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# ON THE $P$-NORM OF THE TRUNCATED $N$-DIMENSIONAL HILBERT TRANSFORM

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**It is shown that a bounded linear operator $T$ from $L^p(\mathbb{R}^n)$ to itself which commutes both with translations and dilatations is a finite linear combination of Hilbert-type transforms. Using this we show that the $p$-norm of the Hilbert transform is the same as the $p$-norm of its truncation to any Lebesgue measurable subset of $\mathbb{R}^n$ with non-zero measure.**

## 1. Preliminaries

For a function $f(x)$ defined on the real line, the Hilbert transform $(Hf)(x)$ is given by the Cauchy principal value:

$$(1.1) \qquad (Hf)(x) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{f(t)}{t-x}\, dt.$$

One of the fundamental results in the subject is that $(Hf)(x)$ exists for almost every $x$ if $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, and $H\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is both continuous and linear, and

$$(1.2) \qquad \|Hf\|_p \leqslant C_p \|f\|_p \quad \text{for } 1 < p < \infty,$$

where $C_p$ is a constant independent of $f$ [15].

An $n$-dimensional Hilbert transform $(Hf)(x)$ for $f \in L^p(\mathbb{R}^n)$, $p > 1$, may be defined as

$$(1.3) \qquad \begin{aligned} (Hf)(x) &= \frac{1}{\pi^n} P \int_{\mathbb{R}^n} \frac{f(t)}{\prod_{i=1}^n (t_i - x_i)}\, dt \\ &= \lim_{\varepsilon \to 0} \frac{1}{\pi^n} \int_{\substack{|t_i - x_i| > \varepsilon_i > 0 \\ i=1,2,\dots,n}} \frac{f(t)}{\prod_{i=1}^n (t_i - x_i)}\, dt \end{aligned}$$

where $\varepsilon = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2}$, $t = (t_1, t_2, \dots, t_n)$ and $dt = dt_1 dt_2 \cdots dt_n$. The existence of the singular integral in (1.3) and its boundedness property

$$(1.4) \qquad \|Hf\|_p \leqslant C_p^n \|f\|_p,$$

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were proved by Kokilashvili [5]. In 1989 Singh and Pandey [13] extended the $n$-dimensional Hilbert transform to the Schwartz distribution space $D'(\mathrm{R}^n)$ [12] and proved that $H$ is an automorphism on the distribution space $D'_{L^p}(\mathrm{R}^n)$, $p > 1$ [7]. They also obtained the following inversion formula

$$(1.5) \qquad \left(H^2 f\right)(x) = (-1)^n f(x) \text{ almost everywhere}$$

for $f \in L^p(\mathrm{R}^n)$. The inversion formula (1.5) is a generalisation of the corresponding one-dimensional result proved by Riesz; see Titchmarsh [15].

Fefferman showed the iterative nature of the double Hilbert transform [3] in 1972. In 1989 Singh and Pandey [13] proved the iterative nature of the $n$-dimensional Hilbert transform over the spaces $L^p(\mathrm{R}^n)$ and $D'_{L^p}(\mathrm{R}^n)$, $p > 1$. In fact, it was shown that

$$(1.6) \qquad H = \prod_{i=1}^{n} H_i$$

where $(H_i f)(t_1, \ldots, t_{i-1}, x_i, t_{i+1}, \ldots, t_n) = \dfrac{1}{\pi} P \displaystyle\int_{\mathrm{R}} \frac{f(t_1, \ldots, t_i, \ldots, t_n)}{t_i - x_i} dt_i.$

The operators $H_i$ and $H_j$ $i, j = 1, 2, \ldots, n$ commute with each other.

During the 1960's O'Neil and Weiss [8], Gohberg and Krupnik [4] tried to obtain the best possible value $C_p^*\left(= \|H\|_p\right)$ of $C_p$ in (1.2). They gave the following upper and lower bounds for $C_p^*$:

$$\nu(p) \leqslant C_p^* \leqslant \frac{q}{\pi^{3/2}} \Gamma\left(\frac{1}{2p}\right) \Gamma\left(\frac{1}{2q}\right),$$

where
$$\nu(p) = \begin{cases} \tan(\pi/2p), & 1 < p \leqslant 2 \\ \cot(\pi/2p), & 2 \leqslant p < \infty, \end{cases}$$

and $1/p + 1/q = 1$. Later Pichorides [10] proved that $C_p^* = \nu(p)$ for $1 < p < \infty$. Recently McLean and Elliott [6] found the best possible constant $C_{p,E}^*\left(= \|H_E\|_p\right)$, $1 < p < \infty$, for the truncated Hilbert transform $H_E$, defined by

$$(1.7) \qquad (H_E f)(x) = \frac{1}{\pi i} P \int_E \frac{f(t)}{t - x} dt, \quad x \in E$$

where $E$ is a measurable subset of $R$. It is obvious that there exists a constant $C_{P,E} < \infty$ such that

$$\|H_E f\|_p \leqslant C_{p,E} \|f\|_p,$$

for every $f \in L^p(\mathrm{R})$ and moreover the best constant $C_{p,E}^* \leqslant C_p^*$. McLean and Elliott [6] proved that

$$(1.8) \qquad C_{p,E}^* = C_p^* = \nu(p) \text{ for } 1 < p < \infty,$$

provided the Lebesgue measure of $E$ is not zero.

In the present paper we will extend the result (1.8) to $n$ dimensions. More precisely, we show that for the $n$-dimensional Hilbert transform $H$ defined in (1.3),

$$C^{*n}_{p,E} = \|H_E\|_p = \|H\|_p = C^{*n}_p = [\nu(p)]^n, \tag{1.9}$$

for every measurable subset $E$ of $\mathrm{R}^n$ with non-zero Lebesgue measure. The $n$-dimensional truncated Hilbert transform $H_E$ is defined by

$$(H_E f)(x) = \frac{1}{\pi^n} P \int_E \frac{f(t)}{\prod_{i=1}^n (t_i - x_i)} dt, \quad x \in E. \tag{1.10}$$

In view of (1.6) and the fact that

$$\|H_i\|_p = C^*_p = \nu(p), \quad 1 \leqslant i \leqslant n,$$

it is easy to see that

$$\|H\|_p = C^{*n}_p = [\nu(p)]^n,$$

thus proving the latter half of (1.9).

## 2. The main results

Let $a = (a_1, a_2, \dots, a_n)$ and $m = (m_1, m_2, \dots, m_n) \in \mathrm{R}^n$ with $m_i > 0$ for each $i$. We define the translation operator

$$\tau_a : L^p(\mathrm{R}^n) \to L^p(\mathrm{R}^n)$$

and the dilatation operators

$$D_m, D_{m^*} : L^p(\mathrm{R}^n) \to L^p(\mathrm{R}^n)$$

by

$$\tau_a f(x) = f(x - a),$$

$$D_m f(x) = \left(\prod_{i=1}^n m_i\right)^{-1/p} f\left(\frac{x_1}{m_1}, \frac{x_2}{m_2}, \dots, \frac{x_n}{m_n}\right),$$

$$D_{m^*} f(x) = \left(\prod_{i=1}^n m_i\right)^{1/p} f(m_1 x_1, m_2 x_2, \dots, m_n x_n). \quad [14, \text{p.50}].$$

Then both $\tau_a$ and $D_m$ are isometric isomorphisms since

$$(\tau_a)^{-1} = \tau_{-a}, \quad (D_m)^{-1} = D_{m^*},$$

and

$$\|\tau_a f\|_p = \|f\|_p, \quad \|D_m f\|_p = \|f\|_p, \quad \text{for every } f \in L^p(\mathrm{R}^n).$$

Let $\mathcal{B}(L^p(\mathbb{R}^n))$ denote the space of all bounded linear operators from $L^p(\mathbb{R}^n)$ into itself. Then $T \in \mathcal{B}(L^p(\mathbb{R}^n))$ is said to commute with translations if $\tau_a T = T\tau_a$ for all $a \in \mathbb{R}$ and similarly it commutes with dilatations if $D_m T = T D_m$ for all $m \in \mathbb{R}^n$ with $m_i > 0$ for $1 \leqslant i \leqslant n$. The following lemma, the proof of which is trivial, characterises an integral operator commuting with translations or dilatations.

**Lemma 2.1.** *Let $K$ in $\mathcal{B}(L^p(\mathbb{R}^n))$ be an integral operator given by*

$$K f(x) = P \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad x \in \mathbb{R}^n.$$

*Then*

(i) *$K$ commutes with translations if and only if $K$ is a difference kernel, that is,*

$$K(x, y) = K(x - y, 0) = K(0, y - x),$$

*and*

(ii) *$K$ commutes with dilatations if and only if $K$ is a Hardy kernel, that is,*

$$K(mx, my) = \left(\prod_{i=1}^{n} m_i\right)^{-1} K(x, y),$$

*where by $mx$ and $my$ we mean $(m_1x_1, m_2x_2, \ldots, m_nx_n)$ and $(m_1y_1, m_2y_2, \ldots, m_ny_n)$ respectively.*

Note that the $n$-dimensional Hilbert transform $H$ commutes with both translations and dilatations, since

$$H = -H_1 H_2 \ldots H_n$$

and each $H_i$ commutes both with translations and dilatations. Actually $H$ is essentially the only integral operator having this property. To prove this we need the following two lemmas.

**Lemma 2.2.** *Let $T \in \mathcal{B}(L^p(\mathbb{R}^n))$, $p > 1$ commute with translations. Then there exists a unique bounded complex-valued Borel measurable function $\sigma(\xi)$ satisfying*

$$\left(\widehat{T\phi}\right)(\xi) = \widehat{\phi}(\xi)\sigma(\xi)$$

*where $\sigma(\xi) \in L_\infty(R^n)$.*

Proof: If $T \in B(L^p(\mathbb{R}^n))$, then $\tau_a T \ (= T\tau_a) \in \mathcal{B}(L^p(\mathbb{R}^n))$ for each $a \in \mathbb{R}^n$. The Schwartz testing functions space $D(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Let $\varphi \in D(\mathbb{R}^n)$ and $g_m$ be a sequence of $C^\infty$ functions with bounded supports such that $\|g_m\|_p = 1$ and

$g_m * \varphi \to \varphi$ as $m \to \infty$, in sup norm as well as in $L^p(\mathbb{R}^n)$ norm [7, pp.6–8]. Since $\varphi$ and $g_m$ are of compact supports, $g_m * \varphi$ are also $C^\infty$ functions with compact supports for all $m$. Therefore in view of the Riesz representation theorem [11, p.131], there exists a bounded complex regular Borel measure $\mu$ on $\mathbb{R}^n$ such that

$$\begin{aligned}[T((g_m * \varphi)(y))](0) &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} g_m(x)\varphi(y-x)dx \right) d\mu(y) \\ &= \int_{\mathbb{R}^n} dx\, g_m(x) \int_{\mathbb{R}^n} d\mu(y)\varphi(y-x) \quad \text{(by Fubini's Theorem)} \\ &= \int_{\mathbb{R}^n} g_m(-x)(T\varphi)(x)dx.\end{aligned}$$

Hence
$$(g_m * T(\cdot))(0) \colon D(\mathbb{R}^n) \to \mathbb{C}$$

is a bounded linear functional. The Riesz representation theorem asserts the existence of a regular Borel measure $\mu_m$ (depending on $g_m$) bounded on $\mathbb{R}^n$ such that

$$(g_m * T\varphi)(0) = \int_{\mathbb{R}^n} \varphi(-x)d\mu_m(x), \; \varphi \in D(\mathbb{R}^n), \quad [11, \text{p.131}].$$

Hence

$$(g_m * T\varphi)(y) = \int_{\mathbb{R}^n} \varphi(y-x)d\mu_m \tag{2.2}$$

for

$$\begin{aligned}\tau_{-y}T(g_m * \varphi)(0) &= (g_m * \tau_{-y}T\varphi)(0) \\ &= (g_m * T\tau_{-y}\varphi)(0).\end{aligned}$$

Since $|\mu_m|(\mathbb{R}^n) \leqslant \|T\|$, we can select a sequence $g_m$ in such a way that

$$\lim_{m\to\infty} (g_m * T\varphi)(y) = (T\varphi)(y) \tag{2.3}$$

in $L^p(\mathbb{R}^n)$ norm as well as in sup norm. Hence from (2.2), and by selecting an appropriate subsequence $\{m_j\}$ of $\{m\}$ and letting $m_j \to \infty$, we have $\lim_{m_j\to\infty} \widehat{\mu}_{m_j} = \sigma(\xi)$, a bounded complex-valued measurable function

$$\left(\widehat{T\varphi}\right)(\xi) = \widehat{\phi}(\xi)\sigma(\xi)\phi \in D(R^{-n}) \tag{2.4}$$

[1, pp.132, 133]. This completes the proof of the lemma. $\square$

**Corollary 2.1.** *For $T \in \mathcal{B}(L^p(\mathbb{R}^n))$ commuting with translations, there exists $\sigma \in L^\infty(\mathbb{R}^n)$ such that*

$$\widehat{Tf}(\xi) = \sigma(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, \quad f \in L^p(\mathbb{R}^n), \tag{2.5}$$

*where* $\widehat{\ }$ *denotes the operator of Fourier transform.*

PROOF: Using the definition of the Fourier transform of $f$ in $L^p(\mathbb{R}^n)$, where $f$ is treated as a regular tempered distribution in $S'(\mathbb{R}^n)$, [1, pp.131–132; 7], it follows that

$$\widehat{f}(\xi) = \lim_{\substack{\min N_j \to \infty \\ 1 \leqslant j \leqslant n}} \int_{|x_j| < N_i} f(x) e^{-ix\cdot\xi} dx,$$

where the above limit is interpreted in the sense of $S'(\mathbb{R}^n)$ and $x \cdot \xi$ is the inner product of $x$ and $\xi$ in $\mathbb{R}^n$. Since $D(R^n)$ is dense in $L^p(R^n)$ the result (2.5) follows from Lemma 2.2, Bergh and Löfström [1, pp.132–133] and Stein [14, p.28]. $\square$

**THEOREM 2.1.** *Let* $1 < p < \infty$ *and* $T \in \mathcal{B}(L^p(\mathbb{R}^n))$. *Suppose* $T$ *commutes both with translations and with dilatations. Then there exist constants* $a$, $a_i$, $a_{i,j}$, $\dots$, $b$ *such that*

$$T = aI + \sum_{i=1}^{n} a_i H_i + \sum_{\substack{i,j=1 \\ i<j}}^{n} a_{ij} H_i H_j + \dots + bH, \tag{2.6}$$

*where* $I$ *is the identity operator on* $L^p(\mathbb{R}^n)$.

PROOF: Let $T \in \mathcal{B}(L^p(\mathbb{R}^n))$, $1 < p < \infty$, commuting both with translations and dilatations. Then from (2.5), we have

$$\widehat{Tf}(\xi) = \sigma(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, \quad f \in L^p(\mathbb{R}^n)$$

for some $\sigma \in L^\infty(\mathbb{R}^n)$. Since

$$\widehat{D_m f}(\xi) = \left(\prod_{i=1}^{n} m_i\right)^{1-\frac{1}{p}} \widehat{f}(m_1\xi_1, \dots, m_n\xi_n)$$

and $T$ commutes with dilatations, we have $\sigma(\xi) = \sigma(m_1\xi_1, m_2\xi_2, \dots, m_n\xi_n)$, for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $m_1, \dots, m_n > 0$.

Hence

$$\sigma(\xi) = \sigma(\operatorname{sgn}\xi_1, \dots, \operatorname{sgn}\xi_n),$$

where

$$\operatorname{sgn}\xi_j = \begin{cases} +1, & \text{if } \xi_j > 0, \\ -1, & \text{if } \xi_j < 0. \end{cases}$$

When $n = 2$, it is easy to see that

$$\begin{aligned}\sigma(\xi_1, \xi_2) = \frac{1}{2^2}\Big[&[\sigma(1,1) + \sigma(1,-1) + \sigma(-1,1) + \sigma(-1,-1)] \\ &+ [\sigma(1,1) + \sigma(1,-1) - \sigma(-1,1) - \sigma(-1,-1)]\operatorname{sgn}\xi_1 \\ &+ [\sigma(1,1) - \sigma(1,-1) + \sigma(-1,1) - \sigma(-1,-1)]\operatorname{sgn}\xi_2 \\ &+ [\sigma(1,1) - \sigma(1,-1) - \sigma(-1,1) + \sigma(-1,-1)]\operatorname{sgn}\xi_1 \operatorname{sgn}\xi_2\Big].\end{aligned}$$

Generalising this we obtain the following in the $n$-dimensional case

$$\begin{aligned}\sigma(\xi) = \frac{1}{2^n}\Bigg[&\sum_{i=1}^{2^n}\sigma(i_1, i_2, \dots, i_n) + \sum_{j=1}^{n}\left(\sum_{i=1}^{2^n} i_j\sigma(i_1, \dots, i_n)\right)\operatorname{sgn}\xi_j \\ &+ \sum_{\substack{j,k=1\\ j<k}}^{n}\left(\sum_{i=1}^{2^n} i_j i_k \sigma(i_1, \dots, i_n)\right)\operatorname{sgn}\xi_j \cdot \operatorname{sgn}\xi_k + \dots \\ &+ \left(\sum_{i=1}^{2^n}\left(\prod_{j=1}^{n} i_j\right)\sigma(i_1, \dots, i_n)\right)\prod_{j=1}^{n}\operatorname{sgn}\xi_j\Bigg] \\ = a + &\sum_{j=1}^{n} a_j \operatorname{sgn}\xi_j + \sum_{\substack{j,k=1\\ j<k}}^{n} a_{jk}\operatorname{sgn}\xi_j \operatorname{sgn}\xi_k + \dots + b\prod_{j=1}^{n}\operatorname{sgn}\xi_j,\end{aligned}$$

where $i_j = +1$ or $-1$ for $j = 1, 2, \dots, n$. Since $\widehat{Hf}(\xi) = \prod_{j=1}^{n}\operatorname{sgn}\xi_j \hat{f}(\xi)$ and $\widehat{H_j f}(\xi) = \operatorname{sgn}\xi_j \hat{f}(\xi)$, we have the desired result (2.6) see [9]. □

REMARK. The $n$ Riesz transforms $R_1, R_2, \dots, R_n$ are defined as

$$(R_j f)(x) = \lim_{\varepsilon\to 0} c_n \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\,dy, \quad j = 1, \dots, n$$

with $$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}, \quad \text{for } f \in L^p(\mathbb{R}^n), \quad 1 \leqslant p < \infty, \quad [14, \text{p.57}].$$

It is easy to see that in general they do not commute with dilatations $D_m$ for $m = (m_1, \dots, m_n) \in \mathbb{R}^n$, $m_1, \dots, m_n > 0$. Hence none of the $R_j$'s can be written in the form (2.6), despite the fact that in the particular case when $m = (m_1, m_1, \dots, m_1)$ with $m_1 > 0$, the $n$-Riesz transforms commute with dilatations. But only when $n = 1$ does the Riesz transform $R$ commute both with translations and with dilatations, so that it can be written in the form (2.6).

For a measurable set $E \subset \mathbb{R}^n$, define

$$\chi_E \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

by $$\chi_E f(x) = \begin{cases} f(x), & \text{if } x \in E, \\ 0, & \text{otherwise.}\end{cases}$$

Since any $f \in L^p(\mathbb{R}^n)$ can be written as

$$f = \chi_E f + (1 - \chi_E) f,$$

the space $L^p(\mathbb{R}^n)$ is the direct sum

$$L^p(\mathbb{R}^n) = L^p(E) \oplus L^p(\mathbb{R}^n - E).$$

Thus the space $L^p(E)$ can be treated as a closed subspace of $L^p(\mathbb{R}^n)$ and for any bounded linear operator $T$ on $L^p(\mathbb{R}^n)$, we define the truncated operator

$$T_E = \chi_E T \chi_E.$$

For $E \subset \mathbb{R}^n$ and $m, a \in \mathbb{R}^n$,

$$\begin{aligned} a + E &= \{a + x : x \in E\}, \\ mE &= \{(m_1 x_1, \ldots, m_n x_n) : x \in E\} \end{aligned}$$

and $$mE = \{(m x_1, \ldots, m x_n) : x \in E\} \text{ whenever } m \in \mathbb{R}.$$

Then we have the following theorem.

**Theorem 2.2.** *Let $E$ be any measurable subset of $\mathbb{R}^n$.*

(i) *If $T$ commutes with translations, then*

$$\|T_{a+E}\|_p = \|T_E\|_p, \quad \text{for all } a \in \mathbb{R}^n.$$

(ii) *If $T$ commutes with dilatations, then*

$$\|T_{mE}\|_p = \|T_E\|_p, \quad \text{for all } m \in \mathbb{R}^n \text{ with } m_1, \ldots, m_n > 0.$$

The proof of the above theorem is similar to the one given by McLean and Elliott [**6**, Theorem 2.2] for the one-dimensional case.

Let $\mu$ be the Lebesgue measure on $\mathbb{R}^n$. Denote by $J_\delta(x)$ the open box centred at $x$, that is,

$$J_\delta(x) = \prod_{i=1}^{n} (x_i - \delta_i, x_i + \delta_i), \; x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$
$$\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n \text{ with each } \delta_i > 0.$$

The density of $E$ at $x$ is defined by

$$d_E(x) = \lim_{\delta \to 0+} \frac{\mu(E \cap J_\delta(x))}{\mu(J_\delta(x))} \tag{2.7}$$

provided the limit exists. Clearly $0 \leqslant d_E(x) \leqslant 1$. When $x \notin \overline{E}$ (the closure of $E$), then $d_E(x) = 0$ whereas if $x \in E^0$ (the interior of $E$) then $d_E(x) = 1$. The Lebesgue Density Theorem [**2**, p.184] asserts that

$$d_E(x) = 1 \text{ for almost every } x \in E. \tag{2.8}$$

LEMMA 2.2. *If $J$ is a bounded box centred at 0 and $m > 0$, then*

$$\lim_{m \to \infty} \mu(J \cap mE) = d_E(0)\mu(J).$$

PROOF: Let $E$ be a measurable subset of $\mathrm{R}^n$. Then for $m > 0$, we have

$$\mu(mE) = \mu\{(mx_1, \ldots, mx_n) : x = (x_1, \ldots, x_n) \in E\} = m\mu(E),$$

and $m(E_1 \cap E_2) = (mE_1) \cap (mE_2)$, for $E_1$, $E_2$ measurable subsets of $\mathrm{R}^n$. Suppose $J = (-M, M) \times \cdots \times (-M, M)$, ($n$ factors) and let $m = M/\delta$, $\delta > 0$; then $mJ_\delta(0) = J$ and hence

$$d_E(0) = \lim_{\delta \to 0+} \frac{\mu(E \cap J_\delta(0))}{\mu(J_\delta(0))} = \lim_{m \to \infty} \frac{\mu(mE \cap J)}{\mu(J)},$$

proving the lemma. $\square$

The following Lemma 2.3 and Theorem 2.3 have proofs similar to that of Lemma 3.2 and Theorem 3.3 of McLean and Elliott [**6**], so we state them without proof.

LEMMA 2.3. *For $1 \leqslant p < \infty$, the following are equivalent:*

(i) $d_E(0) = 1$,

(ii) $\lim_{m \to \infty} \|\chi_{mE} f\|_p = \|f\|_p$ *for all* $f \in L^p(\mathrm{R}^n)$; $m > 0$,

(iii) $\lim_{m \to \infty} \|(1 - \chi_{mE}) f\|_p = 0$ *for all* $f \in L^p(\mathrm{R}^n)$, $m > 0$.

THEOREM 2.3. *Suppose $d_E(0) = 1$. If $T \in \mathcal{B}(L^p(\mathrm{R}^n))$ commutes with dilatations, then*

$$\|T_E\|_p = \|T\|_p.$$

Since the $n$-dimensional Hilbert transform $H$ commutes both with translations and with dilatations, Theorems 2.1, 2.2 and 2.3 are true for $H$.

So, let $E$ be a subset of $\mathrm{R}^n$ such that $\mu(E) \neq 0$. Then there exists an $x \in E$ such that $d_E(x) = 1$, by (2.8). Hence $d_{-x+E}(0) = 1$. Therefore,

$$\|H_E\|_p = \|H_{-x+E}\|_p = \|H\|_p.$$

Thus we have proved the following theorem.

THEOREM 2.4. *If $\mu(E) \neq 0$, then $\|H_E\|_p = \|H\|_p$.*

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