JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **185**, 438–463 (1994)

Characterization of Functions with Fourier Transform Supported on Orthants

J. N. PANDEY*

Department of Mathematics and Statistics, Carleton University, Ottawa, Canada K1S 5B6

AND

O. P. SINGH

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi-221005, India

Submitted by Avner Friedman

Received July 23, 1991

We characterize the functions in $L^p(\mathbf{R}^n)$ and generalized functions in $D'_{L^p}(\mathbf{R}^n)$, $1 < p < \infty$, whose Fourier transform vanishes on one or more orthants of $\mathbf{R}^n$.

1. INTRODUCTION

It is a fairly difficult problem to characterize the functions in $L^p(\mathbf{R}^n)$ whose Fourier transform vanishes in some orthants of $\mathbf{R}^n$. Very little is known concerning this problem except the classical Paley–Wiener Theorem in one dimension which characterizes the functions in $L^2(\mathbf{R})$ having their Fourier transforms vanish for negative values of the variable [14, p. 175]. Later some results for the space $L^2(\mathbf{R}^n)$ were obtained by Stein and Weiss [29, p. 112].

Concerning the Fourier transform of a distribution with compact support, it was shown that the Fourier transform of a distribution f with bounded support is a function $F(z) = f(\exp(-2\pi i z \cdot x))$, which may be continued to all complex numbers z as an entire function of exponential growth. The converse is also true [32, p. 15]. For further references see [1, 3, 4, 5, 7, 8]. But none of those give the explicit characterization of

* Research supported by Natural Sciences and Engineering Research Council of Canada Grant A-5298.

0022-247X/94 $6.00

functions in $L^p(\mathbf{R}^n)$ and distributions in the Schwartz space $D'_{L^p}(\mathbf{R}^n)$, whose Fourier transforms are supported on a given number of orthants in $\mathbf{R}^n$. The aim of the present paper is to give a complete answer to the problem for functions in $L^p(\mathbf{R}^n)$ and distributions in $D'_{L^p}(\mathbf{R}^n)$, $1 < p < \infty$.

For $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, we construct the holomorphic function $F(z)$, $z \in \mathbf{C}^n$, as

$$F(z) = \frac{1}{(2\pi i)^n} \int_{\mathbf{R}^n} f(t) \frac{1}{\prod_{j=1}^n (t_j - z_j)} \, dt, \tag{1.1}$$

where $z_j = x_j + iy_j$ and $y_j \neq 0$ $\forall j = 1, 2, ..., n$. For a distribution $f \in D'_{L^p}(\mathbf{R}^n)$ the corresponding function $F(z)$ is defined as

$$F(z) = \frac{1}{(2\pi i)^n} \left\langle f(t), \frac{1}{\prod_{j=1}^n (t_j - z_j)} \right\rangle, \qquad y_j \neq 0 \ \forall j. \tag{1.2}$$

There are 2^n different ways in which $y \to 0$ depending upon the way the various components y_j of y tend to either 0_+ or 0_-. Thus we get 2^n different boundary values of $F(z)$ as $y \to 0$. To denote them we adopt the following notation:

Let $\sigma_k = \{\sigma_k(1), \sigma_k(2), ..., \sigma_k(n)\}$ be a sequence of length n whose elements are $+$ and $-$ for $1 \leqslant k \leqslant 2^n$. Then 2^n orthants of $\mathbf{R}^n$ are denoted by S_{σ_k}, $1 \leqslant k \leqslant 2^n$, where

$$S_{\sigma_k} = \{x \in \mathbf{R}^n \mid x_j > 0 \text{ if } \sigma_k(j) = + \text{ and } x_j < 0 \text{ if } \sigma_k(j) = -, j = 1, 2, ..., n\}. \tag{1.3}$$

For example, when $n = 2$ the various quadrants of $\mathbf{R}^2$ are denoted by S_{++}, S_{-+}, S_{+-}, and S_{--}, where

$$S_{+-} = \{x \in \mathbf{R}^2 \mid x_1 > 0 \text{ and } x_2 < 0\}, \text{ etc.}$$

Similarly the various limits of $F(z)$ as $y \to 0$ are denoted by

$$F_{\sigma_k}(x) = \lim_{y_1 \to 0_{\sigma_k(1)}, ..., y_n \to 0_{\sigma_k(n)}} F(z), \tag{1.4}$$

where $0_{\sigma_k(j)} = 0_+$ if $\sigma_k(j) = +$; otherwise it is 0_-.

For $f \in D'_{L^p}(\mathbf{R}^n)$ or $L^p(\mathbf{R}^n)$, with the limits taken in the respective spaces, we have proved that

$$f = \sum_{k=1}^{2^n} (-1)^{m_k} F_{\sigma_k} \qquad \text{in } D'_{L^p}(\mathbf{R}^n) \text{ (or } L^p(\mathbf{R}^n)), \ 1 < p < \infty \tag{1.5}$$

and

$$(-1)^{m_k} \hat{F}_{\sigma_k}(\xi) = \begin{cases} \hat{f}(\xi) & \text{for } \xi \in S_{\sigma_k} \\ 0 & \text{elsewhere,} \end{cases} \tag{1.6}$$

where

m_k is the number of minus signs in the sequence σ_k,

and $\hat{f}$ is the Fourier transform of f in the following sense,

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$$
$$= \left(\int_{\mathbf{R}^n} f\hat{\varphi} \text{ if } f \in L^p(\mathbf{R}^n) \right), \qquad \forall \varphi \in S(\mathbf{R}^n), \tag{1.7}$$

where $\hat{\varphi}$ is the classical Fourier transform of φ defined as

$$\hat{\varphi} = \int_{\mathbf{R}^n} \varphi(t)\, e^{it \cdot x}\, dt \qquad [20, 29, 34].$$

The space $S(\mathbf{R}^n)$ is the testing function space of rapid descent [20, 26].

From (1.6), we are able to prove the following Paley–Wiener Theorem for $D'_{L^p}(\mathbf{R}^n)$ (or $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$:

THEOREM. *For* $f \in D'_{L^p}(\mathbf{R}^n)$,

$$\hat{f}(\xi) = 0 \qquad \textit{for} \quad \xi \in \bigcup_{k=1}^{l} S_{\sigma_k},\ 1 \leqslant l \leqslant 2^n$$

iff

$$\sum_{k=1}^{l} (-1)^{m_k} F_{\sigma_k}(\xi) = 0 \qquad \textit{for} \quad \xi \in \bigcup_{k=1}^{l} S_{\sigma_k}, \textit{ in some space } S'_0(\mathbf{R}^n).$$

The space $S_0(\mathbf{R}^n)$ is a subspace of $S(\mathbf{R}^n)$ which is closed with respect to the multiplication by the function sgn x. The sgn is defined as

$$\operatorname{sgn}(x) = \prod_{i=1}^{n} \operatorname{sgn}(x_i). \tag{1.8}$$

In the process we proved the M. Riesz and Titchmarsh Inequality and many related classical results for $L^p(\mathbf{R}^n)$.

As an application of our theory, we characterize the solution space of the following Dirichlet boundary value problem:

$$\Delta u = 0, \qquad \text{where} \quad \Delta = \prod_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) \tag{1.9}$$

with boundary conditions

$$\lim_{y \to 0_{\sigma_k}} u = F_{\sigma_k} \quad \text{in} \quad D'_{L^p}(\mathbf{R}^n) \text{ (or } L^p(\mathbf{R}^n)) \ (1 \leqslant k \leqslant 2^n). \tag{1.10}$$

Here F_{σ_k}, $1 \leqslant k \leqslant 2^n$, are arbitrary elements of $D'_{L^p}(\mathbf{R}^n)$. Incidently for fixed F_{σ_k} $(1 \leqslant k \leqslant 2^n)$ in $D'_{L^p}(\mathbf{R}^n)$ (or $L^p(\mathbf{R}^n)$), the system (1.9) and (1.10) has

$$F(z) = \frac{1}{(2\pi i)^n} \left\langle \sum_{k=1}^{2^n} (-1)^{m_k} F_{\sigma_k}(t), \frac{1}{\prod_{j=1}^n (t_j - z_j)} \right\rangle \qquad (I_m z_j \neq 0 \ \forall j) \tag{1.11}$$

as a unique solution.

2. The Schwartz Distribution Space $D'_{L^p}(\mathbf{R}^n)$

A C^∞ complex valued function $\varphi(x)$ on $\mathbf{R}^n$ belongs to the space $D_{L^p}(\mathbf{R}^n)$ iff $\partial^\alpha \varphi(x)$ belongs to $L^p(\mathbf{R}^n)$ for each $|\alpha| = 0, 1, 2, \ldots$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, α_i's are non-negative integers, and $|\alpha| = \sum_{i=1}^n \alpha_i$. The topology over $D_{L^p}(\mathbf{R}^n)$ is generated by the countable family of separating seminorms [20, 26, 37]

$$\gamma_\alpha(\varphi) = \left[\int_{\mathbf{R}^n} |\partial^\alpha \varphi(x)|^p \, dx \right]^{1/p}.$$

The space $D_{L^p}(\mathbf{R}^n)$ is a sequentially complete, locally convex, Hausdorff topological linear space.

In conformity with the notation used by Laurent Schwartz [26], we will denote $D'_{L^p}(\mathbf{R}^n)$, $p > 1$, as the dual space of $D_{L^q}(\mathbf{R}^n)$ where $1/p + 1/q = 1$. It can be shown [20, p. 173] that for $f \in D'_{L^p}(\mathbf{R}^n)$, there exist measurable functions f_α in $L^p(\mathbf{R}^n)$ and a $k \in \mathbf{N}$ such that

$$f = \sum_{|\alpha| \leqslant k} \partial^\alpha f_\alpha. \tag{2.1}$$

Let $S'(\mathbf{R}^n)$ denote the space of tempered distributions and $S(\mathbf{R}^n)$ the corresponding testing function space of rapid descent [20]. One can see that $S(\mathbf{R}^n) \subset D_{L^p}(\mathbf{R}^n)$ and is dense in $\mathscr{D}_{L^p}(\mathbf{R}^n)$ [20]. Therefore the restriction of $f \in D'_{L^p}(\mathbf{R}^n)$ to $S(\mathbf{R}^n)$ is in $S'(\mathbf{R}^n)$ and each element of $D'_{L^p}(\mathbf{R}^n)$ can be identified with an element of $S'(\mathbf{R}^n)$ in a one-to-one way and hence with this kind of identification $D'_{L^p}(\mathbf{R}^n) \subset S'(\mathbf{R}^n)$. Therefore the Fourier transform $\hat{f}$ of f in $D'_{L^p}(\mathbf{R}^n)$ can be defined by

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle, \qquad \forall \varphi \in S(\mathbf{R}^n).$$

THEOREM 2.1. *Let $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$. Define*

$$F(x, y) = \int_{\mathbf{R}^n} f(t) \prod_{j=1}^{n} \frac{t_j - x_j}{(t_j - x_j)^2 + y_j^2}\, dt, \tag{2.2}$$

where $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $t = (t_1, t_2, ..., t_n)$ are in $\mathbb{R}^n$ and $y_j \neq 0$ $(j = 1, 2, ..., n)$. Then we have

$$\partial_x^\alpha \partial_y^\beta F(x, y) = \int_{\mathbf{R}^n} f(t)\, \partial_x^\alpha \partial_y^\beta \left[\prod_{j=1}^{n} \frac{t_j - x_j}{(t_j - x_j)^2 + y_j^2} \right] dt, \tag{2.3}$$

where

$$|\alpha|, |\beta| = 0, 1, 2, ..., \partial_x^\alpha \equiv \frac{\alpha^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},\ \partial_y^\beta = \frac{\partial^{\beta_1}}{\partial y_1^{\beta_1}} \cdots \frac{\partial^\beta}{\partial y_n^{\beta_n}}$$

and the α_j's and β_j's are non-negative integers. Also $F(x, y)$ and $\partial_x^\alpha \partial_y^\beta F(x, y)$ are continuous functions of x, $y \in \mathbf{R}^n$. Thus $F(x, y) \in C^\infty(\mathbf{R}^{2n})$.

Proof. Set

$$u(x, y) = \int_{\mathbf{R}^n} f(t)\, \partial_x^\alpha \prod_{j=1}^{n} [(t_j - x_j)/((t_j - x_j)^2 + y_j^2)]\, dt$$

$$= \int_{\mathbf{R}^n} f(t + x) \left(\frac{-\partial}{\partial t} \right) \prod_{j=1}^{n} [t_j/(t_j^2 + y_j^2)]\, dt.$$

Thus by Holder's inequality we have

$$|u(x, y)| \leqslant \|f\|_p \left\| \prod_{j=1}^{n} \partial_{t_j}^{\alpha_j} \left(\frac{t_j}{t_j^2 + y_j^2} \right) \right\|_q.$$

Hence the integral representing $u(x, y)$ is uniformly convergent $\forall x$ in $\mathbf{R}^n$ and a fixed $y \in \mathbf{R}^n$ having all non-zero components. By using the mean value theorem, we can prove the continuity of $F(x, y)$ and $u(x, y)$ with respect to both x and y. These results are true for arbitrary α. Hence, using a standard classical theorem [31, p. 59], it follows that

$$\partial_x^\alpha F(x, y) = \int_{\mathbf{R}^n} f(t)\, \partial_x^\alpha \left[\prod_{j=1}^{n} \frac{t_j - x_j}{(t_j - x_j)^2 + y_j^2} \right] dt.$$

Also we have

$$|\partial_y^\beta F(x, y)| \leqslant \|f\|_p \left\| \prod_{j=1}^{n} \partial_{y_j}^{\beta_j} \left(\frac{t_j}{t_j^2 + y_j^2} \right) \right\|_q$$

Using the fact that

$$\left|\frac{y_j}{t_j^2+y_j^2}\right|^q \leqslant \left|\frac{a_j+\delta_j}{t_j^2+(a_j-\delta_j)^2}\right|^q$$

$\forall y_j \in (a_j-\delta_j, a_j+\delta_j)$, we can see that for arbitrary β, the integral representing $\partial_y^\beta F(x, y)$ is uniformly convergent in an appropriately chosen rectangle lying in the region

$$\{y \in \mathbf{R}^n \mid |y_j| > 0, j=1, 2, ..., n\}.$$

Therefore we have

$$\partial_y^\beta F(x, y) = \int_{\mathbf{R}^n} f(t)\, \partial_y^\beta \prod_{j=1}^n \frac{t_j-x_j}{(t_j-x_j)^2+y_j^2}\, dt \qquad [31, \text{p. } 59]. \qquad \text{Q.E.D.}$$

LEMMA 2.1. *Let* $x, y, t \in \mathbf{R}^n$ *be such that* $y_j \neq 0$, $\forall j=1, 2, ..., n$. *For* $f \in D'_{L^p}(\mathbf{R}^n)$ $(1 < p < \infty)$ *define a function*

$$F(x, y) = \left\langle f(t), \prod_{j=1}^n \left[(t_j-x_j) \mid ((t_j-x_j)^2+y_j^2)\right]\right\rangle. \tag{2.4}$$

Then

$$\partial_y^\beta \partial_x^\alpha F(x, y) = \left\langle f(t), \partial_y^\beta \partial_x^\alpha \prod_{j=1}^n \left[(t_j-x_j) \mid ((t_j-x_j)^2+y_j^2)\right]\right\rangle. \tag{2.5}$$

Proof. Since $\prod_{j=1}^n ((t_j-x_j)/((t_j-x_j)^2+y_j^2)) \in L^q(\mathbf{R}^n)$ as a function of t for a fixed x and y, and $f \in D'_{L^p}(\mathbf{R}^n)$, the dual of $L^q(\mathbf{R}^n)$ $(1/p+1/q=1)$, $F(x, y)$ is well defined for each $x, y \in \mathbf{R}^n$ with y having all non-zero components. Using the structure formula (2.1) for $f \in D'_{L^p}(\mathbf{R}^n)$, we see that

$$\begin{aligned} F(x, y) &= \sum_{|\gamma| \leqslant k} \left\langle f_\gamma(t), (-1)^{|\gamma|} \partial_t^\gamma \prod_{j=1}^n \frac{t_j-x_j}{(t_j-x_j)^2+y_j^2}\right\rangle \\ &= \sum_{|\gamma| \leqslant k} \left\langle f_\gamma(t), \partial_x^\gamma \prod_{j=1}^n \frac{t_j-x_j}{(t_j-x_j)^2+y_j^2}\right\rangle, \qquad f_\gamma \in L^p(\mathbf{R}^n). \end{aligned}$$

Then, using Theorem 2.1, we obtain

$$\begin{aligned} \partial_y^\beta \partial_x^\alpha F(x, y) &= \sum_{|\gamma| \leqslant k} \left\langle f_\gamma(t), \partial_y^\beta \partial_x^{\alpha+\gamma} \prod_{j=1}^n \left[(t_j-x_j)/((t_j-x_j)^2+y_j^2)\right]\right\rangle \\ &= \sum_{|\gamma| \leqslant k} \left\langle f_\gamma(t), (-1)^{|\gamma|} \partial_t^\gamma \partial_y^\beta \partial_x^\alpha \right. \\ &\quad \left. \times \prod_{j=1}^n \left[(t_j-x_j)/((t_j-x_j)^2+y_j^2)\right]\right\rangle \\ &= \left\langle f(t), \partial_y^\beta \partial_x^\alpha \prod_{j=1}^n \left[(t_j-x_j) \mid ((t_j-x_j)^2+y_j^2)\right]\right\rangle. \qquad \text{Q.E.D.} \end{aligned}$$

3. An Approximate Hilbert Transform and Its Limits in $L^p(\mathbf{R}^n)$

The authors acknowledge the facts that the results proved by them in Sections 3, 4, and 5 are not entirely new. Some of their results proved in Sections 3, 4, and 5 are proved by Tillmann [33] and Vladimirov [35, Chap. 5]. However, our techniques are different in that we make an extended use of the results proved by Riesz and Titchmarsh [30], thereby making our treatment simpler. Our main results proved in Section 6 are new and are not proved anywhere else. In our analysis we heavily rely upon the result that

$$F(Hf) = i^n \prod_{j=1}^{n} \operatorname{sgn}(x_j)(Ff) \qquad \forall f \in (D_{L^p}(\mathbf{R}^n))^p,\ p > 1$$

in the weak topology of $S_0(\mathbf{R}^n)$. The space $S_0(\mathbf{R}^n)$ is a subspace of the Schwartz testing function space $S(\mathbf{R}^n)$ such that every element of $S_0(\mathbf{R}^n)$ vanishes at the origin along with all its derivatives. The topology of $S_0(\mathbf{R}^n)$ is the same as that induced on $S_0(\mathbf{R}^n)$ by $S(\mathbf{R}^n)$.

Let H be the operator of the classical Hilbert transform from $L^p(\mathbf{R}^n)$, $p > 1$, into itself defined by

$$\begin{aligned}(Hf)(x) &= \lim_{\substack{\max \varepsilon_j \to 0 \\ 1 \leqslant j \leqslant n}} \frac{1}{\pi^n} \int_{|t_j - x_j| > \varepsilon_j} \frac{f(t)}{\prod_{j=1}^{n} (t_j - x_j)}\, dt \\ &= \frac{1}{\pi^n} P \int_{\mathbf{R}^n} \frac{f(t)}{\prod_{j=1}^{n} (t_j - x_j)}\, dt. \end{aligned} \tag{3.1}$$

It is a known fact that the limit exists a.e. [15] and that $(Hf)(x) \in L^p(\mathbf{R}^n)$. Also

$$\|Hf\|_p \leqslant C_p \|f\|_p \qquad [5, 15, 27, 39], \tag{3.2}$$

where C_p is a constant independent of f [15, 28].

Titchmarsh [30] proved that if $f \in L^p(\mathbf{R})$, $p > 1$, then its approximate Hilbert transform

$$(H_y f)(x) = \frac{1}{x} \int_{\mathbf{R}} \frac{f(t) \cdot (t - x)}{(t - x)^2 + y^2}\, dt, \qquad y \neq 0 \tag{3.3}$$

exists a.e. and

$$\lim_{y \to 0} \frac{1}{\pi} \int_{\mathbf{R}} \frac{t - x}{(t - x)^2 + y^2} f(t)\, dt = (Hf)(x) \qquad \text{in } L^p(\mathbf{R}). \tag{3.4}$$

It is also known that

$$\|(H_y f)(x)\|_p \leqslant C_p \|f\|_p, \tag{3.5}$$

where C_p is a constant independent of f and y. Stein and Weiss [29, p. 218] proved similar result for $L^p(\mathbf{R})$ over the Lebesgue set of f. We extend the above results to n-dimensions.

DEFINITION. The n-dimensional approximate Hilbert transform $(H_y f)(x)$ of $f \in L^p(\mathbf{R}^n)$ $(p > 1)$ is defined by

$$(H_y f)(x) = \frac{1}{\pi^n} \int_{\mathbf{R}^n} \prod_{j=1}^{n} \frac{t_j - x_j}{(t_j - x_j)^2 + y_j^2} f(t)\, dt, \qquad y_j \neq 0 \ \forall j = 1, 2, 3, ..., n. \tag{3.6}$$

THEOREM 3.1. *The operator H_y as defined by* (3.6) *is a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself.*

Proof. We will first prove the result for $n = 2$. Let $f \in L^p(\mathbf{R}^2)$. Then we have

$$\|f\|_p = \left(\int_{\mathbf{R}^2} |f|^p \, dx\, dy\right)^{1/p} = \left(\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx |f(x, y)|^p\right)^{1/p}$$

$$= \left(\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |f(x, y)|^p\right)^{1/p} \qquad \text{(by Fubini's theorem [12])}$$

so that

$$\|f\|_p = \|f(\cdot, y)\|_{1, p; 2, p} = \|f(x, \cdot)\|_{2, p; 1, p}, \tag{3.7}$$

where

$$\|f(x, \cdot)\|_{2, p} = \left(\int_{\mathbf{R}} |f(x, y)|^p \, dy\right)^{1/p}$$

$$\|f(\cdot, y)\|_{1, p} = \left(\int_{R} |f(x, y)|^p \, dx\right)^{1/p}$$

$$\|f(x, y)\|_{1, p; 2, p} = L^p \text{ norm of } \|f(\cdot, y)\|_{1, p} \text{ as a function of } y$$

and

$$\|f(x, y)\|_{2, p; 1, p} = L^p \text{ norm of } \|f(x, \cdot)\|_{2, p} \text{ as a function of } x.$$

If one of the expressions in (3.7) exists, the remaining two also exist. Now,

$$(H_y f)(x_1, x_2) = \frac{1}{\pi^2} \int_{\mathbf{R}^2} f(t) \prod_{j=1}^{2} [(t_j - x_j)/((t_j - x_j)^2 + y_j^2)] \, dt.$$

Therefore, by (3.5), we have

$$\begin{aligned} \|(H_y f)(x_1, x_2)\|_p &= \|(H_y f)(x_1, x_2)\|_{1,p;2,p} \\ &\leqslant C_p \left\| \int_{\mathbf{R}} \frac{t_2 - x_2}{(t_2 - x_2)^2 + y_2^2} f(\cdot, t_2) \, dt_2 \right\|_{1,p;2,p}, \end{aligned}$$

where C_p is a constant independent of f and y [30]. But

$$\begin{aligned} \left\| \int_{\mathbf{R}} \frac{t_2 - x_2}{(t_2 - x_2)^2 + y_2^2} f(\cdot, t_2) \, dt_2 \right\|_{2,p;1,p} &\leqslant C_p \|f(t_1, \cdot)\|_{2,p;1,p} \\ &\leqslant C_p^2 \|f\|_p. \end{aligned}$$

Therefore, in view of Fubini's theorem [12]

$$\|H_y f\|_p \leqslant C_p^2 \|f\|_p.$$

Thus the theorem is proved for $n = 2$. Using similar techniques and induction on n, it can be shown that for $f \in L^p(\mathbf{R}^n)$,

$$\|H_y f\|_p \leqslant C_p^n \|f\|_p. \qquad \text{Q.E.D.} \quad (3.8)$$

DEFINITION. The space $X(\mathbf{R}^n)$ is defined to be the collection of $\varphi \in D(\mathbf{R}^n)$ which are finite sums of the form

$$\varphi(x) = \sum \varphi_{m1}(x_1) \, \varphi_{m2}(x_2) \cdots \varphi_{mn}(x_n),$$

where

$$\varphi_{mj}(x_j) \in D(\mathbf{R}), \qquad 1 \leqslant j \leqslant n.$$

The space $X(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$ [36, p. 71].

THEOREM 3.2. *For $f \in L^p(\mathbf{R}^n)$, define $(Hf)(x)$ (the Hilbert transform of f) and $(H_y f)(x)$ (the approximate Hilbert transform of f) as in* (3.1) *and* (3.6), *respectively. Then*

$$\lim_{y_1, y_2, \ldots, y_n \to 0} (H_y f)(x) = (Hf)(x) \text{ in } L^p(\mathbf{R}^n) \text{ norm}.$$

Proof. Let φ_m be a sequence in $X(\mathbf{R}^n)$ converging to f in $L^p(\mathbf{R}^n)$. Then

$$\lim_{m \to \infty} \|f(x) - \varphi_m(x)\|_p = 0.$$

Now

$$H_y f - Hf = H_y f - H_y \varphi_m + H_y \varphi_m - H\varphi_m + (H\varphi_m - Hf).$$

So,

$$\begin{aligned}\|H_y f - Hf\|_p &\leq \|H_y(f - \varphi_m)\|_p + \|H_y \varphi_m - H\varphi_m\|_p + \|H(\varphi_m - f)\|_p \\ &\leq C_p^n \|f - \varphi_m\|_p + \|H_y \varphi_m - H\varphi_m\|_p + C_p^n \|\varphi_m - f\|_p.\end{aligned}$$

It is a simple exercise to show that $\|H_y \varphi_m - H\varphi_m\|_p \to 0$ as $y \to 0$. Letting $y \to 0$, we deduce

$$\lim_{y \to 0} \|H_y f - Hf\|_p \leq 2C_p^n \|f - \varphi_m\|_p.$$

Now letting $m \to \infty$, we obtain

$$\lim_{y \to 0} \|H_y f - Hf\|_p = 0. \qquad \text{Q.E.D.}$$

THEOREM 3.3. *Let $f \in L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ and let $y_1, y_2, \ldots, y_n$ be non-zero real numbers. Then*

$$\text{(i)} \quad (I_y f)(x) = \frac{1}{\pi^n} \int_{\mathbf{R}^n} f(t) \prod_{j=1}^{n} [y_j/((t_j - x_j)^2 + y_j^2)]\, dt, \tag{3.9}$$

which, as a function of x, belongs to $L^p(\mathbf{R}^n)$,

(ii) $\|I_y f\|_p \leq C_p^n \|f\|_p$, *where C_p is a constant independent of f and y,*

(iii) $\|I_y f - f\|_p \to 0$ *as $y \to 0_+$, i.e., $y_1, y_2, \ldots, y_n \to 0_+$.*

Proof. The proof is very similar to that given for Theorem 3.1. One can use the fact that for $g \in L^p(\mathbf{R})$,

$$(I_y g)(x) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{y g(t)}{(t - x)^2 + y^2}\, dt \in L^p(\mathbf{R}), \qquad y \neq 0$$

$$\lim_{y \to 0_+} (I_y g)(x) = g(x) \qquad \text{in } L^p(\mathbf{R})$$

and [30]

$$\|I_y g\| \leq C_p \|g\|_p.$$

The result (i) can also be proved by using [12, p. 400]. It is easy to see that if $f \in L^p(\mathbf{R}^n)$

$$\|I_y f\|_p \leqslant C_p^n \|f\|_p, \qquad \forall f \in L^p(\mathbf{R}^n). \qquad \text{Q.E.D.}$$

THEOREM 3.4. *For $f \in L^p(\mathbf{R}^n)$ $(p > 1)$ and $x, y \in \mathbf{R}^n$ define*

$$(Tf)(x) = \frac{1}{\pi^n} \int_{\mathbf{R}^n} f(t) \left[\prod_{j=1}^{m} \frac{t_j - x_j}{(t_j - x_j)^2 + y_j^2} \right] \left[\prod_{k=m+1}^{n} \frac{y_k}{(t_k - x_k)^2 + y_k^2} \right] dt, \tag{3.10}$$

where $0 \leqslant m \leqslant n$. Then T is a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself,

$$\|Tf\|_p \leqslant C_p^n \|f\|_p, \tag{3.11}$$

and

$$\begin{aligned} \lim_{y_1, y_2, \ldots, y_n \to 0^+} (Tf)(x) &= (H_1 H_2 \cdots H_m I_{m+1} \cdots I_n f)(x) \\ &= (H_1 H_2 \cdots H_m f)(x), \end{aligned} \tag{3.12}$$

where $I_1, I_2, \ldots, I_n$ are all one dimensional identity operators and $H_1, H_2, \ldots, H_n$ are all one dimensional Hilbert transform operators.

Proof. The proof of (3.11) can be given by using the technique followed in Theorems 3.2 and 3.3 and then (3.12) can be proved by using (3.11) and the density of $X(\mathbf{R}^n)$ in $L^p(\mathbf{R}^n)$ [36, p. 71]. Q.E.D.

4. COMPLEX HILBERT TRANSFORM

Let $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, and $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$ s.t. $I_m z_j = y_j \neq 0$ $\forall j = 1, 2, \ldots, n$. We define the complex Hilbert transform $(Hf)(z)$ of f by

$$\begin{aligned} (Hf)(z) &= \frac{1}{\pi^n} \int_{\mathbf{R}^n} \frac{f(t)}{\prod_{j=1}^{n} (t_j - z_j)} dt \\ &= \frac{1}{\pi^n} \int_{\mathbf{R}^n} f(t) \prod_{j=1}^{n} \frac{(t_j - x_j) + i y_j}{(t_j - x_j)^2 + y_j^2} dt. \end{aligned} \tag{4.1}$$

Then we have the following

THEOREM 4.1. *For $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, its complex Hilbert transform $(Hf)(z)$ as a function of x belongs to $L^p(\mathbf{R}^n)$ for a fixed y with all non-zero components. Also*

$$\|Hf\|_p \leqslant (2C_p)^n \|f\|_p \qquad (\textit{Titchmarsh and Riesz inequality}) \tag{4.2}$$

and

$$\lim_{y_1, y_2, \ldots, y_n \to 0_+} (Hf)(z) = \left(\prod_{j=1}^{n} (H_j + iI_j) \right) f(x) \qquad \textit{in} \quad L^p(\mathbf{R}^n), \tag{4.3}$$

where

$$(H_j f)(x) = \frac{1}{\pi} P \int_{\mathbf{R}} \frac{f(x_1, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_n)}{t_j - x_j} dt_j \tag{4.4}$$

and

$$(I_j f)(x) = I_j f(x_1, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_n) = f(x). \tag{4.5}$$

Similarly

$$\lim_{\ldots, y_j \to 0_+, \ldots, y_k \to 0_-, \ldots} (Hf)(z) = (\cdots (H_j + iI_j) \cdots (H_k - iI_k) \cdots) f(x). \tag{4.6}$$

Proof. The proof can be given by using the technique followed in Theorems 3.1, 3.2, and 3.3. Q.E.D.

THEOREM 4.2. *For $f \in L^p(\mathbf{R}^n)$, $p > 1$, define*

$$F(x) = \frac{1}{\pi^n} \int_{\mathbf{R}^n} f(t) \prod_{j=1}^{n} \frac{t_j - x_j}{(t_j - x_j)^2 + y_j^2} dt, \qquad y_j \neq 0 \ \forall j. \tag{4.7}$$

Then

$$\partial^\alpha F(x) \in L^p(\mathbf{R}^n).$$

Proof. We will prove the result for the simple case when $\partial^\alpha = \partial/\partial x_1$ and the general result will follow by induction. Now

$$\frac{\partial}{\partial x_1} F(x_1, x_2, \ldots, x_n) = \frac{1}{\pi^n} \int_{\mathbf{R}} \frac{(t_1 - x_1)^2 - y_1^2}{[(t_1 - x_1)^2 + y_1^2]^2} dt_1 \times \int_{\mathbf{R}^{n-1}} f(t) \prod_{j=2}^{n} \frac{(t_j - x_j)\, dt_j}{(t_j - x_j)^2 + y_j^2}.$$

Therefore,

$$\left\| \frac{\partial}{\partial x_1} F(x) \right\|_{L^p(\mathbf{R}^{n-1})} \leqslant C_p^{n-1} \left\| \frac{t_1^2 - y_1^2}{(t_1^2 + y_1^2)^2} \right\|_{L^1(\mathbf{R})} \|f(t_1, ...)\|_{L^p(\mathbf{R}^{n-1})} \qquad [12, \text{p. } 401]$$

$$\leqslant C_p^{n-1} \frac{1}{|y_1|} \|f\|_{L^p(\mathbf{R}^{n-1})}$$

$$\left\| \frac{\partial}{\partial x_1} F(x) \right\|_p \leqslant C_p^{n-1} \frac{1}{|y_1|} \|f\|_p. \qquad \text{Q.E.D.}$$

COROLLARY 4.1. *For $f \in D'_{L^p}(\mathbf{R}^n)$, $p > 1$, and fixed real numbers $y_1, y_2, ..., y_n$ different from zero, define*

$$F(x) = \frac{1}{\pi^n} \left\langle f(t), \prod_{j=1}^{n} [(t_j - x_j)/((t_j - x_j)^2 + y_j^2)] \right\rangle. \tag{4.8}$$

Then $F(x) \in L^p(\mathbf{R}^n)$.

Proof. Using the structure formula (2.1)

$$f = \sum_{|\alpha| \leqslant k} \partial^\alpha f_\alpha, \qquad \text{where each } f_\alpha \in L^p(\mathbf{R}^n),$$

and Lemma 2.1, we have

$$F(x) = \sum_{|\alpha| = 0}^{k} \partial_x^\alpha \left\langle f_\alpha(t), \prod_{j=1}^{n} [(t_j - x_j)/((t_j - x_j)^2 + y_j^2)] \right\rangle.$$

The result now follows in view of Theorems 3.1 and 4.2. Q.E.D.

THEOREM 4.3. *For $f \in D'_{L^p}(\mathbf{R}^n)$, $p > 1$, and $y_j \neq 0$, $1 \leqslant j \leqslant n$, define*

$$(H_y f)(x) = F(x), \qquad \textit{as defined in } (4.8).$$

Then

$$\lim_{|y| \to 0} (H_y f)(x) = (H_1 \cdots H_n f)(x) = (Hf)(x), \tag{4.9}$$

where the limit is interpreted as the weak limit on $D'_{L^p}(\mathbf{R}^n)$.

Proof. In view of Theorem 4.2 and Corollary 4.1, $(H_y f)(x)$ can be interpreted as a regular distribution on $D_{L^q}(\mathbf{R}^n)$. Therefore, for each $\varphi \in D_{L^q}(\mathbf{R}^n)$,

$$\left\langle\left\langle f(t), \prod_{j=1}^{n} [(t_j-x_j)/((t_j-x_j)^2+y_j^2)]\right\rangle, \varphi(x)\right\rangle$$

$$=\left\langle\left\langle \sum_{|\alpha|=0}^{m} \partial_t^\alpha f_\alpha(t), \prod_{j=1}^{n} [(t_j-x_j)/((t_j-x_j)^2+y_j^2)]\right\rangle, \varphi(x)\right\rangle$$

[20, p. 175]

$$=\sum_{|\alpha|=0}^{m}\left\langle\left\langle f_\alpha(t), \partial_x^\alpha \prod_{j=1}^{n} [(t_j-x_j)/((t_j-x_j)^2+y_j^2)]\right\rangle, \varphi(x)\right\rangle$$

$$=\sum_{|\alpha|=0}^{m}\left\langle \partial_x^\alpha\left\langle f_\alpha(t), \prod_{j=1}^{n} [(t_j-x_j)/((t_j-x_j)^2+y_j^2)]\right\rangle, \varphi(x)\right\rangle$$

[Lemma 2.1]

$$=\sum_{|\alpha|=0}^{m}\left\langle\left\langle f_\alpha(t), \prod_{j=1}^{n} [(t_j-x_j)/((t_j-x_j)^2+y_j^2)]\right\rangle, (-\partial_x)^\alpha \varphi(x)\right\rangle$$

$$=\sum_{|\alpha|=0}^{m} (-1)^{|\alpha|}\left\langle f_\alpha(t), \int_{\mathbf{R}^n} (\partial_x^\alpha \varphi(x)) \times \prod_{j=1}^{n} [(t_j-x_j)/((t_j-x_j)^2+y_j^2)]\, dx\right\rangle. \tag{4.10}$$

Since

$$f_\alpha \in L^p(\mathbf{R}^n) \qquad \text{and} \qquad \partial_x^\alpha \varphi(x) \in D_{L^q}(\mathbf{R}^n),$$

by using the duality theorems and the limiting processes, the switch in the order of integration is justified. Now letting $|y| \to 0$ in (4.10), we obtain

$$\lim_{|y|\to 0} \langle (H_y f)(x), \varphi(x)\rangle = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} \langle f_\alpha(t), (-1)^n H(\partial^\alpha \varphi(t))\rangle. \tag{4.11}$$

The steps in (4.11) can easily be justified in view of Theorem 3.2. Now using the commutativity of the distributional differentiation ∂^α and H [22, 27], we deduce

$$\lim_{|y|\to 0} \langle (H_y f)(x), \varphi(x)\rangle = \left\langle H \sum_{|\alpha|=0}^{m} \partial_t^\alpha f_\alpha(t), \varphi(t)\right\rangle$$
$$= \langle Hf, \varphi\rangle.$$

Therefore,

$$\lim_{|y|\to 0} H_y f = Hf \qquad \text{in } D'_{L^p}(\mathbf{R}^n)$$

Q.E.D.

COROLLARY 4.2. *For $f \in D'_{L^p}(\mathbf{R}^n)$, define the complex Hilbert transform of f by*

$$F(z) = \frac{1}{\pi^n} \left\langle f(t), \frac{1}{\prod_{j=1}^n (t_j - z_j)} \right\rangle, \qquad I_m z_j = y_j \neq 0 \ \forall j. \tag{4.12}$$

Then

$$\lim_{y_1, \ldots, y_n \to 0^+} F(z) = \prod_{j=1}^{N} (H_j + iI_j) f. \tag{4.13}$$

Proof. The proof is similar to the proof of (4.3). Q.E.D.

5. Distributional Representation of Holomorphic Functions

The holomorphic function $F(z)$ given by (4.12) satisfies the uniform asymptotic orders (uniformity with respect to x is assumed here)

$$|F(z)| = O\left(\frac{1}{(y_1 y_2 \cdots y_n)^{(p-1)/p}}\right), \qquad \text{as} \quad y_1, y_2, \ldots, y_n \to \infty.$$

Let us now reverse the problem. Let $F(z)$ be holomorphic in $y_j > 0$ ($j = 1, 2, \ldots, n$), i.e., on $S_{++\ldots+}$ and let it satisfy the relation

$$\sup_{\substack{x_j \in \mathbf{R},\ y_j \geqslant \delta > 0 \\ 1 \leqslant j \leqslant n}} |F(x+iy)| < A_\delta < \infty, \tag{5.1}$$

and the uniform asymptotic order (w.r.t. x)

$$|F(x+iy)| = o(1), \qquad y \to \infty. \tag{5.2}$$

Assume also that

$$\lim_{y_1, y_2, \ldots, y_n \to 0^+} F(z) = F_{++\ldots+}(x) \qquad \text{in } D'_{L^p}(\mathbf{R}^n). \tag{5.3}$$

Then by using the technique of [22], it can be shown that

$$\frac{1}{(2\pi i)^n} \left\langle F_{++\ldots+}(t), \frac{1}{\prod_{j=1}^n (t_j - z_j)} \right\rangle = \begin{cases} F(z), & \text{for } y \in S_{++\ldots+}, \\ 0, & \text{elsewhere}, \end{cases} \tag{5.4}$$

where the positive orthant $S_{++\ldots+} = \{y \in \mathbb{R}^n \mid y_j > 0, j = 1, 2, \ldots, n\}$. Results similar to (5.4) can be obtained by taking $F(z)$ holomorphic in other of the $2^n - 1$ orthants and evaluating the corresponding limits of $F(z)$. Let

$$\Omega = \{z \in \mathbf{C}^n \mid I_m z_j = y_j \neq 0 \ \forall j = 1, 2, \ldots, n\}. \tag{5.5}$$

For $F(z) \in \mathrm{Hol}(\Omega)$, there are 2^n different ways of evaluating $\lim_{y \to 0} F(z)$ depending upon the various components of y going to either 0^+ or 0^-. These limits are denoted by $F_{\sigma_k}(x)$.

EXAMPLE 1. When $n=2$, there are four quadrants S_{++}, S_{-+}, S_{+-}, and S_{--} and four different limits F_{++}, F_{-+}, F_{+-}, F_{--}, where for example

$$S_{-+} = \{ y \in \mathbf{R}^2 \mid y_1 < 0 \text{ and } y_2 > 0 \}$$

and

$$F_{-+}(x) = \lim_{y_1 \to 0_-,\, y_2 \to 0_+} F(z).$$

Let

$$M = \{ F(z) \in \mathrm{Hol}(\Omega) \mid F(z) \text{ satisfies the following conditions (A), (B) and (C)} \}: \tag{5.6}$$

$$\sup_{\substack{x_j \in \mathbf{R},\, |y_j| \geqslant \delta > 0 \\ 1 \leqslant j \leqslant n}} |F(x+iy)| < A_\delta < \infty, \tag{A}$$

$$|F(x+iy)| = o(1), \qquad \text{as} \quad |y_1|, ..., |y_n| \to \infty \tag{B}$$

independently of each other and the asymptotic order is valid uniformly $\forall x \in \mathbf{R}^n$ and

$$\lim_{y \to 0_{\sigma_k}} F(z) = F_{\sigma_k}(x) \qquad \text{in} \quad D'_{L^p}(\mathbf{R}^n),\ k = 1, 2, ..., 2^n, \tag{C}$$

where $y \to 0_{\sigma_k}$ means $y_j \to 0_{\sigma_k(j)}$, $1 \leqslant j \leqslant n$. Then we have the following theorem.

THEOREM 5.1. *For any $F(z) \in M$, we have*

$$F(z) = \frac{1}{(2\pi i)^n} \left\langle \sum_{k=1}^{2^n} (-1)^{m_k} F_{\sigma_k}(t), \frac{1}{\prod_{j=1}^n (t_j - z_j)} \right\rangle, \tag{5.7}$$

where $m_k =$ the number of minus signs present in the sequences σ_k. For example, when $n=2$,

$$F(z) = \frac{1}{(2\pi i)^2} \Big\langle F_{++} + (-1) F_{-+} + (-1) F_{+-} + (-1)^2 F_{--}, \frac{1}{(t_1 - z_1)(t_2 - z_2)} \Big\rangle$$

$$= -\frac{1}{4\pi^2} \left\langle (F_{++} - F_{-+} - F_{+-} + F_{--}), \frac{1}{(t_1 - z_1)(t_2 - z_2)} \right\rangle_t.$$

6. Action of the Fourier Transform on the Hilbert Transform

If $f \in L^2(\mathbb{R})$ then

$$\widehat{(Hf)}(x) = i \operatorname{sgn}(x) \hat{f}(x) \text{ a.e.} \qquad [29, \text{p. } 219], \tag{6.1}$$

where the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(x) = \int_{\mathbf{R}} f(t) \exp(t \cdot x)\, dt. \tag{6.2}$$

Note that in the RHS expression of (6.1) Stein and Weiss use $(-i)$ in place of i as their Hilbert transform differs from ours by a constant factor only. The result (6.1) can easily be extended to $L^2(\mathbf{R}^n)$ as follows:

The space $X(\mathbf{R}^n)$ consisting of finite linear combinations of functions of the type $\varphi_1(x_1)\,\varphi_2(x_2)...\varphi_n(x_n)$, where each $\varphi_j(x_j) \in D(\mathbf{R})$, is dense in $L^2(\mathbf{R}^n)$ [36, p. 71]. Therefore for $f \in L^2(\mathbf{R}^n)$ we can find a sequence ψ_m in $X(\mathbf{R}^n)$ s.t. $\psi_m(x) \to f(x)$ in $L^2(\mathbf{R}^n)$ as $m \to \infty$. Denoting by $\mathscr{F}$, the Fourier transform operator, we have

$$(\mathscr{F}(H(\psi_m)))(x) = i^n \operatorname{sgn}(x)(\mathscr{F}\psi_m)(x), \tag{6.3}$$

where

$$\operatorname{sgn}(x) = \prod_{j=1}^{n} \operatorname{sgn}(x_j).$$

Now letting $m \to \infty$ in (6.3) and interpreting the convergence in $L^2(\mathbf{R}^n)$, we deduce

$$(\mathscr{F} Hf)(x) = i^n \operatorname{sgn}(x)(\mathscr{F} f)(x). \tag{6.4}$$

The question now arises whether or not such a result can be proved for the space $L^p(\mathbf{R}^n)$, $p > 1$. We are able to prove the result (6.4) for $p = 2$ because of the fact that the Fourier transform maps $L^2(\mathbf{R}^n)$ into itself. But such a result is not true in general for $p > 1$, $p \neq 2$. If $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, its Fourier transform can be defined, treating f as a regular tempered distribution, as follows,

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle = \int_{\mathbf{R}^n} f\hat{\varphi}\, dx, \qquad \forall \varphi \in S(\mathbf{R}^n),$$

where $\hat{\varphi}(x)$ is the classical Fourier transform of $\varphi(t)$ given by

$$\hat{\varphi}(x) = \int_{\mathbf{R}^n} \varphi(t) \exp(t \cdot x)\, dt,$$

where $t \cdot x$ is now the inner product of t and x [30, p. 9]. For $f \in L^p(\mathbf{R}^n)$, let ψ_m be a sequence in $X(\mathbf{R}^n)$ tending to f in $L^p(\mathbf{R}^n)$, as $m \to \infty$. Then we have

$$\lim_{m \to \infty} \hat{\psi}_m = \hat{f} \qquad \text{in } S'(\mathbf{R}^n).$$

Since the Hilbert transform H is a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself [25], it follows that

$$\lim_{m \to \infty} \mathscr{F}(H(\psi_m)) = \mathscr{F}(Hf).$$

As $\psi_m \in L^2(\mathbf{R}^n)$, from (6.4) we conclude that

$$\lim_{m \to \infty} i^n \prod_{j=1}^{n} \operatorname{sgn}(x_j)\, \hat{\psi}_m = \mathscr{F}(H(f)), \tag{6.5}$$

i.e.,

$$\langle \mathscr{F} Hf, \varphi \rangle = \lim_{m \to \infty} \langle i^n \operatorname{sgn}(x)\, \hat{\psi}_m(x), \varphi(x) \rangle$$

$$= \lim_{m \to \infty} i^n \int_{\mathbf{R}^n} \operatorname{sgn}(x)\, \hat{\psi}_m \varphi(x)\, dx, \qquad \forall \varphi \in S(\mathbf{R}^n).$$

But still we cannot, in general, say that this limit (6.5) equals $i^n \operatorname{sgn}(x) \hat{f}$, $\forall f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$.

We now construct a testing function space $S_0(\mathbf{R}^n)$ which is a subspace of $S(\mathbf{R}^n)$ closed with respect to multiplication by $\prod_{j=1}^{n} \operatorname{sgn}(x_j)$. The topology of $S_0(\mathbf{R}^n)$ is the same as that induced on it by $S(\mathbf{R}^n)$. $S_0(\mathbf{R}^n)$ is a non-empty subspace of $S(\mathbf{R}^n)$. All functions in $S(\mathbf{R}^n)$ which vanish at the origin along with all of their derivatives are in $S_0(\mathbf{R}^n)$. For example

$$\varphi(x) = \begin{cases} \prod_{j=1}^{n} \exp(-x_j^2 - x_j^{-2}), & \text{when each } x_j \neq 0,\ \forall j = 1, 2, ..., n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\psi(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right)\exp(-|x|^2), & \text{when } |x| > 1, \\ 0, & \text{when } |x| \leqslant 1, \end{cases}$$

are members of $S_0(\mathbf{R}^n)$. The convergence of a sequence to zero in $S_0(\mathbf{R}^n)$ implies its convergence to zero in $D_{L^p}(\mathbf{R}^n)$. Therefore, the restriction of $f \in D'_{L^p}(\mathbf{R}^n)$ to $S_0(\mathbf{R}^n)$ is in $S'_0(\mathbf{R}^n)$. We express this fact by saying that $\mathscr{D}'_{L^p}(\mathbf{R}^n) \subset S'_0(\mathbf{R}^n)$. Elements of $D'_{L^p}(\mathbf{R}^n)$ cannot be identified with the elements of $S'_0(\mathbf{R}^n)$ in a one-to-one manner as $S_0(\mathbf{R}^n)$ is not dense in $\mathscr{D}_{L^p}(\mathbf{R}^n)$. Therefore

$$\mathscr{F}(Hf) = i^n \prod_{j=1}^{n} \operatorname{sgn}(x_j)\, \mathscr{F}f \qquad \text{on } S_0(\mathbf{R}^n),\ \forall f \in L^p(\mathbf{R}^n). \tag{6.6}$$

Because,

$$\begin{aligned} \langle \mathscr{F}(H\psi_m), \varphi \rangle &= \langle i^n \operatorname{sgn}(x)(\mathscr{F}\psi_m)(x), \varphi(x) \rangle \qquad \text{(from (6.4))} \\ &= \langle i^n \hat{\psi}_m(x), \operatorname{sgn}(x)\, \varphi(x) \rangle, \qquad \forall \varphi \in S_0(\mathbf{R}^n). \end{aligned} \tag{6.7}$$

Now taking the limit $m \to \infty$, we obtain

$$\begin{aligned} \langle \mathscr{F}(Hf), \varphi \rangle &= \langle i^n \hat{f}(x), \operatorname{sgn}(x)\, \varphi(x) \rangle \\ &= \langle i^n \operatorname{sgn}(x) \hat{f}, \varphi(x) \rangle, \qquad \forall \varphi(x) \in S_0(\mathbf{R}^n). \end{aligned}$$

DEFINITION 6.1. The Hilbert transform Hf of $f \in D'_{L^p}(\mathbf{R}^n)$ is defined by

$$\langle Hf, \varphi \rangle = \langle f, (-1)^n H\varphi \rangle, \qquad \forall \varphi \in D_{L^q}(\mathbf{R}^n),$$

where $(H\varphi)(x)$ is the Hilbert transform of $\varphi \in D_{L^q}(\mathbf{R}^n)$, given by (3.1).

DEFINITION 6.2. The Fourier transform $\mathscr{F}f(=\hat{f})$ of $f \in D'_{L^p}(\mathbf{R}^n)$ is defined by

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle, \qquad \forall \varphi \in S(\mathbf{R}^n). \tag{6.8}$$

Then we have the following

THEOREM 6.1. *Let $f \in D'_{L^p}(\mathbf{R}^n)$, $1 < p < \infty$. Then*

$$(\mathscr{F}(Hf))(x) = i^n \prod_{j=1}^{n} \operatorname{sgn}(x_j)\, \hat{f}(x) \qquad \textit{on } S_0(\mathbf{R}^n). \tag{6.9}$$

Proof. Let $f \in D'_{L^p}(\mathbf{R}^n)$. Then for every $\varphi \in S_0(\mathbf{R}^n)$, we have

$$\begin{aligned}
\langle \mathscr{F} Hf, \varphi \rangle &= \left\langle \mathscr{F} H \sum_{|\alpha| \leqslant m} \partial_t^\alpha f_\alpha, \varphi \right\rangle \qquad \text{(from (2.1))} \\
&= \left\langle \sum_{|\alpha| \leqslant m} \partial_t^\alpha Hf_\alpha, \hat{\varphi} \right\rangle \qquad [25] \\
&= \sum_{|\alpha| \leqslant m} \langle Hf_\alpha, (-1)^{|\alpha|} \partial_t^\alpha \hat{\varphi} \rangle \\
&= \sum_{|\alpha| \leqslant m} \langle \mathscr{F} Hf_\alpha, (-1)^{|\alpha|} (ix)^\alpha \varphi \rangle \qquad [32, \text{p. } 9] \\
&= \sum_{|\alpha| \leqslant m} \langle i^n \operatorname{sgn}(x) \hat{f}_\alpha(x), (-1)^{|\alpha|} (ix)^\alpha \varphi(x) \rangle \\
&= \sum_{|\alpha| \leqslant m} \langle i^n f_\alpha, (-1)^{|\alpha|} \partial_x^\alpha (\mathscr{F}(\operatorname{sgn}(t) \varphi(t)))(x) \rangle \\
&= \left\langle \sum_{|\alpha| \leqslant m} i^n \partial_x^\alpha f_\alpha(x), (\mathscr{F}(\operatorname{sgn}(t) \varphi(t)))(x) \right\rangle \\
&= \langle i^n \operatorname{sgn}(x) \hat{f}(x), \varphi(x) \rangle \qquad \text{(from (2.1)).}
\end{aligned}$$

Q.E.D.

Another proof of this theorem is given in [24]. In [25] the result

$$F(Hf)(\xi) = i^n \operatorname{sgn}(\xi)(Ff) \qquad \text{in } S_0'(\mathbf{R}^n)$$

is made use to prove the fact that a bounded linear operator T from $L^p(\mathbf{R}^n)$ into itself which commutes with the operators of translation as well as dilatation is a finite linear combination of the identity operator I and the Hilbert transform type operator $H_1, H_2, ..., H_n, H_i H_j, H_i H_j H_k, ..., H$.

For $f \in D'_{L^p}(\mathbf{R}^n)$, define a holomorphic function

$$F(z) = \frac{1}{(2\pi i)^n} \left\langle f(t), \frac{1}{\prod_{j=1}^n (t_j - z_j)} \right\rangle, \qquad y_j \neq 0, j = 1, 2, ..., n, \quad (6.10)$$

where y_j is the imaginary part of z_j. Then we have the following decomposition theorem.

THEOREM 6.2. *For* $f \in D'_{L^p}(\mathbf{R}^n)$, $1 < p < \infty$, *define* $F(z)$ *as in* (6.10). *Then*

$$f = \sum_{k=1}^{2^n} (-1)^{m_k} F_{\sigma_k} \qquad \text{in } D'_{L^p}(\mathbf{R}^n), \qquad (6.11)$$

and

$$(-1)^{m_k}\hat{F}_{\sigma_k}(\xi)=\begin{cases}\hat{f}(\xi), & \text{for } \xi\in S_{\sigma_k}\\ 0, & \text{elsewhere,}\end{cases} \tag{6.12}$$

on $S_0(\mathbf{R}^n)$, *where*

$$F_{\sigma_k}=\lim_{y_1\to 0_{\sigma_k(1)},\ldots,y_n\to 0_{\sigma_k(n)}} F(z). \tag{6.13}$$

Here m_k *stands for the number of negative signs in the sequence* σ_k.

Proof. Without loss of generality, we can take $n=2$. Then

$$F(z)=-\frac{1}{4\pi^2}\left\langle f(t),\frac{1}{(t_1-z_1)(t_2-z_2)}\right\rangle,\qquad y_1,y_2\neq 0.$$

Now

$$F_{++}(x)=\lim_{y_1,y_2\to 0^+}F(z)=-\tfrac{1}{4}((H_1+iI_1)(H_2+iI_2)f)(x)\qquad\text{[Cor. 4.2]}$$

$$F_{--}(x)=\lim_{y_1,y_2\to 0^-}F(z)=-\tfrac{1}{4}((H_1-iI_1)(H_2-iI_2)f)(x).$$

Similarly we have

$$F_{+-}(x)=-\tfrac{1}{4}((H_1+iI_1)(H_2-iI_2)f)(x)$$

and

$$F_{-+}(x)=-\tfrac{1}{4}((H_1-iI_1)(H_2+iI_2)f)(x),$$

so that

$$[F_{++}-F_{+-}-F_{-+}+F_{--}](x)=-\tfrac{1}{4}[-4I]f(x)=f(x).$$

Also

$$F_{++}(x)=-\tfrac{1}{4}[H+i(H_1I_2+H_2I_1)-I]f(x), \tag{6.14}$$

where $H=H_1H_2$ and $I=I_1I_2$. Taking the Fourier transform of Eq. (6.14), we get

$$\begin{aligned}\hat{F}_{++}(\xi)&=-\tfrac{1}{4}[i^2\operatorname{sgn}(\xi_1)\operatorname{sgn}(\xi_2)+i^2(\operatorname{sgn}\xi_1+\operatorname{sgn}\xi_2)-1]\hat{f}(\xi)\\&=\tfrac{1}{4}[\operatorname{sgn}(\xi_1)\operatorname{sgn}(\xi_2)+\operatorname{sgn}(\xi_1)+\operatorname{sgn}(\xi_2)+1]\hat{f}(\xi).\end{aligned}$$

Case 1. $\xi_1, \xi_2 > 0$:

$$\hat{F}_{++}(\xi) = \tfrac{1}{4}[1+1+1+1]\hat{f}(\xi) = \hat{f}(\xi).$$

Case 2. $\xi_1, \xi_2 < 0$:

$$\hat{F}_{++}(\xi) = \tfrac{1}{4}[1-1-1-1]\hat{f}(\xi) = 0.$$

Case 3. $\xi_1 > 0$, $\xi_2 < 0$:

$$\hat{F}_{++}(\xi) = \tfrac{1}{4}[-1+1-1+1]\hat{f}(\xi) = 0.$$

Case 4. $\xi_1 < 0$, $\xi_2 > 0$:

$$\hat{F}_{++}(\xi) = \tfrac{1}{4}[-1-1+1+1]\hat{f}(\xi) = 0.$$

Hence

$$\hat{F}_{++} = \begin{cases} \hat{f}, & \text{for} \quad \xi \in S_{++} = \{\xi \in \mathbf{R}^2 \mid \xi_1, \xi_2 > 0\}, \\ 0, & \text{elsewhere.} \end{cases}$$

Thus we have proved the theorem for $n = 2$. Using induction the proof can be given for any $n > 1$. Q.E.D.

Note that

$$\mathscr{F}(F_{++} + F_{--})(\xi) = \begin{cases} \hat{f}(\xi), & \text{for} \quad \xi \in S_{++} \cup S_{--}, \\ 0, & \text{elsewhere.} \end{cases}$$

Similarly

$$(\hat{F}_{++} - \hat{F}_{+-} + \hat{F}_{--})(\xi) = \begin{cases} \hat{f}(\xi), & \text{for} \quad \xi \in S_{++} \cup S_{+-} \cup S_{--}, \\ 0, & \text{elsewhere.} \end{cases}$$

Similar results hold for all the other possible combinations of $\hat{F}_{\sigma_k}$. Our Theorem 6.2 is analogous to the result proved by Tillmann [33, p. 19] for the space $H'(\bar{\mathbf{R}}^n)$. However, our technique is operator theoretic, i.e., it is based upon properties of the complex Hilbert transform and its limit whereas the techniques used by Tillmann are essentially an outcome of complex integration in $\mathbf{C}^n$ on appropriately chosen Jordan arcs. The space $H(\bar{\mathbf{R}}^n)$ chosen by Tillman [33] is a subspace of $D_{L^p}(\mathbf{R}^n)$ and the convergence of a sequence in $H(\bar{\mathbf{R}}^n)$ to zero necessarily implies its convergence to zero in the space $D_{L^p}(\mathbf{R}^n)$ and as such the restriction of any $t \in (D_{L^p}\mathbf{R}^n))'$ to $H(\bar{\mathbf{R}}^n)$ is in $H'(\hat{\mathbf{R}}^n)$, i.e., [33, p. 19] $(D_{L^p}(\mathbf{R}^n))' \subset H'(\mathbf{R}^n)$. However, the advantage of our space $(D_{L^p}(\mathbf{R}^n))'$ is that it is a Fourier as well as Hilbert transformable space so that using Theorem 6.1 of this paper, we are able to prove a Paley–Wiener type Theorem 6.3. Some special cases of our representation formulas are also proved by Vladimirov [35, Chap. 5].

Analyzing in the same manner yields the following results for $f \in D'_{L^p}(\mathbf{R}^n)$.

LEMMA 6.1. *For $f \in D'_{L^p}(\mathbf{R}^n)$, $1 < p < \infty$, and $F(z)$ defined as in (6.10), we have*

$$\sum_{k=1}^{l} (-1)^{m_k} \hat{F}_{\sigma_k}(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } \xi \in \bigcup_{k=1}^{l} S_{\sigma_k} \\ 0, & \text{elsewhere,} \end{cases} \tag{6.15}$$

for $1 \leqslant l \leqslant 2^n$, equality in the sense of $S'_0(\mathbf{R}^n)$.

Suppose one of the summands, say, $F_{\sigma_{k_0}}(\xi)$, for some $1 \leqslant k_0 \leqslant 2^n$, is zero $\forall \xi \in S_{\sigma_{k_0}}$. Then, since $\hat{f} = \sum_{k=1}^{2^n} (-1)^{m_k} \hat{F}_{\sigma_k}$, Eq. (6.12) implies that $\hat{f}(\xi) = 0$ $\forall \xi \in S_{\sigma_{k_0}}$. Conversely, suppose $\hat{f} = 0$ $\forall \xi \in S_{\sigma_{k_0}}$. Then again Eq. (6.12) gives us $\hat{F}_{\sigma_{k_0}} = 0$, i.e.,

$$\langle \hat{F}_{\sigma_{k_0}}, \varphi \rangle = 0, \qquad \forall \varphi \in S_0(\mathbf{R}^n).$$

So that

$$\langle F_{\sigma_{k_0}}, \hat{\varphi} \rangle = 0, \qquad \forall \varphi \in S_0(\mathbf{R}^n).$$

We can generalize the above argument to obtain the following

THEOREM 6.3 (Paley–Wiener Theorem for $D'_{L^p}(\mathbf{R}^n)$). *Let $f \in D'_{L^p}(\mathbf{R}^n)$, $1 < p < \infty$. Define $F(z)$ by*

$$F(z) = \frac{1}{(2\pi i)^n} \left\langle f(t), \frac{1}{\prod_{j=1}^{n} (t_j - z_j)} \right\rangle, \qquad I_m z_j \neq 0 \ (j = 1, 2, ..., n).$$

Then we have

$$\hat{f}(\xi) = 0 \qquad \text{for} \quad \xi \in \bigcup_{k=1}^{l} S_{\sigma_k} \text{ in } S'_0(\mathbf{R}^n)$$

iff

$$\sum_{k=1}^{l} (-1)^{m_k} F_{\sigma_k} = 0 \qquad \forall \xi \in \bigcup_{k=1}^{l} S_{\sigma_k} \text{ in } \mathscr{F}(S_0(\mathbf{R}^n))',$$

i.e.,

$$\left\langle \sum_{k=1}^{l} (-1)^{m_k} F_{\sigma_k}, \hat{\varphi} \right\rangle = 0,$$

$$\forall \varphi \in S_0(\mathbf{R}^n) \textit{ with support contained in } \bigcup_{k=1}^{l} S_{\sigma_k}.$$

Remark. Lemma 6.1, Theorems 6.2 and 6.3 are also true when we replace $D'_{L^p}(\mathbf{R}^n)$ by $L^p(\mathbf{R}^n)$ and $F(z)$ by

$$\frac{1}{(2\pi i)^n} \int_{\mathbf{R}^n} f(t) \frac{1}{\prod_{j=1}^n (t_j - z_j)} dt, \qquad I_m z_j = 0 \ (j = 1, 2, ..., n),$$

and treating $L^p(\mathbf{R}^n)$ as a subspace of $D'_{L^p}(\mathbf{R}^n)$.

7. The Dirichlet Boundary Value Problem

Let $F(z) \in M$ (defined by (5.6)). Then, by (5.7), we have

$$\Delta F(z) = \Delta \left(\frac{1}{2\pi i}\right)^n \left\langle \sum_{k=1}^{2^n} (-1)^{m_k} F_{\sigma_k}(t), \frac{1}{\prod_{j=1}^n (t_j - z_j)} \right\rangle, \tag{7.1}$$

where

$$\Delta = \prod_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right). \tag{7.2}$$

Using a method similar to that used in proving Lemma 2.1, we see that

$$\Delta F(z) = 0.$$

So we have proved

THEOREM 7.1. $\Delta F(z) = 0$, $\forall F(z) \in M$.

Consider the operator equation

$$\Delta u = 0, \tag{7.3}$$

with the following boundary conditions

$$\lim_{y \to 0_{\sigma_k}} u = F_{\sigma_k} \qquad \text{in } D'_{L^p}(\mathbf{R}^n), \ 1 \leqslant k \leqslant 2^n. \tag{7.4}$$

Then

$$F(z)=\frac{1}{(2\pi i)^n}\left\langle \sum_{k=1}^{2^n} (-1)^{m_k} F_{\sigma_k}(t), \frac{1}{\prod_{j=1}^{n}(t_j-z_j)}\right\rangle,$$

$$I_m z_j \neq 0,\ j=1,2,...,n, \qquad (7.5)$$

is in M (from Theorem 5.1) and it is also a solution of (7.3) with (7.4) as the boundary condition. The fact that $F(z)$ given by (7.5) is a unique solution in M of (7.3) and (7.4) follows from the representation formula (7.5).

REFERENCES

1. K. ANDERSEN, Weighted norm inequalities for Hilbert transforms and conjugate functions of even and odd functions, *Proc. Amer. Math. Soc.* **56** (1976), 99–107.
2. HANS BREMMERMAN, Distributions, complex variables and Fourier transforms, Addison–Wesley, Reading, MA, 1965.
3. S. Y. A. CHENG AND R. FEFFERMAN, Some recent developments in Fourier analysis on product domains, *Bull. Amer. Math. Soc.* (*N.S.*) **12** (1985), 1–44.
4. R. R. COIFMAN AND C. FEFFERMAN, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241–250.
5. M. COTLAR, Some generalizations of Hardy–Littlewood maximal theorem, *Rev. Math., Cuyana* **1** (1955), 85–104.
6. L. EHRENPREIS, Analytic functions and the Fourier transform of distributions, I, *Ann. of Math.* (2) **63** (1956), 129–159.
7. E. B. FABES AND N. M. RIVIERE, Singular integrals with mixed homogeneity, *Studia Math.* **27** (1966), 19–38.
8. C. FEFFERMAN, Estimates for double Hilbert transform, *Studia Math.* **44** (1972), 1–15.
9. C. FEFFERMAN, Recent progress on classical Fourier analysis, *in* "Internat. Congress Math., Vancouver, 1974," pp. 95–118.
10. AVNER FREIDMAN, "Generalized Functions and Partial Differential Equations," Prentice–Hall, Englewood Cliffs, NJ, 1963.
11. I. M. GEL'FAND AND G. E. SHILOV, "Generalized Functions," Vol. II, Academic Press, San Diego, CA, 1968.
12. E. HEWIT AND K. STROMBERG, "Real and Abstract Analysis," Springer-Verlag, New York/Berlin, 1969.
13. R. HUNT, B. MUCKENHAUPT, AND R. WHEEDEN, Weighted norm inequalities for the conjugate functions and Hilbert transform, *Trans. Amer. Math. Soc.* **176** (1973), 227–251.
14. Y. KATZNELSON, "An Introduction to Harmonic Analysis," Wiley, New York, 1968.
15. V. M. KOKILASHVILI, Singular integral operators in weighted spaces, *Colloq. Math. Soc. János Bolyai* **35**, Functions, Series, Operators (1980).
16. H. A. LAUWERIER, The Hilbert problem for generalized functions, *Arch. Rational Mech. Anal.* **13** (1963), 157–166.
17. D. MITROVIC, A distributional representation of analytic functions, *Math. Balkanica* (*N.S.*) **79** (1974), 437–440.

18. B. MUCKENHOUPT, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207–226.
19. N. I. MUCKHELISHVILI, "Singular Integral Equations," Moscow, 1946.
20. J. BARROS-NETO, "An Introduction to the Theory of Distributions," Dekkar, New York, 1973.
21. M. ORTON, Hilbert transforms, Plemelj relations and Fourier transform of distributions, *SIAM J. Math. Anal.* **4** (1973), 656–667.
22. J. N. PANDEY AND M. A. CHAUDHRY, The Hilbert transform of generalized functions and applications, *Canad. J. Math.* **35**, No. 3 (1983), 478–495.
23. J. N. PANDEY, The Hilbert transform of Schwartz distributions, *Proc. Amer. Math. Soc.* **89**, No. 1 (1983), 86–90.
24. J. N. PANDEY, Action of the Fourier transform on the Hilbert transform and vice-versa, submitted for publication.
25. J. N. PANDEY AND O. P. SINGH, On the p-norm of the truncated n-dimensional Hilbert transform, *Bull. Austral. Math. Soc.*, in press.
26. LAURENT SCHWARTZ, "Theorie des distributions," Herman, Paris, 1966.
27. O. P. SINGH AND J. N. PANDEY, The n-dimensional Hilbert transform of distributions, its inversion and applications, *Canad. J. Math.*, in press.
28. K. SOKEL-SOHOLOWSKI, On trignometric series conjugate to Fourier series in two variables, *Fund. Math.* **33** (1945), 166–182.
29. E. STEIN AND G. WEISS, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1971.
30. E. C. TITCHMARSH, "Introduction to the Theory of Fourier Integrals," Oxford Univ. Press, London/New York, 1967.
31. E. C. TITCHMARSH, "The Theory of Functions," 2nd ed., Oxford Univ. Press, London/New York, 1964.
32. A. TERRAS, "Harmonic Analysis on Symmetric Spaces and Applications, I," Springer-Verlag, New York/Berlin, 1985.
33. HEINZ GUNTHER TILLMAN, Distributionen als Randverteilungen analytischer Functionen, II, *Math. Z.* **76** *(1961), 5–21.*
34. F. TREVES, "Topological Vector Spaces Distributions and Kernels," Academic Press, New York/London, 1967.
35. V. S. VLADIMIROV, "Methods of the Theory of Functions of Many Complex Variables," The MIT Press, Cambridge, MA, 1966.
36. J. H. WILLIAMS, "Lebesgue Integration," Holt, Rinehart & Winston, New York, 1962.
37. A. H. ZEMANIAN, "Distribution Theory and Transform Analysis," McGraw–Hill, New York, 1965.
38. A. H. ZEMANIAN, "Generalized Integral Transforms," Interscience, New York, 1968.
39. A. ZIGMUND, On the boundary values of functions of several complex variables, *Fund. Math.* **36** (1949), 201–235.