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Generalized Convex Duality for Multiobjective Fractional Programs

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Egudo derived some duality theorems for Multi-objective programs using the concept of efficiency coupled with some generalized convexity assumptions on the objective and constraint functions. The main results of the present work can be thought of as extensions of the results of Egudo in the context of multi-objective fractional programs.

1. INTRODUCTION

We consider the following multi-objective fractional progamming problem

(VFP)

$$\text{Minimize } \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_P(x)}{g_P(x)}\right) \quad (1)$$
$$\text{subject to } h(x) \leqslant 0.$$

Corresponding to (VFP) we have a parametric multi-objective programming $(\text{VFP})_v$ of the following type

$(\text{VFP})_v$

$$\text{Minimize } (f_1(x) - u_1 g_1(x), f_2(x) - v_2 g_2(x), ..., f_P(x) - v_P g_P(x)) \quad (2)$$
$$\text{subject to } h(x) \leqslant 0.$$

The dual of $(\text{VFP})_v$ can be given as follows

(VFD)

$$\text{Maximize } (v_1, v_2, ..., v_P)$$
$$\text{subject to } \nabla\{r'(f(u) - v'g(u)) + \mu' h(u)\} = 0 \quad (3)$$

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$$f_i(u) - v_i\, g_i(u) \geqslant 0 \tag{4}$$

$$\mu^T h(u) \geqslant 0 \tag{5}$$

$$\tau_i \geqslant 0, \qquad i = 1, 2, ..., P, \qquad \sum_{i=1}^{P} \tau_i = 1, \qquad v \in R_+^P,\ \mu \in R_+^m. \tag{6}$$

The functions f_i: $R^n \to R$, g_i: $R^n \to R$ and h: $R \to R^m$, for $i = 1, 2, 3, ..., P$ are assumed to be differentiable.

We give some preliminary definitions and results from [3] which are used subsequently in our later results.

DEFINITION 1. A feasible solution is efficient for (VFP) if and only if there is no other feasible solution x for (VFP) such that for some $i \in P = 1, 2, ..., P$

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^0)}{g_i(x^0)} \tag{7}$$

and

$$\frac{f_j(x)}{g_j(x)} \leqslant \frac{f_j(x^0)}{g_j(x^0)}, \qquad \text{for all} \quad j \in P. \tag{8}$$

For maximization problem the signs in the inequalities (7) and (8) can be reversed.

The following lemma connecting (VFP) and $(\text{VFP})_v$ has been proved in [1].

LEMMA 1. *Let x_0 be an efficient solution to* (VFP). *Then there exists* $\bar{v} \in R_+^P$ *such that x_0 is efficient to program* $(\text{VFP})_v$.

Geoffrion [4] characterized proper efficiency of a multi-objective programming by the following lemma. Such characterization is used for strong duality results derived later in the sequel.

LEMMA 2. *If for some fixed $\lambda \in 0 \in R^P$, x^0 solves the single objective program*
(P_λ)

$$\textit{Minimize } \lambda^t f(x)$$

$$\textit{subject to } h(x) \leqslant 0;$$

Then x^0 is properly efficient for (VFP), *where* $g_i(x) = 1$, *for* $i = 1, 2, 3, ..., P$.

A necessary and sufficient condition which characterizes efficient solutions of (VFP) with $g_i(x)=1$, for $i=1,2,3,...,P$ can be given as follows (see [2] or [7]).

LEMMA 3. *A point x^0 is an efficient solution for* (VFP) *if and only if x^0 solves*
$P_k(\varepsilon^0)$

$$\text{Minimize } f_k(x)$$

$$\text{subject to } f_j(x) \leqslant f_j(x^0), \quad \text{for all} \quad j \neq k, h(x) \leqslant 0;$$

for each $k=1,2,...,P$.

Remark. For Theorem 3 in the section on strong duality theorem we shall use Lemma 3 with f_i's replaced by f_i/g_i.

2. DUALITY RESULTS

Some duality theorems of weak and strong type connecting $(\text{VFP})_v$ and (VFD) have been derived next with the beginning result which assumes the function $f_i(x)$, $-g_i(x)$, and $h_j(x)$ to be convex for $1 \leqslant i \leqslant P$ and $1 \leqslant j \leqslant m$.

THEOREM 1. *Assume that all feasible x for* $(\text{VFP})_v$ *and all feasible (u, τ, μ) for* (VFD), *$f_i - v_i g_i$, $i=1,2,...,P$ and h_j, $j=1,2,...,m$ are convex functions. If also either* (a) $\tau_i > 0$ all $i=1,2,...,P$ *or* (b) $\sum_{i=1}^{P} \tau_i(f_i(\cdot) - v_i g_i(\cdot)) + \sum_{j=1}^{m} y_j g_j(\cdot)$ *is strictly convex at u, then the following cannot hold*

$$\frac{f_j(x)}{g_j(x)} \leqslant v_j \qquad \text{for all} \quad j \in P = 1, 2, ..., P \tag{9}$$

and

$$\frac{f_j(x)}{g_i(x)} < v_i \qquad \text{for some} \quad i \in P. \tag{10}$$

Proof. Suppose contrary to the result that (9) and (10) hold. Then since x is feasible for $(\text{VFP})_v$ and $y \geqslant 0$, (9) and (10) imply

$$f_j(x) - v_j g_i(x) \leqslant 0 \qquad \text{for all} \quad j \in P$$

and

$$f_j(x) - v_i g_i(x) < 0 \qquad \text{for some} \quad i \in P,$$

respectively. Now hypothesis (a) and $\sum_{i=1}^{P} \tau_i = 1$ imply

$$\sum_{i=1}^{P} \tau_i(f_i(x) - v_i g_i(x)) < 0. \tag{11}$$

Also from (4), (5), (6), and (11) we have

$$\sum_{i=1}^{P} \tau_i(f_i(x) - v_i g_i(x)) + \sum_{j=1}^{m} y_j h_j(x)$$
$$< \sum_{i=1}^{P} \tau_i(f_i(u) - v_i g_i(u)) + \sum_{j=1}^{m} y_j h_j(u). \tag{12}$$

Since $f_i - u_i g_i$ and h_j are convex and $\tau_i > 0$, $i = 1, 2, ..., P$, $y \geqslant 0$, it now follows from (12) that

$$(x-u)^{\tau}\left(\sum \tau_i\{\nabla f_i(u) - u_i \nabla g_i(u)\} + \nabla y^t h(u)\right) < 0, \tag{13}$$

which contradicts (3).

Case (b) can be proved similarly.

Using ρ-convexity we can obtain a weak duality theorem for a fractional programming problem as follows.

DEFINITION. A function f: $R^n \to R$ is said to be ρ—convex [5, 6] if there exists a real number ρ such that for each x, $u \in R^n$ and $0 \leqslant \lambda \leqslant 1$,

$$f(\lambda x + (1-\lambda)\, u) \leqslant \lambda(x) + (1-\lambda) f(u) - \rho\lambda(1-\lambda)\| x - u\|^2.$$

For a differentiable function f: $R^n \to R$, f is ρ-convex if and only if for all x, $u \in R^n$

$$f(x) - f(u) \geqslant (x-u)^t\, \nabla f(u) + \rho\, \| x - u\|^2.$$

For $\rho > 0$ then f is said to be strongly convex; and if ρ is negative then f is said to be weakly convex.

THEOREM 2. (Weak Duality). *Assume that for all feasible c for* $(\text{VFP})_v$ *and all feasible* (u, τ, y) *for* (VFD), $f_i - v_i g_i$ *are* ρ_i—*convex and* h_j, $j = 1, 2, ..., m$ *are* σ_j-*convex. If also either*

(a) $\tau_i > 0$ *for all* $i = 1, 2, ..., m$ *and* $\sum_{i=1}^{P} \tau_i P_i + \sum_{j=1}^{m} y_j \sigma_j \geqslant 0$ *or*

(b) $\sum_{i=1}^{P} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j > 0$.

Then the following cannot hold

$$\frac{f_j(x)}{g_j(x)} \leqslant v_j \qquad \textit{for all} \quad j \in P = \{1, 2, ..., P\} \tag{14}$$

and

$$\frac{f_1(x)}{g_i(x)} < v_i \qquad \textit{for some} \quad i \in P. \tag{15}$$

Proof. Suppose contrary to the result that (14) and (15) hold. Then since x is feasible for $(\mathrm{VFP})_v$ and $y \geqslant 0$, (14) and (15) imply that

$$f_j(x) - v_j\, g_j(x) \leqslant 0, \qquad \text{for all} \quad j \in P \tag{16}$$

and

$$f_{i_0}(x) - v_i\, g_{i_0}(x) < 0, \qquad \text{for some} \quad i_0 \in P. \tag{17}$$

Now if hypothesis (a) holds, then from $\tau_i > 0$ for all $i \in P$, (16) and (17) we obtain

$$\sum \tau_i(f_i(x) - v_i\, g_i(x)) < 0. \tag{18}$$

From (4), (5), (6), and (18) we obtain

$$\sum_{i=1}^{P} \tau_i(f_i(x) - v_i\, g_i(x)) - \sum_{i=1}^{P} \tau_i(f_i(u) - g_i(u)) + \sum_{j=1}^{m} y_j h_j(x) - \sum_{j=1}^{m} y_j h_j(u) < 0. \tag{19}$$

Now from (19), ρ_i—convexity of $f_i - v_i\, g_i$ and σ_j-convexity of h_j's we obtain

$$(x-u)^t \left(\sum \tau_i \nabla(f_i(u) - u_i\, g_i(u) \right) + \nabla \left(\sum y_j h_j(u) \right) + \left(\sum_{i=1}^{P} \tau_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \right) \| x-u \|^2 < 0.$$

By hypothesis (a) this implies that

$$(x-u)^t \left(\sum_{i=1}^{P} \tau_i \nabla(f_i(u) - v_i\, g_i(u_j) + y^t \nabla h(u) \right) < 0 \tag{20}$$

which contradicts (3).

The proof for the case (b) is achieved in a similar fashion.

COROLLARY 1. *Assume that weak duality (Theorem 1 or 2) holds between* $(\mathrm{VFP})_v$ *and* (VFD). *If* (u^0, τ^0, y^0, v_0) *is feasible for* (VFD) *and* u^0 *is feasible for* $(\mathrm{VFP})_v$ *with* $v^0 = f(u^0)/g(u^0)$, *then* u^0 *is efficient for* $(\mathrm{VFP})_v$ *and* (u^0, τ^0, y^0, v^0) *is efficient for* (VFD).

THEOREM 3 (Strong Duality). *Let* x^0 *be a efficient point for* (VFP) *and assume* x^0 *satisfies a constraint qualification for* $P_k(x^0)$ *(see remark following Lemma 3) for at least one* $k = 1, 2, ..., P$; *then there exists* $\tau^0 \in R^P$, $y^0 \in R^m$, *and* $v_0 \in R^P$ *such that* (x^0, τ^0, y^0, v_0) *is feasible for* (VFD). *If also weakly duality (Theorem 1 or 2) holds between* $(\mathrm{VFP})_v$ *and* (VFD) *then* (x^0, τ^0, y^0, v_0) *is efficient for* (VFD).

The proof of Theorem 3 can be given on similar lines as that of [3, Theorem 3].

3. WEAK AND STRONG DUALITY RELATIONS

In this section we give some weak and strong duality relations between programs $(\mathrm{VFP})_v$ and (VFD) under some pseudoconvexity/strictly pseudoconvexity assumption on the objective functions and quasiconvexity assumptions on the constraint functions.

THEOREM 4 (Weak Duality). *Assume that for all feasible* x *for* $(\mathrm{VFP})_v$ *and all feasible* (u, τ, y) *for* (VFD), $y^t h(\cdot)$ *is quasiconvex at* u. *If also any of the following holds*

(a) $\tau_i > 0$, $\forall i \in P = \{1, 2, ..., P\}$, *and* $f_i - v_i g_i$, $i = 1, 2, ..., P$ *are pseudoconvex at* u;

(b) $\tau_i > 0$ *for all* $i \in P$ *and* $\sum \tau_i(f_i(\cdot) - v_i g_i(\cdot))$ *is pseudoconvex at* u;

(c) $\sum_{i=1}^{P} \tau_i(f_i(\cdot) - g_i(\cdot))$ *is strictly pseudoconvex at* u, *then the following cannot hold*:

$$\frac{f_j(x)}{g_j(x)} \leqslant v_j, \qquad \forall j \in P = \{1, 2, ..., P\} \tag{21}$$

and

$$\frac{f_{i_0}(x)}{g_{i_0}(x)} < v_{i_0}, \qquad \textit{for some} \quad i_0 \in P. \tag{22}$$

Proof. For each feasible x for $(\mathrm{VFP})_v$ and each feasible (u, τ, y) for (VFD) we have

$$y^t h(x) - y^t h(u) \leqslant 0.$$

Since $y'h(\cdot)$ is quasiconvex at u, we have

$$(x-u)' \nabla y'h(u) \leqslant 0. \tag{23}$$

Applying (21) to (3), we obtain

$$(x-u)' \sum_{i=1}^{P} \tau_i \nabla (f_i(u) - u_i\, g_i(u)) \geqslant 0. \tag{24}$$

Now suppose contrary to the result of the theorem that (21) and (22) hold. If $\tau_i > 0$, for all $i \in P = 1, 2, ..., P$, then (21) and (22) imply that

$$\tau_i(f_j(x) - u_j\, g_j(x)) \leqslant 0 \qquad \text{for all} \quad j \in P \tag{25}$$

and

$$\tau_{i_0}(f_{i_0}(x) - v_{i_0}\, g_{i_0}(x) < 0, \qquad \text{for some} \quad i_0 \in P. \tag{26}$$

Equations (25) and (26) also imply that

$$\sum_{i=1}^{P} \tau_i(f_i(x) - v_i\, g_i(x) < 0. \tag{27}$$

By (4),

$$\sum_{i=1}^{P} \tau_i(f_i(x) - u_i\, g_i(x) < \sum_{i=1}^{P} \tau_i(f_i(u) - v_i\, g_i(u)). \tag{28}$$

By hypothesis (a) since $f_i(\cdot) - v_i\, g_i(\cdot)$ are pseudoconvex, (28) it implies than

$$(x-u)' \left(\sum_{i=1}^{P} \tau_i \nabla (f_i(u) - v_i\, g_i(u) - v_i\, g_i(u) \right) < 0, \tag{29}$$

contradicting (24).

By hypothesis (b), i.e., $\sum_{i=1}^{P} \tau_i(f_i(\cdot) - u_i\, g_i(\cdot))$ is pseudoconvex at u, (28) implies (29), again contradicting (24).

From $\tau_i \geqslant 0$, $i = 1, 2, ..., P$, (25) and (26) we obtain

$$\sum_{i=1}^{P} \tau_i(f_i(x) - v_i\, g_i(x)) \leqslant \sum_{i=1}^{P} \tau_i(f_i(u) - v_i\, g_i(u)) \tag{30}$$

and by hypothesis (c), i.e., $\sum_{i=1}^{P} \tau_i(f_i(\cdot) - v_i\, g_i(\cdot))$ is strictly pseudoconvex, (28) which implies (29), again contradicting (24).

COROLLARY 4. *Assume weak duality (Theorem 4) holds between* $(\text{VFP})_v$ *and* (VFD). *If* (u^0, τ^0, y^0, v_0) *is feasible for* (VDP) *such that* u^0 *is feasible for* $(\text{VFP})_v$ *(where* $v_0 = (f(u^0)/g(u^0)/g(u^0)))$ *then* u^0 *is efficient for* $(\text{VFP})_v$ *and* (u^0, τ^0, y^0, v_0) *is efficient for* (VDP).

THEOREM 5 (Strong Duality). *Let* x^0 *be efficient for* (VFP) *and assume that* x^0 *satisfies a constraint qualification* [3] *for* $P_k(\varepsilon^0)$ *for at least one* $k = 1, 2, ..., p$ *(see also remark following Lemma 3). Then there exists* $\tau^0 \in R^P$, $y^0 \in R^m$, *and* $v_0 \in R^P_+$ *such that* (x^0, τ^0, y^0, v_0) *is feasible for* (VDP). *If also weak duality (Theorem 4) holds between* $(\text{VFP})_v$ *and* (VDP) *then* (x^0, τ^0, y^0, v_0) *is efficient for* (VDP).

The proof of Theorem 5 is analogous to Theorem 3.

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