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Mixed Type Duality for Multiobjective Variational Problems

R. N. Mukherjee and Ch. Purnachandra Rao

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 221 005, India

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The concept of mixed-type duality has been extended to the class of multiobjective variational problems. A number of duality relations are proved to relate the efficient solutions of the primal and its mixed-type dual problems. The results are obtained for ρ-convex (generalized ρ-convex) functions. These studies have been generalized to the case of ρ-invex (generalized ρ-invex) functions. Our results apparently generalize a fairly large number of duality results previously obtained for finite-dimensional nonlinear programming problems under various convexity assumptions. © 2000 Academic Press

Key Words: mixed-type dual; efficient solution; ρ-convex; ρ-invex; multiobjective programming.

1. INTRODUCTION

The duality theory has been studied extensively in the nonlinear programming literature. This theory may be regarded as the most delicate subject in the theory of nonlinear programming, and its theoretical importance cannot be questioned (e.g., in the theory of computational algorithms of linear programming and in the theory of prices and markets in economics). The main question which is investigated in the duality is as follows: under which assumptions is it possible to associate an equivalent maximization (dual) problem to a given minimization (primal) problem? For this purpose, in the recent past various duality models have appeared. Among many such models, two well-known duality models are Wolfe dual and Mond–Weir dual, which were widely used in the area of finite-dimen-

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sional smooth and nonsmooth nonlinear programming problems. Quite recently, Zengkun Xu [6] introduced a mixed-type duality model which contains the above two models as special cases and establishes various duality results by relating "efficient" solutions of his mixed-type dual pair of problems.

On-the other hand, another basic concept in the theory of nonlinear programming is the generalization of convexity, which assumes a central role in many aspects of mathematical programming, including sufficient optimality conditions, duality relations, theorems of alternatives, and convergence of the optimization algorithms. Various generalizations of convexity, for example, invexity, quasiinvexity, and pseudoinvexity (see, e.g., [3]), are quite close to convexity in the sense that they preserve some of the important properties of convexity. Another generalization of convexity known as ρ-convexity, in which the defining inequality for convex holds approximately, to within a term depending on a parameter ρ which may be zero (convex), positive (strongly convex), or negative (weakly convex), was introduced by Vial [4], whose role in the construction and convergence analysis of algorithms in nonlinear programming is well known. The notion of ρ-convexity has been further generalized to the notion of ρ-invexity by Jeyakumar [2]. Quite recently, many papers have been devoted to the study of these functions for the class of variational and control problems; for example, one may consult [1].

The purpose of this paper is to introduce a continuous analog of the (static) mixed-type dual introduced quite recently by Zengkun Xu [6], in a class of variational multiobjective programming problems, and to establish a fairly large number of duality results by relating efficient solutions between this mixed-type dual pair. The results are obtained for differentiable ρ-convex (generalized ρ-convex) functions and ρ-invex (generalized ρ-invex) functions in their continuous version. These duality results contain as special cases the counterparts of most well-known results originally obtained for conventional nonlinear programming problems with differentiable data.

2. NOTATION AND PRELIMINARIES

Let $I = [a, b]$ be a real interval and let $\{1, 2, \ldots, p\} = P$ and $\{1, 2, \ldots, m\} = M$. In this paper we assume the following: $x(t)$ is an n-dimensional piecewise smooth function of t, and $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t in $[a, b]$.

For notational simplicity, we shall write, as and when necessary, $x(t)$ and $\dot{x}(t)$ as x and $\dot{x}$, respectively, and so on. We denote the partial derivatives of f with respect to t, x and $\dot{x}$, respectively, by f_t, f_x, and $f_{\dot{x}}$ such that $f_x = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)$ and $f_{\dot{x}} = (\partial f/\partial \dot{x}_1, \dots, \partial f/\partial \dot{x}_n)$. Similarly, the partial derivatives of the vector function g can be written, using matrices with m rows instead of one. Let S denote the space of n-dimensional piecewise smooth functions x with norm $\|x\| = \|x\|_\infty = \|Dx\|_\infty$, where the differentiation operator D is $u = Dx \leftrightarrow x(t) = \int_{a_0}^{t} u(s)\, ds$, where a_0 is a given boundary value. Therefore, $D = d/dx$, except at discontinuities.

Remark 1. For notational simplicity, no notational distinction is made between row and column vectors. Subscripts denote partial derivatives, and superscripts denote vector components. Unless otherwise specified, for any indexed set $T = \{1, 2, 3, \dots, t\}$, Σ_T means the sum over all $i \in T$.

We now give some definitions which will be used subsequently in our later results. Let $F[x]$: $S \to \mathbb{R}$, denoted by $F[x] = \int_a^b f(t, x, \dot{x})\, dt$, be Frechet differentiable. Let ρ be a real number. At a point u in S, we define a functional F to be

(a) ρ-convex if $\exists$ a real number ρ such that $\forall x$ ($\neq u$) in S, $F[x] - F[u] \geq \int_a^b \{(x - u) f_u(t, u, \dot{u}) + (D(x - u)) f_{\dot{u}}(t, u, \dot{u})\}\, dt + \rho \|x - u\|^2$ or strictly ρ-convex if strict inequality holds;

(b) ρ-pseudoconvex if $\exists$ a real number ρ such that $\forall x$ ($\neq u$) in S, $\int_a^b \{(x - u) f_u(t, u, \dot{u}) + (D(x - u)) f_{\dot{u}}(t, u, \dot{u})\}\, dt \geq -\rho \|x - u\|^2 \to F[x] \geq F[u]$ or strictly ρ-pseudoconvex if strict inequality holds in the right-hand inequality of the above implication;

(c) ρ-quasiconvex if $\exists$ a real number ρ such that, $\forall x$ ($\neq u$) in S, $F[x] \leq F[u] \to \int_a^b \{(x - u) f_u(t, u, \dot{u}) + (D(x - u)) f_{\dot{u}}(t, u, \dot{u})\}\, dt \leq -\rho \|x - u\|^2$.

From this point on we use the term "generalized ρ-convexity" to indicate ρ-pseudoconvexity, ρ-quasiconvexity, etc. Now the most immediate way to extend the ρ-convexity (generalized ρ-convexity) to the vector functions requires the ρ-convexity (generalized ρ-convexity) of the single components. For this purpose let $h = (h^1, h^2, \dots, h^n)$ be an n-dimensional vector function and each of its components be ρ-convex (generalized ρ-convex) at the same point u. Also, let $k = (k_1, k_2, \dots, k_n)$ be a vector constant such that $k_i \geq 0$ for all $i = 1, 2, \dots, n$. Then

(a) $\Sigma_N h^i(t, ., .)$ is $\Sigma_N \rho_i$-convex at u,

(b) Each $k_i f^i(t, ., .)$ is $k_i\, \rho_i$-convex at u, and hence

(c) $h(t, ., .)$ is $\Sigma_N \rho_i$-convex at u.

These properties will be used frequently throughout the paper without being specified.

3. FORMULATION OF THE MAIN PROBLEM

We consider the following multiobjective variational programming problem:

$$\text{Min} \int_a^b f(t, x, \dot{x})\, dt \tag{MP}$$

subject to

$$x \in X = \{x \in S \,|\, x(a) = a_0,\, x(b) = b_0,\, g(t, x, \dot{x}) \leq 0,\, t \in I\},$$

where $f = (f^1, f^2, \dots, f^p)$; $I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^P$, each component function is a continuously differentiable real scalar function, and $g = (g^1, g^2, \dots, g^m)$: $I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is an m-dimensional continuously differentiable vector function.

Since the objectives in multiobjective programming problems generally conflict with one another, an optimal solution is chosen from the set of "efficient" solutions in the following sense, and Min means finding n-dimensional piecewise smooth efficient solution $x = x(t)$, $t \in I$, for the problem (MP).

DEFINITION 1. An n-dimensional piecewise smooth function$u$ in the feasible region of the problem (MP) is said to be an efficient solution for the problem (MP) if $\forall x \in X$ and $\forall i \in P$:

$$\int_a^b f^i(t, u, \dot{u}) \geq \int_a^b f^i(t, x, \dot{x})\, dt \to \int_a^b f^i(t, u, \dot{u}) = \int_a^b f^i(t, x, \dot{x})\, dt.$$

DEFINITION 2. An n-dimensional piecewise smooth function u in the feasible region of the problem (MP) is said to be a weak minimum for the problem (MP) if there exists no other x in X for which $\int_a^b f(t, u, \dot{u}) > \int_a^b f(t, x, \dot{x})\, dt$.

From the above two definitions it follows that if x in X is an efficient solution for (MP), then it is also a weak minimum for (MP). Before presenting the mixed-type dual to (MP) we state, in the form of the following proposition, the continuous version of Theorem 2.2 of [5], which will be needed in the proof of the Strong Duality Theorem.

PROPOSITION 1. *Let $\bar{x}$ be a weak minimum for (MP) at which the Kuhn–Tucker constraint qualification is satisfied. Then there exist α in $\mathbb{R}^p$*

and a piecewise smooth $\beta(.)$: $I \to \mathbb{R}^m$ *such that*

$$\left[\alpha f_x(t, \bar{x}, \dot{\bar{x}}) + \beta(t) g_x(t, \bar{x}, \dot{\bar{x}})\right] = D\left[\alpha f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \beta(t) g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right]$$

$$\int_a^b \beta(t) g(t, \bar{x}, \dot{\bar{x}})\, dt = 0$$

$$\beta(t) \geq 0, \qquad \alpha e = 1, \qquad \alpha \geq 0,$$

where e is the vector of $\mathbb{R}^p$, *the components of which are all ones.*

We divide the index set M of the constraint functions of the problem (MP) into two disjoint subsets, namely J and K, such that $JUK = M$, and let

$$\beta_J(t) g^J(t, x, \dot{x}) = \Sigma_J\, \beta_i(t) g^i(t, x, \dot{x})$$

$$\beta_K(t) g^K(t, x, \dot{x}) = \Sigma_K\, \beta_i(t) g^i(t, x, \dot{x}).$$

Now we introduce the continuous analog of the static mixed-type dual [6], for the primal problem (MP).

$$\text{Max} \int_a^b \left\{ f(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u}) e \right\} dt \tag{MD}$$

subject to

$$\left[\alpha f_u(t, u, \dot{u}) + \beta(t) g_u(t, u, \dot{u})\right] = D\left[\alpha f_{\dot{u}}(t, u, \dot{u}) + \beta(t) g_{\dot{u}}(t, u, \dot{u})\right] \tag{1}$$

$$\int_a^b \beta_K(t) g^K(t, u, \dot{u}) \geq 0 \tag{2}$$

$$\beta(t) \geq 0, \qquad \alpha e = 1, \qquad \alpha \geq 0 \tag{3}$$

$$x(a) = a_0, \qquad x(b) = b_0,$$

where e is the vector of $\mathbb{R}^p$, the components of which are all ones.

4. DUALITY THEOREMS

In this section, we present and discuss a fairly large number of duality results between (MP) and (MD) by imposing various ρ-convexity (generalized ρ-convexity) conditions upon the objective and constraint functions. We begin with a situation in which all of the functions are ρ-convex.

Subsequently, we formulate more general duality criteria in which the generalized ρ-convexity requirements are placed on certain combinations of the objective and constraint functions.

Let Y denote the set of all feasible solutions of (MD). The theorems that follow are weak duality theorems in which we prove that

$$\int_a^b f^i(t,x,\dot{x})\,dt \leq \int_a^b \{f^i(t,u,\dot{u}) + \beta_J(t)g^J(t,u,\dot{u})\}\,dt \tag{4}$$

cannot hold for x in X and u in Y, for all i in P, and for some i in P,

$$\int_a^b f^i(t,x,\dot{x})\,dt < \int_a^b \{f^i(t,u,\dot{u}) + \beta_J(t)g^J(t,u,\dot{u})\}\,dt. \tag{5}$$

THEOREM 1. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(1a) *for each $i \in P$, $f^i(t,.,.)$ is ρ_i-convex, and for each $j \in M$, $g^j(t,.,.)$ is γ_j-convex; then* (4) *and* (5) *cannot hold if either of the following hold.*

(1b) *For each $i \in P$, $\alpha_i > 0$ with $\Sigma_P \alpha_i \rho_i + \Sigma_M \beta_j(t)\gamma_j \leq 0$*

(1c) $\Sigma_P \alpha_i \rho_i + \Sigma_M \beta_j(t)\gamma_j > 0$.

Proof. If $x = u$, then a weak duality theorem trivially holds, so assume that $x \neq u$. From the duality constraint (1) we have

$$\int_a^b (x-u)[\alpha f_u(t,u,\dot{u}) + \beta(t)g_u(t,u,\dot{u})]\,dt$$
$$= \int_a^b (x-u)D[\alpha f_{\dot{u}}(t,u,\dot{u}) + \beta(t)g_{\dot{u}}(t,u,\dot{u})]\,dt. \tag{6}$$

Suppose, on the contrary, that (4) and (5) hold. These inequalities imply, in view of the feasibility of x for (MP), that

$$\int_a^b \{f^i(t,x,\dot{x}) + \beta_J(t)g^J(t,x,\dot{x})\}\,dt$$
$$\leq \int_a^b \{f^i(t,u,\dot{u}) + \beta_J(t)g^J(t,u,\dot{u})\}\,dt$$

for all $i \in P$, and for some $i \in P$,

$$\int_a^b \{f^i(t,x,\dot{x}) + \beta_J(t)g^J(t,x,\dot{x})\}\,dt$$
$$< \int_a^b \{f^i(t,u,\dot{u}) + \beta_J(t)g^J(t,u,\dot{u})\}\,dt.$$

From the strict positivity of each component α_i of α and the fact that $\alpha e = 1$, it follows that

$$\int_a^b \{\alpha f(t,x,\dot{x}) + \beta_J(t) g^J(t,x,\dot{x})\}\, dt$$

$$< \int_a^b \{\alpha f^i(t,u,\dot{u}) + \beta_J(t) g^J(t,u,\dot{u})\}\, dt. \tag{7}$$

Now by the definitions of ρ_i-convexity of $f^i(t,.,.)$, $i \in P$, and γ_j-convexity of $g^j(t,.,.)$, $j \in M$, we have

$$\int_a^b \{f^i(t,x,\dot{x}) - f^i(t,u,\dot{u})\}\, dt$$

$$\geq \int_a^b \{(x-u) f_u^i(t,u,\dot{u}) + (D(x-u)) f_{\dot{u}}^i(t,u,\dot{u})\}\, dt$$

$$+ \rho_i \|x-u\| \qquad \text{for all } i \in P \tag{8}$$

$$\int_a^b \{g^j(t,x,\dot{x}) - g^j(t,u,\dot{u})\}\, dt$$

$$\geq \int_a^b \{(x-u) g_u^j(t,u,\dot{u}) + (D(x-u)) g_{\dot{u}}^j(t,u,\dot{u})\}\, dt$$

$$+ \gamma_j \|x-u\|^2 \qquad \text{for all } j \in M. \tag{9}$$

On multiplying each inequality of (8) by each α_i of $\alpha \in \mathbb{R}_+^p$ and each inequality of (9) by each $\beta_j(t)$ of $\beta(t) \in \mathbb{R}_+^m$, and adding the inequalities (among $i \in P$ and $j \in M$) we obtain

$$\int_a^b \{\alpha f(t,x,\dot{x}) + \beta(t) g(t,u,\dot{u}) - \alpha f(t,u,\dot{u}) - \beta(t) g(t,u,\dot{u})\}\, dt$$

$$\geq \int_a^b \{(x-u)[\alpha f_u(t,u,\dot{u}) + \beta(t) g_u(t,u,\dot{u}))]$$

$$+ (D(x-u))[\alpha f_{\dot{u}}(t,u,\dot{u}) + \beta(t) g_{\dot{u}}(t,u,\dot{u})]\}\, dt$$

$$+ \left(\Sigma_P \alpha_i \rho_i + \Sigma_M \beta_j(t) \dot{\gamma}_j\right) \|x-u\|^2.$$

By integration by parts, the right-hand side reduces to the following, via (1b):

$$\int_a^b \{(x-u)[\alpha f_u(t,u,\dot{u}) + \beta(t) g_u(t,u,\dot{u})]\}\, dt$$

$$+ \{[\alpha f_{\dot{u}}(t,u,\dot{u}) + \beta(t) g_{\dot{u}}(t,u,\dot{u})](x-u)\}_{t=a}^{t=b}$$

$$- \int_a^b \{(x-u) D[\alpha f_{\dot{u}}(t,u,\dot{u}) + \beta(t) g_{\dot{u}}(t,u,\dot{u})]\, dt.$$

On making use of the boundary conditions (6), the above yields

$$\int_a^b \{ \alpha f(t,x,\dot{x}) + \beta(t) g(t,x,\dot{x}) - \alpha f(t,u,\dot{u}) - \beta(t) g(t,u,\dot{u}) \} \, dt \geq 0 \tag{10}$$

Since $M = JUK$,

$$\beta(t) g(t,.,.) = \beta_J(t) g^J(t,.,.) + \beta_K(t) g^K(t,.,.), \tag{11}$$

and hence the above inequality implies, along with (7), that

$$\int_a^b \{ \beta_K(t) g^K(t,x,\dot{x}) - \beta_K(t) g^K(t,u,\dot{u}) \} \, dt > 0. \tag{12}$$

Now, since $(u, \alpha, \beta(t) \in Y$, from (2), $\int_a^b \beta_K(t) g^K(t, x, \dot{x}) > 0$, which is a contradiction of the fact that x is feasible for (MP), and hence (4) and (5) cannot hold.

(1c) In this case the multipliers α_i of the objective functions $f^i(t,.,.)$ need not be strictly positive, and it gives $\leq$ in place of $<$ of (7). If we assume the condition in (1c), we get $>$ in place of $\geq$ of (10). Hence we get (12) and we conclude the theorem as in the case of (1b). This completes the proof.

Evidently, the above theorem has a number of important special cases which can readily be identified by the suitable algebraic properties of the ρ-convex functions. We shall state some of these as corollaries. The static function analogs of these weak duality results are well known in the area of nonlinear programming.

COROLLARY 1. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(a) *for each $i \in P$, $f^i(t,.,.)$ is ρ_i-convex, and for each $j \in M$, $\beta_j(t) g^j(t,.,.)$ is γ_j-convex then* (4) *and* (5) *cannot hold if either of the following holds.*

(b) *For each $i \in P$, $\alpha_i > 0$ with $\Sigma_P \alpha_i \rho_i + \Sigma_M \gamma_j \geq 0$ or*

(c) $\Sigma_P \alpha_i \rho_i + \Sigma_M \gamma_j > 0$.

Proof. Since $g^j(t,.,.)$ is γ_j-convex whenever $\beta_j(t) g^j(t,.,.)$ is $\beta_j(t)\gamma_j$-convex and $\beta_j(t) \geq 0$, the proof is similar to Theorem 1.

COROLLARY 2. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$ and assume as in Corollary* 1, *except that instead of $\beta_j(t) g^j(t,.,.)$ being γ_j-convex*, (b) *and* (c), *the function $(t, u, \dot{u}) \to \Sigma_M \beta_j(t) g^j(t, u, \dot{u})$ is γ-convex*, (b) *for each $i \in P \alpha_i > 0$ with $\Sigma_P \alpha_i \rho_i + \gamma \geq 0$ and* (c) $\Sigma_P \alpha_i \rho_i + \gamma > 0$, *respectively. Then* (4) *and* (5) *cannot hold.*

Note that in Theorem 1, each constraint function $g^j(t,.,.)$ is assumed to be γ_j-convex, whereas in Corollary 2 they are aggregated into one γ-convex function. We observe that it is also possible to consider a situation intermediate between these two extreme cases (keeping in view the partition of the constraint function in the objective function of the dual problem (MD)), in which some of the constraint functions can be combined into a γ-convex function while the rest are individually γ-convex. Situations of this type are presented in the next two corollaries.

COROLLARY 3. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(a) *for each $i \in P$, $f^i(t,.,.)$ is ρ_i-convex and $\beta_J(t)g^J(t,.,.)$ is γ_J-convex, whereas for each $j \in K$, $\beta_j(t)g^j(t,.,.)$ is γ_j-convex. Then* (4) *and* (5) *cannot hold if either of the following holds.*

(b) *For each $i \in P$, $\alpha_i > 0$ with $\Sigma_P \alpha_i \rho_i + \gamma_J + \Sigma_K \gamma_j \geq 0$ or*

(c) $\Sigma_P \alpha_i \rho_i + \gamma_J + \Sigma_K \gamma_j > 0$.

COROLLARY 4. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(a) *For each $i \in P$, $f^i(t,.,.)$ is ρ_i-convex and $\beta_J(t)g^J(t,.,.)$ is γ_J-convex, whereas $\beta_K(t)g^K(t,.,.)$ is γ_K-convex. Then* (4) *and* (5) *cannot hold if either of the following holds.*

(b) *For each $i \in P$, $\alpha_i > 0$ with $\Sigma_P \alpha_i \rho_i + \gamma_J + \gamma_K \geq 0$ or*

(c) $\Sigma_P \alpha_i \rho_i + \gamma_J + \gamma_K > 0$.

The next corollary is the situation in which all of the objectives and constraints are aggregated into a single one.

COROLLARY 5. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(a) *$\alpha f(t,.,.) + \beta(t)g(t,.,.)$ is γ-convex, then* (4) *and* (5) *cannot hold if either of the following holds.*

(b) *For each $i \in P$, $\alpha_i > 0$ with $\rho \geq 0$ or*

(c) $\rho > 0$.

In the rest of this section we use the generalized ρ-convexity. From this point on we will try to restrict ourselves in most of the cases to situations in which only scalarizations of the objective and constraint functions are considered. And we remark here that the immediate consequences in each of those situations in the form of corollaries can easily be seen, just as in the case of Theorem 1. We do not explicitly state these corollaries.

THEOREM 2. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(2a) *$\beta_K(t)g^K(t,.,.)$ is ρ-quasiconvex.*

(2b) *For each $i \in P$, $\alpha_i > 0$ and $f^i(t,.,.) + \beta_J(t)g^J(t,.,.)$ is both γ_i-quasiconvex and γ_i-pseudoconvex with $\Sigma_P \alpha_i \rho_i + \rho \geq 0$. Then* (4) *and* (5) *cannot hold.*

Proof. If $x = u$, then a weak duality theorem trivially holds, so assume that $x \neq u$. Since $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, we have

$$\int_a^b \beta_K(t) g^K(t, x, \dot{x})\, dt \leq 0 \leq \int_a^b \beta_K(t) g^K(t, u, \dot{u})\, dt. \tag{13}$$

ρ-Quasiconvexity in (2a), in view of the above, implies that

$$\int_a^b \{(x-u)\beta_K(t) g_u^K(t,u,\dot{u}) + [D(x-u)\beta_K(t) g_{\dot{u}}^K(t,u,\dot{u})]\}$$
$$\leq -\rho\|x-u\|^2. \tag{14}$$

The substitution of the duality constraint (1) in the first term of the above implication gives us, along with (11),

$$\int_a^b [(x-u)\{D[\alpha f_{\dot{u}}(t,u,\dot{u}) + \beta_J(t) g_{\dot{u}}^J(t,u,\dot{u}) + \beta_K(t) g_{\dot{u}}^K(t,u,\dot{u})]$$
$$-\alpha f_u(t,u,\dot{u}) + \beta_J(t) g_u^J(t,u,\dot{u})\}]\, dt$$
$$+\int_a^b \{D(x-u)\beta_K(t) g_{\dot{u}}^K(t,u,\dot{u})\}\, dt \leq -\rho\|x-u\|^2. \tag{15}$$

On using the boundary conditions after integration by parts,

$$\int_a^b (x-u)\{\alpha f_u(t,u,\dot{u}) + \beta_J(t) g_u^J(t,u,\dot{u}) + (D(x-u))$$
$$\times[\alpha f_{\dot{u}}(t,u,\dot{u}) + \beta_J(t) g_{\dot{u}}^J(t,u,\dot{u})]\}\, dt \geq \rho\|x-u\|^2. \tag{16}$$

That is, on making use of the condition $\Sigma_P \alpha_i \gamma_i + \rho \geq 0$ and the fact that $\alpha e = 1$, we have the following:

$$\Sigma_P \alpha_i \bigg[\int_a^b \{(x-u)[f_u^i(t,u,\dot{u}) + \beta_J(t) g_u^J(t,u,\dot{u})]$$
$$+[D(x-u)][f_{\dot{u}}^i(t,u,\dot{u}) + \beta_J(t) g_{\dot{u}}^J(t,u,\dot{u})]\}\, dt$$
$$\geq -(\Sigma_P \alpha_i \gamma_i)\|x-u\|^2.$$

Since $\alpha_i > 0$, $i \in P$, it follows from the above that

$$\int_a^b \{(x-u)[f_u^i(t,u,\dot{u}) + \beta_J(t) g_u^J(t,u,\dot{u})]$$
$$+[D(x-u)][f_{\dot{u}}^i(t,u,\dot{u}) + \beta_J(t) g_{\dot{u}}^J(t,u,\dot{u})]\}\, dt$$
$$\geq -\gamma_i\|x-u\|^2 \tag{17}$$

for all $i \in P$, and for some $i \in P$,

$$\begin{aligned}\int_a^b \{ &(x-u)\left[f_u^i(t,u,\dot{u}) + \beta_J(t) g_u^J(t,u,\dot{u})\right] \\ &+ \left[D(x-u)\right]\left[f_{\dot{u}}^i(t,u,\dot{u}) + \beta_J(t) g_{\dot{u}}^J(t,u,\dot{u})\right]\} \, dt \\ &> -\gamma_i \|x-u\|^2. \end{aligned} \tag{18}$$

Suppose (17) holds; then the γ_i-pseudoconvexity assumption in (2b) gives, along with the feasibility of x for (MP), for all $i \in P$,

$$\int_a^b f^i(t,x,\dot{x}) \, dt \geq \int_a^b \{ f^i(t,u,\dot{u}) + \beta_J(t) g^J(t,u,\dot{u}) \} \, dt. \tag{19}$$

Now suppose (18) holds; then the equivalent form of the γ_i-quasiconvexity assumption in (2b) gives, along with the feasibility of x for (MP), for some $i \in P$,

$$\int_a^b f^i(t,x,\dot{x}) \, dt > \int_a^b \{ f^i(t,u,\dot{u}) + \beta_J(t) g^J(t,u,\dot{u}) \} \, dt. \tag{20}$$

Obviously (19) and (20) can show that (4) and (5) cannot hold, which completes the proof.

The following theorem is stated without proof. It will be established in a manner very similar to that of Theorem 2.

THEOREM 3. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(3a) *$\beta_K(t) g^K(t,.,.)$ is ρ-quasiconvex.*

(3b) *For each $i \in P$, $\alpha_i > 0$ and $f^i(t,.,.) + \beta_J(t) g^J(t,.,.)$ is γ_i-quasiconvex and there exists some $k \in P$ such that it is strictly γ_k-pseudoconvex (with the corresponding component α_k of α positive) with $\Sigma_P \alpha_i \gamma_i + \rho \geq 0$. Then (4) and (5) cannot hold.*

THEOREM 4. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(4a) *$\beta_K(t) g^K(t,.,.)$ is ρ-quasiconvex.*

(4b) *For each $i \in P$, $\alpha_i > 0$ and $\alpha f(t,.,.) + \beta_J(t) g^J(t,.,.)$ is γ-pseudoconvex with $\rho + \gamma \geq 0$. Then (4) and (5) cannot hold.*

Proof. As in the case of Theorem 2, assume $x \neq u$ and get (16). By making use of the condition $\rho + \gamma \geq 0$ and by our γ-pseudoconvexity assumption in (4b) we obtain

$$\int_a^b \{\alpha f(t, x, \dot{x}) + \beta_J(t) g^J(t, x, \dot{x})\}\, dt$$
$$\geq \int_a^b \{\alpha f(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u})\}\, dt.$$

Now the feasibility of x for (MP) and the fact that $\alpha e = 1$ imply

$$\alpha \int_a^b f(t, x, \dot{x}) \geq \alpha \int_a^b \{f(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u})\}\, dt.$$

Clearly, this concludes the theorem, since $\alpha_i > 0$ for each $i \in P$.

The assumption that $\beta_K(t) g^K(t, ., .)$ is ρ-quasiconvex is very important, as we see in the previous theorems (2–4). Of course, to get the desired results without this condition, other conditions should be enforced, which leads to the following theorem.

THEOREM 5. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(5a) *For each $i \in P$, $\alpha_i > 0$ and $f^i(t, ., .) + \beta(t) g(t, ., .)$ is both ρ-pseudoconvex and ρ-quasiconvex with $\Sigma_P \alpha_i \rho_i \geq 0$. Then* (4) *and* (5) *cannot hold.*

Proof. Assume $x \neq u$. From the duality constraint (1) we get (6). Now by integration by parts,

$$\int_a^b (x - u)[\alpha f_u(t, u, \dot{u}) + \beta(t) g_u(t, u, \dot{u})]\, dt$$
$$= \{(x - u)[\alpha f_{\dot{u}}(t, u, \dot{u}) \beta(t) g_{\dot{u}}(t, u, \dot{u})]\}_{t=a}^{t=b}$$
$$- \int_a^b (D(x - u))[\alpha f_{\dot{u}}(t, u, \dot{u}) + \beta(t) g_{\dot{u}}(t, u, \dot{u})]\, dt.$$

Since each $\alpha_i > 0$, $i \in P$, from the condition $\alpha e = 1$, we have

$$\Sigma_P \alpha_i \int_a^b \{(x - u)[f_u^i(t, u, \dot{u}) + \beta_J(t) g_u(t, u, \dot{u})$$
$$+ [D(x - u)][f_{\dot{u}}^i(t, u, \dot{u}) + \beta(t) g_{\dot{u}}(t, u, \dot{u})]]\, dt\} = 0. \quad (21)$$

Given that $\Sigma_P \alpha_i \rho_i \geq 0$ and $\|x - u\|^2$ is always positive, Eq. (21) $\geq -\Sigma_P \alpha_i \rho_i \|x - u\|^2$.

Again using the nonnegativity of each α_i for $i \in P$, and ρ-pseudoconvexity and the equivalent form of ρ-quasiconvexity in (5a), it follows from the above inequality that

$$\int_a^b \{f^i(t, x, \dot{x}) + \beta(t) g(t, x, \dot{x})\}\, dt \geq \int_a^b \{f^i(t, u, \dot{u}) + \beta(t) g(t, u, \dot{u})\}\, dt$$

for all $i \in P$, and for some $i \in P$,

$$\int_a^b \{f^i(t, x, \dot{x}) + \beta(t) g(t, x, \dot{x})\}\, dt$$
$$> \int_a^b \{f^i(t, u, \dot{u}) + \beta(t) g(t, u, \dot{u})\}\, dt.$$

Now the feasibilities of x for (MP) and $(u, \alpha, \beta(t))$ for (MD) lead us to the desired conclusion that (4) and (5) cannot hold.

Next we state without proof the last weak duality theorem. It will be proved in a similar manner.

THEOREM 6. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and*

(6a) *For each $i \in P$, $\alpha_i > 0$ and $\alpha f(t, ., .) + \beta(t) g(t, ., .)$ is ρ-convex with $\rho \geq 0$, or*

(6b) *$\alpha f(t, ., .) + \beta(t) g(t, ., .)$ is strictly ρ-convex with $\rho \geq 0$. Then* (4) *and* (5) *cannot hold.*

The assumption (6b) that $\alpha f(t, ., .) + \beta(t) g(t, ., .)$ is strictly ρ-convex can be replaced by much weaker conditions. This idea leads us to the following corollary.

COROLLARY 6. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and assume as in Theorem* 6, *except that instead of* (6*b*) *for each $i \in P$, $f^i(t, ., .) + \beta(t) g(t, ., .)$ is ρ_i-convex, and for at least one $k \in P$, $f^k(t, ., .) + \beta(t) g(t, ., .)$ is strictly ρ_k-convex* (*with the corresponding component α_k of α positive*) *with $\Sigma_P \alpha_i \rho_i \geq 0$. Then* (4) *and* (5) *cannot hold.*

We next turn our attention to a discussion of strong duality. The following lemma is for that purpose.

LEMMA 1. *Assume that weak duality* (*any of the Theorems* 1–6 *or any of the Corollaries* 1–6 *holds between* (MP) *and* (MD). *If $(\bar{u}, \bar{\alpha}, \bar{\beta}(t))$ is feasible for* (MD) *with $\int_a^b \bar{\beta}(t) g(\bar{t}, \bar{u}, \dot{\bar{u}})\, dt = 0$ and $\bar{u}$ is feasible for* (MP), *then $\bar{u}$ is efficient for* (MP) *and $(\bar{u}, \bar{\alpha}, \bar{\beta}(t))$ is efficient for* (MD).

Proof. Suppose, on the contrary, that $\bar{u}$ is not efficient for (MP); then there exists a feasible x for (MP) such that

$$\int_a^b f^i(t, x, \dot{x})\, dt \leq \int_a^b f^i(t, \bar{u}, \dot{\bar{u}})\, dt$$

for all $i \in P$, and for some $i \in P$,

$$\int_a^b f^i(t, x, \dot{x})\, dt < \int_a^b f^i(t, \bar{u}, \dot{\bar{u}})\, dt.$$

Since $\int_a^b \bar{\beta}(t) g(t, \bar{u}, \dot{\bar{u}})\, dt = 0$ $(\to \int_a^b \bar{\beta}_J(t) g^J(t, \bar{u}, \dot{\bar{u}})\, dt = 0)$, we can write $f^i(t, \bar{u}, \dot{\bar{u}}) + \bar{\beta}_J(t) g^J(t, \bar{u}, \dot{\bar{u}})$ in place of $f^i(t, \bar{u}, \dot{\bar{u}})$ in the right-side terms of the above two equations. Then, since $(\bar{u}, \bar{\alpha}, \bar{\beta}(t) \in Y$ and $x \in X$, we get a contradiction to the weak duality. Hence $\bar{u}$ is efficient for (MP). In a similar way we can easily show that $(\bar{u}, \bar{\alpha}, \bar{\beta}(t))$ is efficient for (MD).

Now utilizing this lemma in conjunction with the necessary optimality conditions (Proposition 1) of Section 3, we obtain the following strong duality theorem.

THEOREM 7. *Let $\bar{x}$ be an efficient solution for* (MP) *and assume that $\bar{x}$ satisfies the Kuhn–Tucker constraint qualification for* (MP). *Then there exist $\bar{\alpha} \in \mathbb{R}^p$ and a piecewise smooth function $\beta(t)$: $I \to \mathbb{R}^m$ such that $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ is feasible for* (MD), *along with the condition $\int_a^b \bar{\beta}(t) g(t, \bar{x}, \dot{\bar{x}})\, dt = 0$. Furthermore, if any weak duality (any of the theorems* 1–6 *or any of the Corollaries* 1–6) *also holds between* (MP) *and* (MD), *then $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ is efficient for* (MD).

Proof. We have

$$\int_a^b \bar{\beta}(t) g(t, \bar{x}, \dot{\bar{x}})\, dt = 0 \qquad \left(\to \int_a^b \bar{\beta}_J(t) g^J(t, \bar{x}, \dot{\bar{x}})\, dt = 0\right). \quad (22)$$

Since $\bar{x}$ is an efficient solution for (MP) and since every efficient solution for (MP) is also a weak minimum, all of the conditions of Proposition 1 are satisfied, and hence there exist $\bar{\alpha} \in \mathbb{R}^p$ and a piecewise smooth function $\bar{\beta}(t)$: $I \to \mathbb{R}^m \ni$ that

$$\left[\bar{\alpha} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{\beta}(t) g_x(t, \bar{x}, \dot{\bar{x}})\right] = D\left[\bar{\alpha} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{\beta}(t) g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right]$$

$$\int_a^b \bar{\beta}(t) g(t, \bar{x}, \dot{\bar{x}})\, dt = 0$$

$$\bar{\beta}(t) \geq 0, \qquad \alpha e = 1, \qquad \alpha \geq 0.$$

The above three equations show along with (22) that $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ is feasible for (MD). Now since $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ is a feasible solution for (MD), its efficiency follows from Lemma 1.

Since we believe that the previous observations about the case of ρ-convexity (generalized ρ-convexity) are still valid in the case of ρ-invexity (generalized ρ-invexity), this topic could be the object of further investigations.

5. FURTHER EXTENSIONS

In the previous section it has been shown that, by means of generalizations (i.e., ρ-convexity and generalized ρ-convexity), it is possible to achieve the unification of the most well-known duality results (namely Wolfe and Mond–Weir) in the class of variational problems.

In this section our purpose is to show that the requirements of objective and constraint functions (of the primal and dual problems of Section 3), e.g., to be ρ-convex, ρ-pseudoconvex, or ρ-quasiconvex, can be further weakened to be required to be ρ-invex, ρ-pseudoinvex, and ρ-quasiinvex respectively. For this purpose we reconsider here the problems (MP) and (MD) and recall a few definitions and concepts pertaining to certain types of ρ-invex (generalized ρ-invex) functions which are used frequently throughout this section.

If there exist vector functions $\eta(t, x, u)$: $I \times S \times S \to \mathbb{R}$ (with $\eta(t, x, u) = 0$ at $x = u$) and θ: $I \times S \times S \to \mathbb{R}$ and a real number ρ such that the functional $F[x]$ (as in Section 2) satisfies

$$F[x] - F[u] \geq \int_a^b \{\eta(t, x, u) f_u(t, u, \dot{u}) + D\eta(t, x, u) f_{\dot{u}}(t, u, \dot{u})\}\, dt$$
$$+ \rho \|\theta(t, x, u)\|^2,$$

then F is said to be ρ-invex at $u \in S$ with respect to η and θ.

Here $D\eta(t, x, u)$ is the vector whose ith component is $D\eta^i(t, x, u)$. In a similar way the definitions of strict ρ-invexity, ρ-pseudoinvex, strict ρ-pseudoinvex, and ρ-quasiinvex can easily be obtained. From this point on we use the term "generalized ρ-invexity" to indicate ρ-pseudoinvexity, ρ-quasiinvexity, etc. From the above definitions it is clear that every ρ-invex function (generalized ρ-invex) is a ρ-convex function (generalized ρ-convex) with $\eta(t, x, u) = (x + u) = \theta(t, x, u)$.

In parallel with the results presented in Section 4, we can easily establish analogous theorems (two of them without proof are presented next as an example) for ρ-invex (generalized ρ-invex) functions, since only reinterpretation of ρ-convexity is involved.

WEAK DUALITY THEOREM. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$, and $\beta_K(t) g^K(t, ., .)$ is ρ-quasiinvex, and for each $i \in P$, $\alpha_i > 0$ and $f^i(t, ., .) + \beta_j(t) g^J(t, ., .)$ is both γ_i-quasiinvex and γ_i-pseudoinvex with $\Sigma_P\, \alpha_i \gamma_i + \rho \geq 0$. Then* (4) *and* (5) *cannot hold.*

STRONG DUALITY THEOREM. *Let $\bar{x}$ be an efficient solution for* (MP) *and assume that $\bar{x}$ satisfies the Kuhn–Tucker constraint qualification for* (MP). *Then there exist $\overline{\alpha} \in \mathbb{R}^p$ and a piecewise smooth function $\beta(t)$: $I \to \mathbb{R}^m$ such that $(\bar{x}, \overline{\alpha}, \overline{\beta}(t))$ is feasible for* (MD) *along with the condition $\int_a^b \overline{\beta}(t) g(t, \bar{x}, \dot{\bar{x}})\, dt = 0$. Furthermore, if a weak duality also holds between* (MP) *and* (MD) *then $(\bar{x}, \overline{\alpha}, \overline{\beta}(t))$ is efficient for* (MD).

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