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# Duality for Multiobjective Fractional Variational Problems

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A class of multiobjective fractional variational problems is considered and duals are formulated. Under concavity assumptions on the functions involved, duality theorems are proved through a parametric approach to relate efficient solutions of the primal and dual problems. We generalize those results for control problems also. © 1994 Academic Press, Inc.

## 1. INTRODUCTION

Duality for multiobjective variational problems has been of much interest in recent years, and contributions have been made to its development (Bector *et al.* [2]). Using parametric equivalence, Bector and Husain [1] formulated a dual program for a multiobjective fractional program having continuously differentiable convex functions.

The purpose of the present paper is to consider the duality of multiobjective fractional variational problems by relating the primal problem to a parametric multiobjective variational problem.

## 2. NOTATIONS AND PRELIMINARIES

Let $I = [a, b]$ be a real interval and $f: I \times R^n \times R^n \to R^p$, $g: I \times R^n \times R^n \to R^p$, and $h: I \times R^n \times R^n \to R^m$ be continuously differentiable functions. For $x, y \in \mathbb{R}^n$ by $x \leq y$ we mean $x_i \leq y_i$, $\forall\, i$. All vectors will be taken as column vectors. The symbol $(\ )^T$ will stand for the transpose. $x : I \to R^n$.



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Let $C(I, \mathbb{R}^n)$ denote the space of piecewise smooth functions $x$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator $D$ is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\, ds,$$

where $\alpha$ is a given boundary value. Therefore $D = d/dt$ except at discontinuities. Let $S \subseteq \mathbb{R}^n$ be open.

We now consider the multiobjective fractional variational primal problem as

$$\text{(P)}\quad \text{Minimize } \frac{\int_a^b f(t, x(t), \dot{x}(t))\, dt}{\int_a^b g(t, x(t), \dot{x}(t))\, dt}$$

$$= \left( \frac{\int_a^b f^1(t, x(t), \dot{x}(t))\, dt}{\int_a^b g^1(t, x(t), \dot{x}(t))\, dt}, \ldots, \frac{\int_a^b f^p(t, x(t), \dot{x}(t))\, dt}{\int_a^b g^p(t, x(t), \dot{x}(t))\, dt} \right),$$

subject to

$$x(a) = \alpha \qquad x(b) = \beta \tag{1}$$

$$h_j(t, x(t), \dot{x}(t)) \leqslant 0, \qquad \forall t \in I, \qquad j = 1, \ldots, m,$$

$$x \in S, \qquad \dot{x} \in C(I, \mathbb{R}^n). \tag{2}$$

We assume that $g^i(t, x(t), \dot{x}(t)) > 0$ and $f^i(t, x(t), \dot{x}(t)) \geqslant 0$ whenever $g^i(x)$ is not linear for all $i = 1, \ldots, p$.

Let $X$ denote the set of all feasible solutions of (P) and

$$\phi(t, x(t), \dot{x}(t)) = (\phi^1(t, x(t), \dot{x}(t), \ldots, \phi^p(t, x(t), \dot{x}(t)),$$

where

$$\phi^i(t, x(t), \dot{x}(t)) = \frac{\int_a^b f^i(t, x(t), \dot{x}(t)\, dt}{\int_a^b g^i(t, x(t), \dot{x}(t))\, dt}, \qquad i = 1, \ldots, p.$$

DEFINITION 1. A point $x^* \in X$ is said to be an efficient solution of (P) if for all $x \in X$

$$\frac{\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\,dt}{\int_a^b g^i(t, x^*(t), \dot{x}^*(t))\,dt} \geqslant \frac{\int_a^b f^i(t, x(t), \dot{x}(t))dt}{\int_a^b g^i(t, x(t), \dot{x}(t))\,dt}, \qquad i = 1, \ldots, p.$$

$$\Rightarrow \frac{\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\,dt}{\int_a^b g^i(t, x^*(t), \dot{x}^*(t)\,dt} = \frac{\int_a^b f^i(t, x(t), \dot{x}(t))\,dt}{\int_a^b g^i(t, x(t), \dot{x}(t))\,dt}, \qquad i = 1, \ldots, p.$$

Let $D = \{d \in \mathbb{R}^n;\, h_x^j\,(t, x^*(t), \dot{x}(t))d \leqslant 0\}$, where $J$ is the set of all active constraints at $x^*$, i.e.,

$$J = \{j : h^j(t, x^*(t), \dot{x}^*(t)) = 0, j = 1, 2, \ldots, n\}.$$

For details, see Bector *et al.* [1].

## 3. DUAL PROBLEM: FORMULATION AND MOTIVATION

We consider the following parametric vector variational problem $(\mathrm{P_V})$ for each $v \in \mathbb{R}^p_+$, where $\mathbb{R}^p_+$ denotes the non-negative orthant of $\mathbb{R}^p$.

$(\mathrm{P_V})$ Minimize

$$\left(\int_a^b \left\{f^1(t, x(t), \dot{x}(t)) - v_1\, g^1(t, x(t), \dot{x}(t))\right\} dt, \ldots,\right.$$

$$\left.\int_a^b \left\{f^p(t, x(t), \dot{x}(t)) - v_p\, g^p(t, x(t), \dot{x}(t))\,dt\right\} dt\right),$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta,$$

$$h^j(t, x(t), \dot{x}(t)) \leqslant 0, \qquad j = 1, \ldots, m, x \in S, \dot{x} \in C.$$

We now prove the following.

LEMMA 1. *Let $x^*$ be efficient for* (P). *Then there exists $v^* \in \mathbb{R}^p_+$ such that $x^*$ is also efficient for* $(\mathrm{P_V})$.

*Proof.* For $i = 1, \ldots, p$ let

$$v_i^* = \frac{\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt}{\int_a^b g^i(t, x^*(t), \dot{x}^*(t)\, dt}$$

and

$$v^* = (v_1^*, \ldots, v_p^*).$$

Let $x^*$ be inefficient for $(\mathrm{P}_{\mathrm{V}^*})$. This implies that there is an $x$ feasible for $(\mathrm{P}_{\mathrm{V}^*})$ such that

$$\int_a^b \{f^i(t, x(t), \dot{x}(t)) - v_i^* g^i(t, x(t), \dot{x}(t))\}\, dt$$
$$\leq \int_a^b \{f^i(t, x^*(t), \dot{x}^*(t)) - v_i^* g^i(t, x^*(t), \dot{x}^*(t))\}\, dt, \qquad i = 1, \ldots, p,$$

and

$$\int_a^b \{f^{i_0}(t, x(t), \dot{x}(t)) - v_i^* g^{i_0}(t, x(t), \dot{x}(t))\}\, dt$$
$$< \int_a^b \{f^{i_0}(t, x^*(t), \dot{x}^*(t)) - v_{i_0}^* g^{i_0}(t, x^*(t), \dot{x}^*(t))\}\, dt$$

for at least one $i$.

$$v_i^* = \frac{\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt}{\int_a^b g^i(t, x^*(t), \dot{x}^*(t))\, dt};$$

therefore,

$$\int_a^b \{f^i(t, x^*(t), \dot{x}^*(t)) - v_i^* g^i(t, x^*(t), \dot{x}^*(t))\}\, dt = 0$$

for $i = 1, \ldots, p$.

Hence from the above we have

$$\frac{\int_a^b f^i(t, x(t), \dot{x}(t))\, dt}{\int_a^b g^i(t, x(t), \dot{x}(t))\, dt} \leqslant v_i^* = \frac{\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt}{\int_a^b g^i(t, x^*(t), \dot{x}^*(t))\, dt}, \qquad \forall i,$$

and

$$\frac{\int_a^b f^{i_0}(t, x(t), \dot{x}(t))\, dt}{\int_a^b g^{i_0}(t, x(t), \dot{x}(t))\, dt} < v_{i_0}^* = \frac{\int_a^b f^{i_0}(t, x^*(t), \dot{x}^*(t))\, dt}{\int_a^b g^{i_0}(t, x^*(t), \dot{x}^*(t))\, dt}$$

for at least one $i$(say $i = i_0$),
This contradicts that $x^*$ is efficient for (P). Hence we are done.

*Remark* 1. The converse of Lemma 1 also holds provided we assume

$$v_i^* = \frac{\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt}{\int_a^b g^i(t, x^*(t), \dot{x}^*(t))\, dt},$$

e.i.,

$$\int_a^b \{f^i(t, x^*(t), \dot{x}^*(t)) - v_i^* g^i(t, x^*(t), \dot{x}^*(t))\}\, dt = 0$$

for $i = 1, 2, ..., p$.

Now, in view of Lemma 1 above and in analogy with the traditions in fractional duality, we introduce the following problem (D) as the dual of (P).

(D) Maximize

$$\int_a^b \{(f(t, x(t), \dot{x}(t)) - vg(t, x(t), \dot{x}(t)) + \mu^{\mathrm{T}} h(t, x(t), \dot{x}(t))\}\, dt,$$

i.e.,

$$\left(\int_a^b \{f^1(t,x(t),\dot{x}(t)) - v_i g^1(t,x(t),\dot{x}(t)) + \mu^{\mathrm{T}} h(t,x(t),\dot{x}(t))\}\,dt, \ldots,\right.$$

$$\left.\int_a^b \{f^p(t,x(t),\dot{x}(t)) - v_p g^p(t,x(t),\dot{x}(t)) + \mu^{\mathrm{T}} h(t,x(t),\dot{x}(t))\}\,dt\right).$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{3}$$

$$\begin{aligned} &f_x(t,x(t),\dot{x}(t)) - v\, g_x(t,x(t),\dot{x}(t)) + \mu^{\mathrm{T}} h_x(t,x(t),\dot{x}(t)) \\ &\quad = D[f_{\dot{x}}(t,x(t),\dot{x}(t)) - v g_{\dot{x}}(t,x(t),\dot{x}(t)) + \mu^{\mathrm{T}} h_{\dot{x}}(t,x(t),\dot{x}(t))] \end{aligned} \tag{4}$$

$$\mu(t) \geqslant 0, \tag{5}$$

where

$$f(t,x(t),\dot{x}(t)) = (f^1(t,x(t),\dot{x}(t)), \ldots, f^p(t,x(t),\dot{x}(t)))$$

and

$$h(t,x(t),\dot{x}(t)) = (h^1(t,x(t),\dot{x})(t), \ldots, h^p(t,x(t),\dot{x}(t))).$$

Let $Z$ be the set of feasible solutions of (D).

## 4. Duality Theorems

We shall now prove that problems (P) and (D) are a dual pair subject to concavity conditions on the objective and constraint functions.

THEOREM 1 (Weak Duality). *If*

$$\int_a^b f(t,x(t),\dot{x}(t))\,dt, \qquad \int_a^b -vg(t,x(t),\dot{x}(t))\,dt$$

*and*

$$\int_a^b \mu^{\mathrm{T}} h(t,x(t),\dot{x}(t))\,dt, \qquad \textit{for any } \mu(t) \in \mathbb{R}, \mu \geqslant 0,$$

*are all convex with respect to the same function* $\eta$, *then* $\inf(\mathrm{P}) \geqslant \sup(\mathrm{D})$.

*Proof.* Let $(x^*, \dot{x}^*)$ satisfy (1) and (2) and let $(x, \dot{x}, \mu, v)$ satisfy (3)–(5). Then

$$\begin{aligned}
&\int_a^b \{f(t, x(t), \dot{x}(t)) - vg(t, x(t), \dot{x}(t)\} \, dt \\
&\qquad - \int_a^b \{f(t, x^*(t), \dot{x}^*(t) - vg(t, x^*(t), \dot{x}^*(t) \\
&\qquad + \mu \, h(t, x^*(t), x^*(t))\} \, dt \\
&\quad \geqslant \int_a^b \Big\{ \eta(t, x(t), \dot{x}^*(t)) \, (f_x(t, x(t), \dot{x}(t)) - vg_x(t, x(t), \dot{x}(t)) \\
&\qquad + \frac{d\eta}{dt}(t, x(t), \dot{x}^*(t))(f_x(t, x(t), \dot{x}(t)) - vg_x(t, x(t), \dot{x}(t))) \Big\} \, dt \\
&\qquad - \int_a^b \mu \, h(t, x^*(t), \dot{x}^*(t)) \, dt
\end{aligned}$$

by concavity of

$$\int_a^b f(t, x(t), \dot{x}(t)) \, dt \qquad \text{and} \qquad \int_a^b \{-vg(t, x(t), \dot{x}(t))\} \, dt.$$

Using integration by parts, concavity of $\int \mu^{\mathrm{T}} h$, and boundary conditions (3), we obtain

$$\begin{aligned}
&\int_a^b \{f(t, x(t), \dot{x}(t) - vg(t, x(t), \dot{x}(t)) + \mu^{\mathrm{T}} h(t, x(t), \dot{x}(t))\} \, dt \\
&- \int_a^b \{f(t, x^*(t), \dot{x}^*(t)) - vg(t, x^*(t), \dot{x}^*(t)) + \mu^{\mathrm{T}} h(t, x^*(t), \dot{x}^*(t))\} \, dt \geqslant 0,
\end{aligned}$$

which in view of (2) and (5) yields

$$\begin{aligned}
&\int_a^b \{f(t, x(t), \dot{x}(t)) - vg(t, x(t), \dot{x}(t))\} \, dt \\
&\qquad \geqslant \int_a^b \{f(t, x^*(t), \dot{x}^*(t)) - vg(t, x^*(t), \dot{x}^*(t)) \\
&\qquad + \mu^T h(t, x^*(t), \dot{x}^*(t)\} \, dt,
\end{aligned}$$

i.e., inf (P) $\geqslant$ sup (D).

Assuming the constraint conditions for the existence of multiplier $\mu(t)$ at extrema of (P) hold, the necessary conditions for $(x^*)$ to be optimal for (P) are:

There exists a piecewise smooth

$\mu_0 : I \to R^m, \qquad \mu(t)$ s.t.

$$F = \mu_0(f(t, x(t), \dot{x}(t)) - vg(t, x(t), \dot{x}(t)) - \mu^T(t)h(t, x(t), \dot{x}(t))$$

satisfies

$$f_x(t, x(t), \dot{x}(t)) - vg_x(t, x(t), \dot{x}(t) - \mu^T h_x(t, x(t), \dot{x}(t))$$
$$= \frac{d}{dt}[f_{\dot{x}}(t, x(t), \dot{x}(t)) - vg_x(t, x(t), \dot{x}(t)) - \mu^T h_{\dot{x}}(t, x(t), \dot{x}(t))]$$

$$\mu_j h_j = 0, \qquad j = 1, \ldots, m \tag{7}$$

$$\mu \geqslant 0. \tag{8}$$

It is assumed from now on that the minimizing solution $x^*$ of (P) is normal, that is, $\mu_0$ is non-zero, so that without loss of generality, we can take $\mu_0 = 1$.

THEOREM 2 (Strong Duality). *Under the concavity conditions of Theorem* 1, *if the function $x^*(t)$ is an efficient solution for* (P), *then there exists a piecewise smooth $\mu(t): I \to \mathbb{R}^m$ such that $(x^*(t), \mu(t), v)$ is an efficient solution of* (D) *and the extreme values of* (P) *and* (D) *are equal.*

*Proof.* Since $x^*$ is an efficient solution of (P) and concavity conditions of Theorem 1 are satisfied, $\exists a, \mu: I \to \mathbb{R}^m$ such that for $t \in I$,

$$f_x(t, x^*(t), \dot{x}^*(t)) - vg_x(t, x^*(t), \dot{x}^*(t)) + \mu(t)^T h_x(t, x^*(t), \dot{x}^*(t))$$
$$= D[f_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) - vg_x(t, x^*(t), \dot{x}^*(t)) + \mu(t)^T h_x(t, x^*(t), \dot{x}^*(t))] \tag{9}$$

$$\mu(t)^T h(t, x^*(t), \dot{x}^*(t)) = 0 \tag{10}$$

$$\mu(t) \geqslant 0. \tag{11}$$

From (9) and (11) it follows that $(x^*, v, \mu) \in Z$; (10) and Lemma 1 imply that $(x^*, v, \mu)$ is efficient for (D).

For the converse duality theorem (Theorem 3), we make the assumption that $X_2$ denotes the space of the piecewise differentiable function $x: I \to \mathbb{R}^n$ for which $x(a) = 0 = x(b)$ equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$.

Problem (D) may be rewritten in the form
Minimize $-\phi(x, v, \mu) = (-\phi^1(x, \nu, \mu), \ldots, -\phi^p(x, \nu, \mu))$, subject to

$$x(a) = \alpha, \qquad x(b) = \beta$$

$$\theta(t, x(t), \dot{x}(t), \ddot{x}(t), \nu, \mu(t), \dot{\mu}(t)) = 0, \qquad t \in I$$

$$\mu(t) \geq 0, \qquad t \in I,$$

where

$$\phi^i(x, \nu, \theta) = \int_a^b [(f^i(t, x(t), \dot{x}(t)) - v_i g^i(t, x(t), \dot{x}(t))) + \mu_j^{\mathrm{T}} h_j(t, x(t), \dot{x}(t))]\, dt, \qquad i = 1, \ldots, p, j = l, \ldots, m$$

and

$$\begin{aligned}\theta &= \theta(t, x(t), \dot{x}(t), \ddot{x}(t), v, \mu(t), \dot{\mu}(t)) \\ &= f_x(t, x(t), \dot{x}(t)) + v g_x(t, x(t), \dot{x}(t)) + \mu^{\mathrm{T}} h_x(t, x(t), \dot{x}(t)) \\ &\quad - D[f_{\dot{x}}(t, x(t), \dot{x}(t)) - \nu g_{\dot{x}}(t, x(t), \dot{x}(t)) \\ &\quad + \mu^{\mathrm{T}} h_{\dot{x}}(t, x(t), \dot{x}(t))], \qquad t \in I,\end{aligned}$$

with

$$\ddot{x}(t) = D^2 x(t).$$

Following Bector *et al.* [1], we are in position to state the converse duality theorem as follows:

THEOREM 3 (Converse Duality). *If* $(x^*, \dot{\mu}, v)$ *is an efficient solution of* (D) *and if*

(I) $\psi'$ *have a (weak*) closed range,*

(II) $f$, $g$, *and* $h$ *are twice continuously differentiable,*

(III) $\int\{f_x^i - v_i g_x^i) - D(f_{\dot{x}}^i - v_i g_{\dot{x}}^i)\}\, dt$, $i = 1, \ldots, p$ *is linearly independent, and*

(IV) $(\beta(t)^{\mathrm{T}} \theta_x - D\beta(t)^{\mathrm{T}} \theta_{\dot{x}} + D^2 \beta(t)^{\mathrm{T}} \theta_{\ddot{x}})\beta(t) = 0$

$\Rightarrow \beta(t) = 0, \qquad t \in I,$

*then* $x^*$ *is an efficient solution of* (P) *and the corresponding objective values are equal.*

*Proof.* See Bector and Husain [2]. ∎

## 5. Formulation of Control Problem

Now we are in position to propose duality for multiobjective fractional variational control problems. Following Mishra and Mukherjee [3], we give the multiobjective fractional control problem as

P (Primal) Minimize

$$\frac{\int_a^b f(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\, dt}{\int_a^b g(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\, dt}$$

$$= \left( \frac{\int_a^b f^1(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\, dt}{\int_a^b g^1(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\, dt}, \ldots, \right.$$

$$\left. \frac{\int_a^b f^p(t, x(t), \dot{x}(t) u(t), \dot{u}(t))\, dt}{\int_a^b g^p(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\, dt} \right),$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{12}$$

$$h_j(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = \dot{x}, \qquad j = 1, \ldots, m \tag{13}$$

$$k_l(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \geq 0, \qquad l = 1, \ldots, m. \tag{14}$$

The equivalent parametric form of the problem is

($P_V$) Minimize

$$\int_a^b \{f(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - vg(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\}\, dt$$

$$= \left( \int_a^b \{f^1(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - v_1 g^1(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\}\, dt, \ldots, \right.$$

$$\left. \int_a^b \{f^p(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - v_p g^p(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\}\, dt \right),$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{15}$$

$$h_j(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = \dot{x}, \qquad j = 1, \ldots, m \tag{16}$$

$$k_l(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \geqslant 0, \qquad l = 1, \ldots, m. \tag{17}$$

D (Dual) Maximize

$$\int_a^b \{(f(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - vg(t, x(t), \dot{x}(t), u(t), \dot{u}(t))) - \lambda(t)[h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - \dot{x}] - \mu(t)k(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\}\,dt,$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta, \tag{18}$$

$$\begin{aligned} &\Big[f_x(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - vg_x(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &\qquad - \lambda(t)h_x(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &\qquad - \mu(t)k_x(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\Big] \\ &= \frac{d}{dt}\Big[\{f_{\dot{x}}(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - vg_{\dot{x}}(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\} \\ &\qquad - \lambda(t)h_{\dot{x}}(t, x(t), \dot{x}(t)u(t), \dot{u}(t)) \\ &\qquad - \mu(t)k_{\dot{x}}(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} &f_u(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - vg_u(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &\qquad - \lambda(t)h_u(t, x(t), \dot{x}(t), u(t), \dot{u}(t) \\ &\qquad - \mu(t)k_u(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = 0 \end{aligned} \tag{20}$$

$$\mu(t) \geqslant 0. \tag{21}$$

## 6. Duality Theorems (Control)

Now we are ready to state analogous results of [4] for the control case.

THEOREM 4 (Weak Duality). *If*

$$\int_a^b f(t,\cdot,\cdot,\cdot,\cdot)\,dt, \qquad \int_a^b -vg(t,\cdot,\cdot,\cdot,\cdot)\,dt,$$

$$\int_a^b -\lambda^{\mathrm{T}} h(t,\cdot,\cdot,\cdot,\cdot)\,dt, \qquad \text{and} \int_a^b -\mu^{\mathrm{T}} k(t,\cdot,\cdot,\cdot,\cdot)\,dt,$$

*for any* $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$, $\mu \geqslant 0$, *are all convex with respect to the same function* $\eta$ *and* $\mathcal{G}$, *then*

$$\frac{\int_a^b f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt}{\int_a^b g^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt}$$

$$\leqslant \frac{\int_a^b \{f^i(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) - \lambda^{\mathrm{T}} h(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t) - \mu^{\mathrm{T}} k(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\}\,dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\,dt}$$

$\forall i \in \{1,\ldots,p\}$

*and*

$$\frac{\int_a^b f^j(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt}{\int_a^b g^j(t,x(t),\dot{x}(t),u(t),\dot{u}(t))\,dt}$$

$$< \frac{\int_a^b \{f^j(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) - \lambda^{\mathrm{T}} h(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t)) - \mu^{\mathrm{T}} k(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\}\,dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t),u^*(t),\dot{u}^*(t))\,dt}$$

*cannot hold for at least one* $j$.

*Proof.* See Bector and Husain [2] and Mishra and Mukherjee [3]. ∎

Once weak duality has been established, strong and converse duality follow as in [3].

For completeness, we restate the results for strong and converse duality. We assume that the necessary constraints for the existence of multipliers at an extremal of (P) are satisfied. Thus for every efficient $(x^*, u^*)$ to (P) there exists $\lambda_0$,

$$F \equiv \lambda_0(f(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - vg(t, x(t), \dot{x}(t), u(t), \dot{u}(t))) \\ - \lambda(t)^{\mathrm{T}}[h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - \dot{x}] \\ - \mu(t)^{\mathrm{T}} k(t, x(t), \dot{x}(t), u(t), \dot{u}(t))$$

satisfying

$$\begin{aligned} f_x(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) &- vg_x(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &- \lambda^{\mathrm{T}} h_x(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &- \mu^{\mathrm{T}} k_x(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &= \frac{d}{dt}[\{f_{\dot{x}}(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &- vg_{\dot{x}}(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\} \\ &- \lambda^{\mathrm{T}} h_{\dot{x}}(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &- \mu^{\mathrm{T}} k_{\dot{x}}(t, x(t), \dot{x}(t), u(t), \dot{u}(t))], \end{aligned} \tag{22}$$

$$\begin{aligned} f_u(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) &- vg_u(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &- \lambda^{\mathrm{T}} h_u(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ &- \mu^{\mathrm{T}} k_u(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = 0 \end{aligned} \tag{23}$$

$$\mu^{\mathrm{T}} k_l = 0, \qquad l = 1, \ldots, m \tag{24}$$

$$\mu(t) \geqslant 0. \tag{25}$$

THEOREM 5 (Strong Duality). *Under the concavity conditions of Theorem 4, if the $(x^*(t), u^*(t))$ is an efficient solution for* (P), *then there exists a piecewise smooth $\lambda(t)$: $I \to \mathbb{R}^m$ such that $(x^*(t), u^*(t), \lambda(t), \mu(t), v)$, $\mu(t)$: $I \to \mathbb{R}$ is an efficient solution of* (D) *and the extreme values of* (P) *and* (D) *are equal.*

*Proof.* Very similar to that of Theorem 2. ∎

For the converse duality theorem (Theorem 6), we make the assumption as before (see Theorem 3), i.e., $X_2$ denotes the space of piecewise differentiable functions $x$: $I \to \mathbb{R}^n$ for which $x(a) = 0 = x(b)$ equipped with the norm $\|x\| = \|x\|_\infty + \|D_x\|_\infty + \|D^2 x\|_\infty$, and $U_2$ denotes the space of

$u\colon I \to \mathbb{R}^m$, with $\|u\| = \|u\|_\infty$. The dual of the primal control problem may be rewritten as

Minimize

$$-\phi(x^*, u^*, \lambda, \mu) = (-\phi^1(x^*, u^*, \lambda, \mu), \ldots, -\phi^p(x^*, u^*, \lambda, \mu))$$

subject to

$$\begin{aligned} x^*(a) &= \alpha, \qquad x^*(b) = \beta, \\ \theta(t, x^*(t), \dot{x}^*(t), \ddot{x}^*(t), \lambda, \mu, u^*(t), \dot{u}^*(t)) &= 0, \qquad t \in I, \\ \mu(t) &\geq 0, \qquad t \in I, \end{aligned}$$

where

$$\begin{aligned} \phi^i(x^*, u^*, \lambda, \mu) = \int_a^b \{ & f^i(t, x^*(t), \dot{x}^*(t), \lambda, \mu, u^*(t), \dot{u}^*(t)) \\ & - v_i g^i(t, x^*(t), \dot{x}^*(t), \lambda, \mu, u^*(t), \dot{u}^*(t)) \\ & - \lambda^{\mathrm{T}}(t)\,[h(t, x^*(t)\dot{x}^*(t), \lambda, \mu, u^*(t), \dot{u}^*(t) - \dot{x}] \\ & - \mu^{\mathrm{T}}(t)\, k(t, x^*(t), \dot{x}^*(t), \lambda, \mu, u^*(t), \dot{u}^*(t))\}\, dt, \\ & i = 1, \ldots, p \end{aligned}$$

and

$$\begin{aligned} \theta \equiv\ & \theta(t, x^*(t), \dot{x}^*(t), \ddot{x}^*(t), \lambda, \mu, u^*(t), \dot{u}^*(t)) \\ =\ & (f_x(t, x^*(t), \dot{x}^*(t), u^*(t), u^*(t)) \\ & - v g_x(t, x^*(t), \dot{x}^*(t), u^*(t), \dot{u}^*(t)) \\ & - D\,[f_{\dot{x}}(t, x^*(t)\dot{x}^*(t), u^*(t), \dot{u}^*(t)) \\ & - v g_{\dot{x}}(t, x^*(t), \dot{x}^*(t), u^*(t), \dot{u}^*(t)) \\ & - \lambda(t)^{\mathrm{T}} h_{\dot{x}}(t, x^*(t), \dot{x}^*(t), u^*(t), \dot{u}^*(t) \\ & - \mu(t)^{\mathrm{T}} k_{\dot{x}}(t, x^*(t), \dot{x}^*(t), u^*(t), \dot{u}^*(t))]. \end{aligned}$$

Now, following Bector *et al.* [2] analogously for the control case, we are in position to deal with the converse duality theorem.

THEOREM 6 (Converse Duality). *If $(x^*, u^*, \lambda, \mu)$ is an efficient solution of* (D) *and if*

(i) *$\psi'$ have a (weak*) closed range,*

(ii) *f, g, and h are twice continuously differentiable,*

(iii) $\int_a^b \{(f_x^i - v_i g_x^i) - D(f_{\dot{x}}^i - v_i g_{\dot{x}}^i)\}\, dt,\ i = 1, \ldots, p$

*is linearly independent, and*

(iv) $(\beta(t)^{\mathrm{T}}\theta_{\dot{x}} - D\beta(t)^{\mathrm{T}}\theta_{\dot{x}} + D^2\beta(t)^{\mathrm{T}}\theta_{\ddot{x}})\beta(t) = 0$

$\Rightarrow \beta(t) = 0, \quad t \in I,$

*then $x^*$ is an efficient solution of* (P) *and the corresponding objective values are equal.*

*Proof.* Very similar to that of Theorem 3. ∎

## SUFFICIENCY

Once duality results have been established in the presence of concavity, sufficiency follows as in [3]. For the sake of completeness we state the result without proof.

THEOREM 7. *If there exists* $(x^*, u^*, \lambda, \mu)$ *such that conditions* (22)–(24) *hold with* $(x^*, u^*)$ *feasible for* (P), *and*

$$\int_a^b f\, dt, \int_a^b -vg\, dt, \qquad \int_a^b -\lambda^{\mathrm{T}}(h - \dot{x})dt, \qquad \text{and} \qquad \int_a^b -\mu^{\mathrm{T}}k\, dt$$

*are all concave with respect to the same functions $\eta$ and $\xi$, then $(x^*, u^*)$ is efficient for (P).*

*Proof.* See Mishra and Mukherjee [3]. ∎

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