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THE FOURIER-BESSEL SERIES REPRESENTATION OF THE PSEUDO-DIFFERENTIAL OPERATOR $(-x^{-1}D)^\nu$

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ABSTRACT. For a certain Fréchet space F consisting of complex-valued C^∞ functions defined on $I = (0, \infty)$ and characterized by their asymptotic behaviour near the boundaries, we show that:

(I) The pseudo-differential operator $(-x^{-1}D)^\nu$, $\nu \in \mathbb{R}$, $D = d/dx$, is an automorphism (in the topological sense) on F;

(II) $(-x^{-1}D)^\nu$ is almost an inverse of the Hankel transform h_ν in the sense that

$$h_\nu \circ (x^{-1}D)^\nu(\varphi) = h_0(\varphi), \quad \forall \varphi \in F, \ \forall \nu \in \mathbb{R};$$

(III) $(-x^{-1}D)^\nu$ has a Fourier-Bessel series representation on a subspace $F_b \subset F$ and also on its dual F_b'.

1. INTRODUCTION

Let F be the space of all C^∞ complex-valued function $\varphi(x)$ defined on $I = (0, \infty)$ such that

$$\varphi(x) = \sum_{i=0}^{k} a_i x^{2i} + o(x^{2k}) \tag{1.1}$$

near the origin and is *rapidly decreasing as* $\mathbf{x} \to \infty$.

For $\nu > -\frac{1}{2}$, we define a νth order Hankel transform h_ν on F by

$$\Phi(y) = [h_\nu \varphi(x)](y) = \int_0^\infty \varphi(x) \mathcal{J}_\nu(xy)\, dm(x), \tag{1.2}$$

where

$$dm(x) = m'(x)\, dx = [2^\nu \Gamma(\nu+1)]^{-1} x^{2\nu+1}\, dx,$$
$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} J_\nu(x),$$

and $J_\nu(x)$ is the Bessel function of order ν. The inversion formula for (1.2) is given by [1, 3, 4],

$$\varphi(x) = \int_0^\infty \Phi(y) \mathcal{J}_\nu(xy)\, dm(y). \tag{1.3}$$

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In this paper we will show that for every real ν:

(I) The pseudodifferential operator $(-x^{-1}D)^{\nu}$ is a topological automorphism on F.

(II) The Hankel transform h_{ν} is also an automorphism on F.

(III) On F, $(-x^{-1}D)^{\nu}$ is almost an inverse of h_{ν} in the sense that

$$[h_{\nu} \circ (-x^{-1}D)^{\nu}](\varphi) = h_0(\varphi), \qquad \varphi \in F.$$

(IV) On a certain subspace $F_b \subset F$ and on its dual F_b', $(-x^{-1}D)^{\nu}$ has Fourier-Bessel series representations.

In the sequel all automorphisms are topological automorphisms.

2. Preliminaries

For any real $\nu \neq -\frac{1}{2}$, F_{ν} is the space of all C^{∞} complex-valued function $\varphi(x)$ defined on I such that

$$\gamma^{\nu}_{m,k}(\varphi) = \sup_{x\in I} |x^m \Delta^k_{\nu,x}\varphi(x)| < \infty, \tag{2.1}$$

for each $m, k = 0, 1, 2\dots$, where

$$\Delta_{\nu,x} = D^2 + x^{-1}(2\nu+1)D.$$

F_{ν} is a Fréchet space. Its topology is generated by the countable family of separating seminorms $\{\gamma^{\nu}_{m,k}\}_{m,k=0,1,2,\dots,}$ [5; 7, p. 8].

Theorem 2.1(i) of Lee [3, p. 429] shows that $F_{\nu} = F_{\mu} = F$ (as a set) for each ν, $\mu(\neq -\frac{1}{2}) \in \mathbb{R}$. Hence for each $\nu \neq -\frac{1}{2}$, we have a topology T_{ν} on F generated by the countable family of seminorms $\gamma^{\nu}_{m,k}$. Hence (F, T_{ν}) is a Fréchet space. When $\nu = -\frac{1}{2}$, $F_{-1/2} \neq F$, since the factor $x^{-1}(2\nu+1)D$ in $\Delta_{\nu,x}$, responsible for the even nature of $\varphi(x) \in F_{\nu}(x)$ near the origin, vanishes. For example $e^{-x} \in F_{-1/2}$ but $e^{-x} \notin F_{\nu}$; $\nu \neq -\frac{1}{2}$.

Definition. Zemanian [7, 8] defined a Hankel transform $\bar{h}_{\nu}$ $(\nu \geq -\frac{1}{2})$ by

$$\Psi(y) = [\bar{h}_{\nu}\psi(x)](y) = \int_0^{\infty} \psi(x)\sqrt{xy}\,J_{\nu}(xy)\,dx. \tag{2.2}$$

He proved that $\bar{h}_{\nu}$ is an automorphism on the space H_{ν} that consists of complex-valued C^{∞} functions defined on I and satisfies the relation

$$\bar{\gamma}^{\nu}_{m,k}(\psi) = \sup_{x\in I} |x^m(x^{-1}D)^k[x^{-\nu-1/2}\psi(x)]| < \infty, \tag{2.3}$$

for each $m, k = 0, 1, 2, \dots$, where $D = d/dx$.

The following theorem is a key result for the latter development of our theory.

Theorem 2.1. *Let ν, μ be real number $\neq -\frac{1}{2}$. Then*

(I) *The operation $\varphi \to x^{\nu+1/2}\varphi$ is an homeomorphism from F onto H_{ν}.*

(II) *$(x^{-1}D)^n \colon F \to F$ is an automorphism on F.*

(III) *(F, T_{ν}) and (F, T_{μ}) are equivalent topological spaces.*

(IV) *$h_{\nu}(\varphi) = (-1)^n[h_{\nu+n}(x^{-1}D)^n]\varphi$, for $\varphi \in F$, $\nu \geq -\frac{1}{2}$, and $n = 0, 1, 2, \dots$.*

Notation. In view of the Theorem 2.1(I, III), we will always write $x^{\nu+1/2}\varphi(x) = \bar{\varphi}(x) \in H_\nu$ for $\varphi \in F$ and drop the suffix ν from T_ν. So henceforth the topological linear space (F, T) will be denoted by F.

Proof. (I) By induction on n and noting that

$$\Delta_{\nu,x} = x^2(x^{-1}D)^2 + 2(\nu+1)(x^{-1}D), \tag{2.4}$$

it can be proved that

$$\Delta^n_{\nu,x} = x^{2n}(x^{-1}D)^{2n} + a_1 x^{2(n-1)}(x^{-1}D)^{2n-1} + \cdots + a_n(x^{-1}D)^n, \tag{2.5}$$

where a_i's are the constants depending on ν. Now $\varphi \in F$ iff $\bar{\varphi} \in H_\nu$ (follows from Remark II of Lee [3]) and taking $a_0 = 1$, it follows from (2.5) that

$$\gamma^\nu_{m,k}(\varphi) \le \sum_{i=k}^{2k} a_{2k-i}\bar{\gamma}^\nu_{m+2(i-k),i}(\bar{\varphi}), \tag{2.6}$$

proving the continuity of the inverse operation $\bar{\varphi} \to x^{-\nu-1/2}\varphi$. Invoking the Open Mapping Theorem [6, p. 172], F being Fréchet space, we complete the proof.

(II) Let φ_j be a sequence tending to zero in F. Then $\bar{\varphi}_j \to 0$ in H_ν for arbitrary $\nu \ne -\frac{1}{2}$. Hence

$$\begin{aligned} \gamma^\nu_{m,k}[(x^{-1}D)^n\varphi_j(x)] &\le \sum_{i=k}^{2k} a_{2k-i}\bar{\gamma}^\nu_{m+2(i-k),i+n}(\bar{\varphi}_j) \quad \text{(from (2.6))} \\ &\to 0 \qquad \text{as } j \to \infty. \end{aligned}$$

It remains to be shown that $(x^{-1}D)^n$ is bijective. It is enough to prove this for $n = 1$. So, let $x^{-1}D\varphi_1(x) = x^{-1}D\varphi_2(x)$ for $\varphi_1, \varphi_2 \in F$. Hence $\varphi_1(x) - \varphi_2(x) =$ constant. But $\varphi_1(x)$ and $\varphi_2(x)$ are of rapid descent as $x \to \infty \Rightarrow \varphi_1(x) = \varphi_2(x)$. Now let $\psi(x) \in F$. Then $\varphi(x) = -\int_x^\infty t\psi(t)\,dt$, defined uniquely (since ψ is of rapid descent as $x \to \infty$) in F, is such that $x^{-1}D\varphi(x) = \psi(x)$. So we see that $(x^{-1}D)^n$ is a continuous bijection on F. The space F being a Fréchet space, the Open Mapping Theorem shows that $(x^{-1}D)^n$ is a bicontinuous bijection on (F, T_ν) for each $\nu \in \mathbb{R} - \{\frac{1}{2}\}$.

(III) Let $\nu = \mu + a$, $a \in \mathbb{R}$, and φ_n be a sequence tending to zero in (F, T_μ). Then

$$\begin{aligned} \gamma^\nu_{m,k}(\varphi_n) &= \sup_{x\in I} |x^m[\Delta_{\mu,x} + 2a(x^{-1}D)]^k\varphi_n(x)| \\ &\le \sup_{x\in I} x^m \Bigg[\left| \sum_{i=0}^k \Delta^{k-i}_{\mu,x}(2ax^{-1}D)^i\varphi_n(x) \right| + \sum_{i=0}^k |(2ax^{-1}D)^{k-i}\Delta^i_{\mu,x}\varphi_n(x)| \\ &\qquad + \text{terms of the type } |\Delta^{i_1}_{\mu,x}(2ax^{-1}D)^{i_2}\Delta^{i_3}_{\mu,x}\cdots\varphi_n(x)| \\ &\qquad \text{and } |(2ax^{-1}D)^{j_1}\Delta^{j_2}_{\mu,x}(2ax^{-1}D)^{j_3}\cdots\varphi_n(x)| \Bigg] \\ &\qquad (\text{where } i_1 + i_2 + i_3 + \cdots = j_1 + j_2 + j_3 \cdots = k) \\ &\to 0 \text{ as } n \to \infty, \text{ for each } m, k = 0, 1, 2 \ldots, \end{aligned}$$

since $\Delta^i_{\mu,x}$ and $(x^{-1}D)^i$ are continuous on (F, T_μ).

(IV) follows from integration by parts and induction on n.

Remark 1. It can be shown that on F

$$\Delta^k_{\nu,x} \circ (x^{-1}D)^n = (x^{-1}D)^n \circ \Delta^k_{\nu-n,x}.$$

The proof follows by induction on k.

Definition. In view of Theorem 2.1(IV), we define the Hankel transform h_ν formally for any $\nu \in \mathbb{R}$, as

$$h_\nu(\varphi) = h_{\nu+n} \circ (-x^{-1}D)^n \varphi, \qquad \varphi \in F, \tag{2.7}$$

where n is so chosen that $\nu + n > -\frac{1}{2}$.

This is a well-defined definition as $(x^{-1}D)^n$ is an automorphism.

Definition. Let F' be the dual space of F. Then for $f \in F'$, define the generalized Hankel transform $h_\nu f (= \hat{f})$ of f by

$$\langle h_\nu f, h_\nu \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in F, \ \nu \in \mathbb{R}.$$

Theorem 2.2. *For $\nu \in \mathbb{R}$, h_ν is an automorphism on F and hence on F'.*

Proof. Let $\varphi(x) \in F$. Then

$$\begin{aligned} h_\nu(\varphi) = \Phi(y) &= \int_0^\infty (x^{-1}D)^{2n} \varphi(x) \mathscr{J}_{\nu+2n}(xy)\, dm(x) \\ &= y^{-\mu-1/2} \bar{h}_\mu(\bar{\psi}(x))(y), \quad \text{where } \mu = \nu + 2n > -\tfrac{1}{2}, \end{aligned} \tag{2.8}$$

where

$$\bar{\psi}(x) = x^{\mu+1/2} \psi(x) = x^{\mu+1/2} (x^{-1}D)^{2n} \varphi(x).$$

Let

$$\begin{aligned} \varphi_m(x) \to 0 \quad \text{in } F \quad &\Rightarrow \quad \bar{\psi}_m(x) \to 0 \quad \text{in } H_\mu \quad \text{(Theorem 2.1(I))}, \\ &\Rightarrow \quad \bar{h}_\mu(\bar{\psi}_m) \to 0 \quad \text{in } H_\mu, \\ &\Rightarrow \quad h_\nu(\varphi_m) \to 0 \quad \text{in } F. \end{aligned}$$

Now $\bar{h}_\mu$, the Zemanian Hankel transform, being bijective, (2.8) shows that h_ν is a bijection. Hence use of the Open Mapping Theorem completes the proof.

Writing $\nu = 0$ in (2.7) we get

$$h_0(\varphi) = h_n \circ (-x^{-1}D)^n(\varphi), \qquad \varphi \in F.$$

The above equation motivates us to propose the following

Definition. For $\nu \in \mathbb{R}$, define $(-x^{-1}D)^\nu$ by

$$(-x^{-1}D)^\nu(\varphi) = h_\nu^{-1} \circ h_0(\varphi), \qquad \varphi \in F. \tag{2.9}$$

Then $(-x^{-1}D)^\nu$ is clearly an automorphism on F for each real ν. From equation (2.9) we get

$$(-x^{-1}D)^\nu \varphi(x) = \int_0^\infty dm(y) \mathscr{J}_\nu(xy) \int_0^\infty dm(x) \varphi(x) \mathscr{J}_0(xy). \tag{2.10}$$

For distributions $f \in F'$, define $(-x^{-1}D)^\nu$ by

$$\langle (-x^{-1}D)^\nu f, \varphi\rangle = \langle f, (-x^{-1}D)^\nu \varphi\rangle, \qquad \varphi \in F. \tag{2.11}$$

So we modify Theorem 2.1(II) to give our main result.

Theorem 2.3. *The pseudodifferential operator $(-x^{-1}D)^\nu$ is an automorphism on F and hence on F' for each $\nu \in \mathbb{R}$.*

3. The Fourier-Bessel series expansion of $(-x^{-1}D)^\nu$

Equation (2.10) gives the integral representation of the operator $(-x^{-1}D)^\nu$. To get the Fourier-Bessel series expansion, we modify our leading function space F suitably as follows (similar to the ones as in Zemanian [7, 9]).

For $b > 0$, define

$$F_b = \{\varphi \in F | \varphi \equiv 0 \text{ for } x > b\}. \tag{3.1}$$

The topology of F_b is generated by a countable family of seminorms

$$\gamma_k^\nu(\varphi) = \sup_{0<x<b} |\Delta_{\nu,x}^k \varphi(x)| < \infty, \qquad k = 0, 1, 2, \dots. \tag{3.2}$$

Clearly all the topologies obtained by choosing different ν's are equivalent.

Remark 2. Without loss of generality, we may take $\nu > -\frac{1}{2}$.

Definition. We define finite Hankel transform h_ν by

$$\Phi(z) = [h_\nu\varphi](z) = \int_0^b \varphi(x)\mathscr{J}_\nu(xz)\,dm(x). \tag{3.3}$$

Then $\Phi(z)$ is an even entire function by Griffith's Theorem [2, 9]. Let $z = y + iw$ and $G_b = \{\Phi(z)|\Phi(z)$ is an even entire function satisfying (3.4)$\}$.

$$\alpha_b^k(\Phi) = \sup_{z\in\mathbb{C}} |e^{-b|w|}z^{2k}\Phi(z)| < \infty, \tag{3.4}$$

for $k = 0, 1, 2, \dots$. Then G_b is a linear topological space with α_b^k as seminorms.

Both the spaces F_b and G_b are Hausdorff, locally convex topological linear spaces satisfying the axiom of first countability. They are sequentially complete spaces.

Theorem 3.1. *h_ν is an homeomorphism from F_b onto G_b.*

Proof. Let $\varphi \in F_b$. Then

$$\Phi(z) = h_{\nu+2m}[(x^{-1}D)^{2m}\varphi(x)], \quad \text{for } m \in \mathbb{N}.$$

Hence

$$z^{2m}\Phi(z) = \int_0^b x^{2\nu+2m+1}[(x^{-1}D)^{2m}\varphi(x)](xz)^{-\nu}J_{\nu+2m}(xz)\,dz.$$

From the asymptotic formula

$$J_\nu(z) \sim \sqrt{2/\pi z}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \qquad |z| \to \infty,\ |\arg z| < \pi,$$

and from the fact that $z^{-\nu}J_{\nu+m}(z)$ is an entire function, it follows that for all x and z,

$$|e^{-b|w|}(xz)^{-\nu}J_{\nu+2m}(xz)| < C_{m\nu} \quad \text{(a constant)}.$$

Hence

$$\alpha_b^m(\Phi) \le C_{m\nu} b^{2(m+\nu+1)} \gamma_0^\nu[(x^{-1}D)^{2m}\varphi(x)] < \infty. \tag{3.5}$$

$(x^{-1}D)^{2m}$ being an automorphism (also on F_b), (3.5) implies the continuity of h_ν. h_ν is clearly injective. For any $\Phi(z) \in G_b$, take

$$\varphi(x) = \int_0^\infty \Phi(y)\mathscr{J}_\nu(xy)\,dm(y).$$

Then it follows from Griffith's Theorem [2] that φ is zero almost everywhere for $x > b$. Also,

$$\begin{aligned}\gamma_k^\nu(\varphi) &= \sup_{0<x<b}\left|\Delta_{\nu,x}^k \int_0^\infty \Phi(y)\mathscr{J}_\nu(xy)\,dm(y)\right| \\ &= \sup_{0<x<b}\left|\int_0^\infty \Phi(y)(-1)^k y^{2\nu+2k+1}(xy)^{-\nu}J_\nu(xy)\,dy\right| \\ &< \infty, \quad \text{for each } k = 0, 1, 2, \dots,\end{aligned}$$

since $\Delta_{\nu,x}^k[(xy)^{-\nu}J_\nu(xy)] = (-1)^k y^{2k}(xy)^{-\nu}J_\nu(xy)$, $\Phi(y)$ is of rapid descent as $y \to \infty$, and $(xy)^{-\nu}J_\nu(xy)$ is bounded for $0 < y < \infty$. Therefore, $\varphi \in F_b$. Hence h_ν is surjective. Now the Open Mapping Theorem completes the proof.

Theorem 3.2. *Let $\varphi \in F_b$. Then*

$$\varphi(x) = \lim_{\varepsilon\to 0^+} \frac{2}{b^2}\sum_{n=1}^\infty \lambda_\varepsilon(x)\left(\frac{\lambda_n}{x}\right)^\nu \frac{J_\nu(x\lambda_n)}{J_{\nu+1}^2(b\lambda_n)}\Phi(\lambda_n), \tag{3.6}$$

where the λ_n's are the positive roots of $J_\nu(bz) = 0$ arranged in the ascending order and for $0 < \varepsilon < b/4$,

$$\lambda_\varepsilon(x) = \begin{cases} E(x/2\varepsilon), & 0 < x < 2\varepsilon, \\ 1, & 2\varepsilon \le x \le b - 2\varepsilon, \\ 1 - E\left(\dfrac{x - b + 2\varepsilon}{2\varepsilon}\right), & b - 2\varepsilon < x < b, \\ 0, & x \ge b, \end{cases}$$

and $E(u) = \int_0^u \exp[1/x(x-1)]\,dx / \int_0^1 \exp[1/x(x-1)]\,dx$.

Proof. Trivial. See also [5].

Theorem 3.2 gives the required Fourier-Bessel Series expansion for the pseudo-differential operator $(-x^{-1}D)^\nu$, which we obtain in the following

Theorem 3.3 (The Fourier-Bessel Series). *For $\varphi \in F_b$, we have*

$$[(-x^{-1}D)^\nu]\varphi(x) = \lim_{\varepsilon\to 0^+} \frac{2}{b^2}\sum_{n=1}^\infty \lambda_\varepsilon(x)\left(\frac{\lambda_n}{x}\right)^\nu \frac{J_\nu(x\lambda_n)}{J_{\nu+1}^2(b\lambda_n)}\Phi_0(\lambda_n), \tag{3.7}$$

where $\Phi_0(y) = h_0[\varphi(x)](y)$.

Proof. Equation (2.9) along with Theorem 3.2 gives the required proof.

Note that

$$|\lambda_n^{\nu+1/2}\Phi_0(\lambda_n)| \le A_{k\nu}\lambda_n^{(\nu+1/2)-2k},$$

$A_{k\nu}$ constants and $[J_\nu(x\lambda_n)/x^\nu \lambda_n^{1/2} J_{\nu+1}^2(b\lambda_n)]$ is smooth and bounded on $0 < x < b$, $0 < \lambda_n < \infty$.

Hence the truncation error

$$E_N = \lim_{\varepsilon\to 0^+} \frac{2}{b^2} \sum_{n=N+1}^{\infty} \lambda_\varepsilon(x) \left(\frac{\lambda_n}{x}\right)^\nu \frac{J_\nu(x\lambda_n)}{J_{\nu+1}^2(b\lambda_n)} \Phi_0(\lambda_n)$$

has exponential decay for large N.

The Theorem 3.3 gives the Fourier-Bessel series representation of the operator $(-x^{-1}D)^\nu$ on the testing function space F_b. We wish to investigate the nature of the Fourier-Bessel series for the pseudodifferential operator $(-x^{-1}D)^\nu$ on the distribution space F_b'.

The spaces F_b' and G_b' are dual spaces of F_b and G_b, respectively. They are assigned the weak topologies generated by the seminorms

$$P_\varphi(f) = |\langle f, \varphi\rangle|, \qquad \varphi \in F_b,\ f \in F_b',$$

and

$$P_\Phi(h_\nu f) = |\langle h_\nu f, h_\nu \varphi\rangle|, \qquad h_\nu\varphi \in G_b,\ h_\nu f \in G_b',$$

respectively.

Both the spaces are sequentially complete.

Definition. For $f \in F_b'$, $\varphi \in F_b$, we define the generalized finite Hankel transform $h_\nu f$ by

$$\langle h_\nu f, h_\nu\varphi\rangle = \langle f, \varphi\rangle. \tag{3.8}$$

Theorem 3.4. *For $\nu \in \mathbb{R}$, h_ν is an homeomorphism from F_b' onto G_b'.*

Theorem 3.5. *For every $\varepsilon \in (0, b/4)$ and each $f \in F_b'$, the function*

$$\hat{f}_\varepsilon(y) = \langle f(x), y^{-\nu-1/2}\lambda_\varepsilon(x) m'(y)\mathscr{J}_\nu(xy)\rangle, \tag{3.9}$$

where $\lambda_\varepsilon(x)$ is defined as in Theorem 3.2, is a smooth function of slow growth, and defines a regular generalized function in G_b'.

Proof. Note that $(x^{-1}D)^k\lambda_\varepsilon(x)$ is bounded on $0 < x < b$ for each k. Using (2.6), it is easy to see that $y^{-\nu-1/2}\lambda_\varepsilon(x)m'(y)\mathscr{J}_\nu(xy) \in F_b$. Hence (3.9) is well defined. The rest of the proof is similar to that of Zemanian [8, Lemma 12].

Theorem 3.6. *The finite Hankel transform $h_\nu f$ of a generalized function f in F_b' is the distributional limit, as $\varepsilon \to 0^+$, of the family $\hat{f}_\varepsilon(z)$ defined by* (3.9).

Proof. Trivial.

Theorem 3.7. *Let $f \in F_b'$ and $\hat{f} = h_\nu f$. Then in the sense of convergence in F_b', we have*

$$f(x) = \lim_{N\to\infty} \frac{2}{b^2} \sum_{n=1}^{N} \frac{x^{\nu+1}}{\sqrt{\lambda_n}} [J_\nu(x\lambda_n)/J_{\nu+1}^2(b\lambda_n)] \cdot \hat{f}(\lambda_n). \tag{3.10}$$

Proof. The proof follows easily from Theorems 3.2 and 3.6.

Remark 3. For $f \in F_b'$, such that either f is regular or $\operatorname{supp} f \subset (0, b]$, the limit of $\hat{f}_\varepsilon(z)$ as $\varepsilon \to 0^+$ exists as an ordinary function and is equivalent to the finite Hankel transform of f [5].

A consequence of the above theorem is the following

Theorem 3.8. *Let f, $g \in F_b'$. If $(h_\nu f)(\lambda_n) = (h_\nu g)(\lambda_n)$, for $n = 1, 2, 3, \ldots$, then $f = g$ and $h_\nu f = h_\nu g$.*

Definition. For $f \in F_b'$, define $(-x^{-1}D)^\nu f$ by

(3.11) $$\langle (-x^{-1}D)^\nu f, \varphi \rangle = \langle f, (-x^{-1}D)^\nu \varphi \rangle, \qquad \varphi \in F_b, \ \nu \in \mathbb{R}.$$

From equations (2.9), (3.8), and (3.11), it follows that

$$\begin{aligned} \langle (-x^{-1}D)^\nu f, \varphi \rangle &= \langle f, (-x^{-1}D)^\nu \varphi \rangle \\ &= \langle h_0^{-1} h_\nu f, \varphi \rangle, \qquad f \in F_b', \ \varphi \in F_b. \end{aligned}$$

Hence

(3.12) $$(-x^{-1}D)^\nu f = h_0^{-1} h_\nu f \quad \text{on } F_b'.$$

Applying Theorem 3.7 to equation (3.12) we get

Theorem 3.10 (The Fourier-Bessel Series). *Let $f \in F_b'$ and $\hat{f} = h_\nu f$. Then in the sense of convergence in F_b', we have*

(3.13) $$(-x^{-1}D)^\nu f(x) = \lim_{N \to \infty} \frac{2}{b^2} \sum_{n=1}^{N} \frac{x}{\sqrt{\lambda_n}} [J_0(x\lambda_n)/J_1^2(b\lambda_n)] \hat{f}(\lambda_n).$$

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