$v_0 \geq 0. \tag{23}$$

For any $x \in X \subseteq R$ satisfying the constraints of (FP), then we must have

$$\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} \geq \frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla \frac{f(\bar{x})}{g(\bar{x})} + \rho \|\theta(\bar{x}, u)\|^2.$$

Since $\frac{f(x)}{g(x)}$ is ρ-invex with respect to $\frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x})$ at $\bar{x}$ and $\theta' = (\frac{1}{g(\bar{x})})(1 - \frac{f(\bar{x})}{g(\bar{x})})^{1/2}\theta$, we have

$$\begin{aligned} \frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} &= -\frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \nabla v_0 h(\bar{x}) - v^t h(\bar{x}) \\ &\quad + \rho \left\| \left\{ \left(\frac{1}{g(x)} \right) \left(1 - \frac{f(\bar{x})}{g(\bar{x})} \right) \right\}^{1/2} \right\|^2 \\ &\geq -v_0 h(x) - v_0 h(\bar{x}). \end{aligned}$$

Since $h_i(x)$ is ρ'-invex with respect to $\frac{g(\bar{x})}{g(x)} \eta^t(x, \bar{x}) \rho$ at $\bar{x}$ with η and θ'

$$\begin{aligned} &\geq (\rho + v_0 \rho') \\ &\geq 0, \end{aligned}$$

we get $\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} \geq 0$.

Thus $\bar{x}$ is a minimum for (FP). This completes the proof of theorem. ∎

Remark. As noted in the last part of the Introduction, the difficulty faced is to extend optimality and duality concepts in the context of multiobjective fractional programming problems, which has been depicted as follows. For example, a given multiobjective fractional programming can be described in the following manner

$$v\text{-max}\left\{\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_p(x)}{g_p(x)}\right\}$$
$$\text{Subject to } h_j(x) \leq 0,\ j = 1, 2, \dots, m,$$

where suitable conditions are put on the functions f_i, g_i, and h_j ($i = 1, \dots, p$; $j = 1, 2, \dots, m$). In the final analysis for optimality and duality we require some suitable ratio invexity coefficients to be derived for the functions of the type $\sum_{i=1}^{p} \lambda_i f_i / g_i$ which ultimately validate the analysis as described in the treatment shown for the case of single objective fractional program. It seems that the method as adopted in the case of single objective fractional programming does not extend easily in the case of multiobjective fractional programming. Therefore, the problem remains very much open and requires further investigation.

ACKNOWLEDGMENT

The authors are indebted to the referee for his valuable comments regarding the revision. The remarks, as has been forwarded for the case of further extensions of our results, have evolved due to the observations made earlier by the referee.

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