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Multiobjective Programming with Semilocally Convex Functions

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1. INTRODUCTION

The present work has been motivated by the paper of Weir [9] in which the author has dealt with semilocally convex functions and described their role in mathematical programming. Multiobjective programming, in recent times, has become an important area of investigation. We try to show naturally some of the uses of the class of semilocally convex functions in dealing with optimality conditions and duality theory as applied for treating a multiobjective mathematical program. In the next paragraph we give a brief description regarding the cited class of functions as dealt with in earlier references.

The concept of semilocal convexity was introduced by Ewing [3] who applied it to derive sufficient optimality conditions for variational and control problems. Semilocally convex functions have certain important convex type properties, e.g., local minima of semilocally convex functions defined on locally star-shaped sets are also global minima, and nonnegative linear combinations of semilocally convex functions are also semilocally convex. Kaur [6] and Kaul and Kaur [5] have made further generalization of semilocally convex functions.

The paper is organized as follows. In Section 2 we introduce certain notations and mention the key lemmas which were relevant for single objective programming problems from Ref. [9]. In the same section some lemmas have been given as statements in a different format. Finally, we state the alternative theorem derived by Weir which is an important proposition for deducing necessary optimality conditions. In Section 3 we introduce a multiobjective programming problem and we derive necessary

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saddle-point criteria for optimality for a semilocally convex program. Section 4 deals with Kuhn–Tucker necessary optimality conditions, sufficient optimality conditions, and duality for a semilocally convex multiobjective program. Some applications have been given in Section 5 for a multiobjective program with fractional objectives.

2. NOTATION AND PRELIMINARIES

This section contains a comprehensive account of materials from Ref. [9] which we give for their usefulness for the treatment which follows in the subsequent sections.

E^n will denote n-dimensional Euclidean space with norm $|\cdot|$, $E_+ = [0, \infty)$. A set $X \subset E^n$ is a convex cone if $X + X \subset X$ and $\alpha X \subset X$ for $\alpha \in E_+$. E_+^n will denote the nonnegative orthant of E^n. Cl(X) and int(X) denote the closure of X and the interior of X, respectively.

A subset C of E^n is called locally star-shaped at $x_0 \in C$, if corresponding to $x_0 \in C$ and each $x \in C$ there exists a positive number $a(x_0, x) \leq 1$ such that $wX + (1 - w)x_0 \in C$ for $0 < w < a(x_0, x)$. The set C is said to be locally star-shaped if it is star shaped at each of its points.

Let C be a locally star-shaped set in E^n. A scalar-valued function f: $C \to R$ is said to be semilocally convex on C if for each $x, y \in C$ there exist a positive number $d(x, y) \leq a(x, y)$ such that

$$f(wx + (1 - w)y) \leq wf(x) + (1 - w)f(y), \qquad 0 < w < d(x, y).$$

There is an obvious extension for an E^m-valued function on a set (locally star-shaped set) of E^n to be S-semi-locally convex, where S is a convex cone in E^n. In particular, if $S = E_+^m$ then f: $C \to E^m$ is said to be semilocally convex on C if each component of f is semilocally convex on C.

A vector-valued function f: $T \to Y$, where T and Y are subsets of E^n and E^m, respectively, is (one-sided) directionally differentiable at the point $x_0 \in T$ in the direction $x - x_0$ if the following limit exists:

$$f^+(x_0; x - x_0) = \lim_{w \to 0^+} w^{-1}\left[f(x_0 + w(x - x_0)) - f(x_0)\right].$$

When $Y = E$, this reduces to the usual definition of directional differentiability. The necessary condition for a directionally differentiable function to be semilocally convex on a locally star-shaped set T can be derived as follows. Since such an f satisfies

$$w\left[f(x) - f(x_0)\right] \geq f(x_0 + w(x - x_0)) - f(x_0)$$

and since f is directionally differentiable, dividing the above inequality by w and taking the limit gives

$$f(x) - f(x_0) \geq f^+(x_0; x - x_0).$$

For a convex cone $S \subset E^m$, if g: $T \to E^m$ is semilocally convex on the locally star-shaped set T and g is directionally differentiable then

$$g(x) - g(x_0) - g^+(x_0; x - x_0) \in S.$$

The following properties of a locally star-shaped set are, in order:

(1) If C is a closed locally star-shaped set in E^n, then C is convex.

(2) If C is a locally star-shaped set in E^n; then $\mathrm{cl}(C)$ is convex.

The following is an important separation theorem which is fundamental in establishing the theorem of the alternatives.

LEMMA 2.1. *Let S be a locally star-shaped set in E^n and let T be a convex set in E^n with a nonempty interior. If S and T are disjoint then there exists a nonzero continuous linear functional P defined on E^n and a scalar β such that*

$$\sup\{P[x]: x \in T\} \leq \beta \leq \inf\{P[x]: x \in S\}.$$

The following lemma depicts the locally star-shaped property of the range set of a function with respect to a convex cone S in E^m.

LEMMA 2.2. *Let S be a convex cone in E^m, let T be a locally star-shaped set in E^n, and let f: $T \to E^m$ be S-semilocally convex. Then the set $Z = f(T) + S$ is locally star-shaped.*

An alternative theorem in the context of semilocally convex functions on a locally star-shaped set can be stated as follows:

THEOREM 2.3. *Let S be a convex cone with nonempty interior in E^m; let T be a locally star-shaped set in E^n and let F: $T \to E^m$ be S-semilocally convex. Then exactly one of the following two systems has a solution*:

(i) $-f(x) \in \mathrm{int}\, S$, $x \in T$,

(ii) $(p \circ f)(T) \subset R_+$, $0 \neq P \in S^*$.

In fact for a multiobjective program (P) (a typical example as given above) using the efficiency set $\xi(Y: D)$ in [8] Sawaragi, Nakayama, and Tanino introduced efficient solutions in the following way.

DEFINITION. A point $\hat{x} \in X$ is said to be an efficient solution to the multiobjective program (P) with respect to the domination structure D if

$f(\hat{x}) \in \xi(Y, D)$; i.e., if there is no $x \in X$ such that $f(\hat{x}) \in f(x) + D(f(x))$ and $f(x) \neq f(\hat{x})$ (i.e., such that $f(\hat{x}) \in f(x) + D(f(x)) \setminus \{0\}$.

When $D = E^P_+$ a subclass of the set of efficient solutions of a multiobjective program (P) is known as the property-efficient solutions. The concept of this particular definition is due to Geoffrion (see [8]).

DEFINITION. A point $\hat{x}$ is said to be a properly efficient solution of (D) if it is efficient and if there is some real $M > 0$ such that for each i and each $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$, there exists at least one j such that $f_j(\hat{x}) < f_j(x)$ and

$$\left(f_i(\hat{x}) - f_i(x)\right) / \left(f_j(x) - f_j(\hat{x})\right) \leq M.$$

In what follows we use the same definition as above for properly coefficient solutions.

3. MULTIOBJECTIVE OPTIMIZATION, SADDLE-POINT NECESSARY CONDITIONS, AND DUALITY

We consider a nonlinear multiobjective optimization problem formulated as

$$\text{(PVP)} \qquad S\text{-minimize}\{ f(x) \colon x \in X \},$$

where

$$X = \left\{ x \in X' \colon g(x) \leq_Q 0 \colon X' \subset E^n \right\}.$$

We make the following assumptions:

(i) X' is a nonempty compact locally star-shaped set in E^n.

(ii) S and Q are pointed closed convex cones with nonempty interiors of E^p and E^m, respectively.

(iii) $f \colon E^n \to E^p$ is continuous and S-semilocally convex.

(iv) $g \colon E^n \to E^m$ is continuous and D-semilocally convex.

There are two types of optimum solution (see [8]) for (P) known as an efficient solution and a properly efficient solution. Under the assumptions (i)–(iv) it can be readily seen that for every $z \in E^m$, both sets

$$X(z) = \left\{ x \in X' \colon g(x) \leq_Q z \right\}$$

and

$$Y(z) = f[X(z)] = \{Y \in R^p: y = f(x), x \in X', g(x) \leq_Q z\} \quad (3.1)$$

are compact, $X(z)$ is locally star-shaped, and $Y(z)$ is S-semilocally convex (the proofs are similar to those for Lemma 4.1 of [9]).

Further, we consider the primal problem (P) by embedding it in a family of perturbed problems with $Y(z)$ given by (3.1):

(P_2) S-minimize $Y(z)$.

Clearly primal problem (P) is identical to problem (P_z) with $z = 0$. Now define

$$\Gamma = \{z \in R^m: X(z) \neq \varnothing\}.$$

It can be readily seen that Γ is locally star-shaped.

DEFINITION 3.1 (Perturbation, or the primal map). The point to set map W: $\Gamma \to E^p$ defined by

$$W(z) = \min_S Y(z)$$

is called a perturbation (or primal) map.

DEFINITION 3.2. W is a S-semilocally convex point to set map on Γ.

Proof. We need to show that corresponding to each z^1, $z^2 \in \Gamma$, there exists a positive number $d(z, z_2) \leq a(z, z_2)$ such that

$$wY(z_1) + (1 - w)Y(z_2) \subset Y(wz_1 + (1 - w)z_2) + S,$$

$0 < w < d(z_1, z_2)$. Here $a(z_1, z_2)$ is attached to the local star-shapedness condition on the set Γ. If we suppose that

$$y \in wY(z_1) + (1 - w)Y(z_2),$$

then there exists $x_1, x_2 \in X'$ such that $g(x_1) \leq_Q z_1$ and $g(x_2) \leq_Q z_2$ and $y = wf(x_1) + (1 - w)f(x_2)$. Since X' is locally star-shaped $wx^1 + (1 - w)x^2 \in X'$, for $0 < w < a(z_1, z_2)$. Furthermore, from the C-semilocally convex property of g

$$\begin{aligned} g(wx' + (1 - w)x^2) &\leq_Q wg(x') + (1 - w)g(x^2) \\ &\leq_Q wz_1 + (1 - w)z_2, \end{aligned}$$

where $0 < w < d(z_1, z_2)$ which implies

$$wx' + (1-w)x^2 \in X\big(wz_1 + (1-w)z_2\big)$$

and, thus,

$$f\big(wx' + (1-w)x^2\big) \in Y\big(wz_1 + (1-w)z_2\big),$$

$0 < w < d(z_1, z_2)$. On the other hand, from the S-semilocally convex property of f,

$$wf(x') + (1-w)f(x^2) \in f\big(wx^1 + (1-w)x^2\big) + S,$$

which implies $y \in Y(wz' + (1-w)z^2) + S'$ for $0 < w < d(z_1, z_2)$, which completes the proof.

DEFINITION 3.3 (Vector-valued Lagrangian function). A vector-valued Lagrangian function for problem (P) is defined on $X' \times \varepsilon$ by

$$L(x, \Lambda) = f(x) + \Lambda g(x),$$

where ε is a family of $p \times m$ matrices Λ such that $\Lambda_Q \subset s$. It is readily seen that for given $u \in S^0 \setminus \{0\}$ and $\lambda \in Q^0$, there exist $\Lambda \in \varepsilon$ such that $\Lambda^T u = \lambda$.

DEFINITION 3.4 (Saddle point for vector-valued Lagrangian functions). A point $(\hat{x}, \hat{\Lambda}) \in X' \times \varepsilon$ is said to be a saddle point for the vector-valued Lagrangian $L(x, \Lambda)$ if the following holds:

$$L(\hat{x}, \hat{\Lambda}) \in \operatorname*{Min}_S \big\{L(x, \hat{\Lambda}) \colon x \in X'\big\} \cap \operatorname*{Max}_S \{L(\hat{x}, \Lambda) \colon \Lambda \in \varepsilon\}.$$

THEOREM 3.5 [8]. *The following three conditions are necessary and sufficient for a pair $(\hat{x}, \hat{\Lambda}) \in X' \times \varepsilon$ to be a saddle point for the vector-valued Lagrangian function (x, Λ):*

(i) $L(\hat{x}, \hat{\Lambda}) \in \operatorname{Min}_S\{L(x, \hat{\Lambda}) \colon x \in X'\}$

(ii) $g(\hat{x}) \leq_Q 0$

(iii) $\hat{\Lambda} g(\hat{x}) = 0$.

Proof. Necessity. Condition (i) is the same as part of the definition of the saddle points for $L(x, \Lambda)$. To prove (ii) note that $L(\hat{x}, \hat{\Lambda}) \in \operatorname{Max}_S\{f(\hat{x}) + \Lambda g(\hat{x}) \colon \Lambda \in \varepsilon\}$ implies that

$$f(\hat{x}) + \hat{\Lambda} g(\hat{x}) \nleq_S f(\hat{x}) + \Lambda g(\hat{x}) \qquad \text{for } \Lambda \in \varepsilon. \tag{3.2}$$

For some $\hat{\mu} \in S^0 \setminus \{0\}$ and for any $\Lambda \in S$, suppose that $g(\hat{x}) \nleq_Q 0$. Then there is $\hat{\lambda} \in Q^0$ such that $\langle \hat{\lambda}, g(\hat{x}) \rangle > 0$. Making $\|\hat{\lambda}\|$ sufficiently large and taking $\Lambda \in \varepsilon$ such that $\hat{\mu}^T \Lambda = \hat{\lambda}^T$, we obtain the relation

$$\langle \hat{\mu}, \Lambda g(\hat{x}) \rangle - \langle \hat{\mu}, \hat{\Lambda} g(\hat{x}) \rangle > 0,$$

which contradicts (3.3). Thus $g(\hat{x}) \leq_Q 0$. Using this, $\hat{\Lambda} g(\hat{x}) \leq 0$ for $\hat{\Lambda} \in s$. On the other hand, substituting $\Lambda = 0$ in (3.2) yields $\hat{\Lambda} g(\hat{x}) \nleq_S 0$. Hence, $\hat{\Lambda} g(\hat{x}) = 0$.

Sufficiency. Since $\Lambda g(\hat{x}) \in -S$ for any $\Lambda \in \varepsilon$ as long as $g(\hat{x}) \leq_Q 0$, it follows that

$$\operatorname*{Max}_S \{ \Lambda g(\hat{x}) \colon \Lambda \in \varepsilon \} = \{0\}.$$

Thus from $\hat{\Lambda} g(\hat{x}) = 0$, we have

$$L(\hat{x}, \hat{\Lambda}) \in \operatorname*{Max}_S \{ f(\hat{x}) + \Lambda g(\hat{x}) \colon \Lambda \in \varepsilon \}.$$

This result and condition (i) imply that the pair $(\hat{x}, \hat{\Lambda})$ is a saddle point of $L(x, \Lambda)$. ∎

THEOREM 3.5. *If $\hat{x}$ is a properly efficient solution to problem* (P), *and if Slater's constraint qualification holds* (*i.e., there exists $x \in X'$ such that $g(x') <_Q 0$*), *then there exists a $p \times m$ matrix $\hat{\Lambda}$ such that $\hat{\Lambda}_Q \subset S$ and*

$$f(\hat{x}) \in \operatorname*{Min}_S \left\{ f(x) + \hat{\Lambda} g(x) \colon x \in X' \right\}$$

$$\hat{\Lambda} g(\hat{x}) = 0.$$

Proof. Let $X = \{x \in R^n \colon g(x) \leq_Q 0\} \cap X'$. Since $\hat{x}$ is a properly efficient solution of $f(x)$ with respect to $\leq_S$, there exists $\hat{\mu} \in \operatorname{int} S^0$ such that

$$\langle \hat{\mu}, f(\hat{x}) \rangle \leq \langle \hat{\mu}, f(x) \rangle \qquad \text{for any } x \in X.$$

It is readily seen that $\langle \hat{\mu}, f(x) \rangle$ is semilocally convex in X'. Therefore from the Lagrangian multiplier theorem (Theorem 4.3 in [8]) in a scalar semilocally convex program, there is a vector $\hat{\Lambda} \in Q^0$ such that

$$\langle \hat{\mu}, f(\hat{x}) \rangle + \langle \hat{\lambda}, g(\hat{x}) \rangle \leq \langle \hat{\mu}, f(x) \rangle + \langle \hat{\lambda}, g(x) \rangle, \tag{3.3}$$

for any $x \in X'$ and $\langle \hat{\lambda}, g(\hat{x}) \rangle = 0$.

Now for such $\hat{\mu}$ and $\hat{\lambda}$, take a $p \times m$ matrix $\hat{\Lambda}$ with $\hat{\Lambda}^T \hat{\mu} = \hat{\lambda}$ in such a way that

$$\hat{\Lambda} = \left\{ \hat{\lambda}_1 e, \hat{\lambda}_2 e, \dots, \hat{\lambda}_m e \right\},$$

where e is a vector of S with $\langle \hat{\mu}, e \rangle = 1$. Then clearly $\hat{\Lambda}_Q \subset S$ and $\hat{\Lambda} g(\hat{x}) = 0$. If we suppose that for this $\hat{\Lambda}$,

$$f(\hat{x}) \notin \operatorname*{Min}_S \{ f(x) + \hat{\Lambda} g(x) \colon x \in X' \},$$

there exists $\bar{x} \in X'$ such that

$$f(\hat{x}) - f(\bar{x}) - \hat{\Lambda} g(\bar{x}) \in S \setminus \{0\}.$$

Hence,

$$\begin{aligned} \langle \hat{\mu}, f(\hat{x}) \rangle &> \langle \hat{\mu}, f(\bar{x}) \rangle + \langle \hat{\mu}, \lambda g(\hat{x}) \rangle \\ &= \langle \hat{\mu}, f(\bar{x}) \rangle + \langle \hat{\lambda}, g(\bar{x}) \rangle, \end{aligned}$$

which contradicts (3.3), and the proof is complete. ∎

COROLLARY 3.7. *Suppose $\hat{x}$ is a properly efficient solution to problem* (P) *and that Slater's constraint qualification is satisfied. Then there exists a $p \times m$ matrix $\hat{\Lambda} \in \varepsilon$ such that $(\hat{x}, \hat{\Lambda})$ is a saddle point for the vector-valued Lagrangian function $L(x, \Lambda)$.*

Proof. It immediately follows from Theorem 3.5 and, conversely, one can derive the following sufficient condition for optimality, without any convexity assumptions on the function defining a multiobjective program in terms of saddle points of the Lagrangian function $L(x, \Lambda)$.

THEOREM 3.9. *If $(\hat{x}, \hat{\Lambda}) \in X' \times \varepsilon$ is a saddle point for the vector-valued Lagrangian function $L(x, \Lambda)$, then $\hat{x}$ is an efficient solution to problems* (P).

The proof is easy.

4. DUAL MAP AND DUALITY THEORY

Regarding the aspects concerned with a duality map and the associated duality theory connected with a multiobjective program involving semilocally convex functions, the treatment follows a course analogous to that of the convex programs (see [8]). Here we state the results. Proofs are omitted because they are similar to those in Ref. [8].

DEFINITION 4.1 (Dual map). Define for any $\Lambda \in \varepsilon$,

$$\Omega(\Lambda) = \{ L(x, \Lambda) \colon x \in X' \} = \{ f(x) + \Lambda g(x) \colon x \in X' \}$$

and

$$\phi(\Lambda) = \operatorname*{Min}_{S} \Omega(\Lambda).$$

The point-to-set map ϕ: $\varepsilon \to E^p$ is called a dual map.

We can now define a dual problem associated with the primal problem (P) as follows:

$$\text{(D)} \qquad S\text{-Maximize} \bigcup_{\Lambda \in z} \phi(\Lambda).$$

THEOREM 4.2 (Weak duality). *For any $x \in X$ and $y \in (\Lambda)$*

$$Y \not\geq_S f(x).$$

THEOREM 4.3. (i) *Suppose that $\hat{x} \in X$ and $\hat{\Lambda} \in \varepsilon$ and $f(\hat{x}) \in \phi(\hat{\Lambda})$. Then $\hat{y} = f(\hat{x})$ is an efficient point to the primal problem* (P) *and also to the dual problem* (D).

(ii) *Suppose that x is a properly efficient solution to problem* (P) *and that Slater's constraint qualification is satisfied. Then*

$$f(\hat{x}) \in \operatorname*{Max}_{S} \bigcup_{\Lambda \in z} \phi(\Lambda).$$

5. OPTIMALITY CONDITIONS AND DUALITY

Scalarization of multiobjective programs and the Alternative theorem 2.3 of Section 2 play a crucial role in deriving the Fritz John [4] and Kuhn−Tucker [7] necessary condition of optimality for a semilocally convex case of the program. In the treatment which follows, we derive such a necessary condition for optimality. Subsequently we also derive sufficient optimality conditions. Further, defining a Wolf [10] type of dual for such programs we give duality results in the same context:

For simplifying the primal formulation of Section 3 we set $S = E_+^p$, the positive orthant of the Euclidean space E^p. The set X can be written as

$$X = \{x \in X' \colon -g(x) \in Q, X' \subset E^n\},$$

where Q is a pointed closed convex cone in E^m.

THEOREM 5.1. *Assume that x_0 is an efficient solution of primal problem* (P). *Then there exists $\tau \in E_+^p$ and $y \in Q$ both nonzero such that*

$$\left(\tau^t f + y^t g\right)^+ (x_0; x - x_0) \geq 0 \text{ for all } x \in X, \tag{5.1}$$

$$y^t g(x_0) = 0. \tag{5.2}$$

Proof. Since $-g(x) \in Q$ implies $f_i(x_0) - f_i(x^0) \leq 0$ for $i = 1, 2, \dots, p$ are for some $i = i_0$, $f_i(x_0) < f_i(x)$, $\forall x \in X$, then there is no solution to the system

$$-\left(f(x) - f(x_0), g(x)\right) \in \operatorname{int}\left(E_+^p \times Q\right).$$

By Theorem 2.3, there exists $\tau \in E_+^p$ and $y \in Q^*$ not both zero, such that for all $x \in X$,

$$\tau^t f(x) + y^t g(x) \geq \tau^t f(x_0).$$

Since $-g(x_0) \in Q$, $y^t g(x_0) = 0$. Therefore, for all $x \in X$

$$\tau^t f(x) + y^t g(x) - \left[\tau^t f(x_0) + y^t g(x_0)\right] \geq 0. \tag{5.3}$$

Now, since X is locally star-shaped, $x_0 + \lambda(x - x_0) \in X$ for $0 < \lambda < a(x, x_0)$ and any $x \in X$; this from (5.3) implies

$$\left(\tau^t f + y^t g\right)^+ (x_0, x - x_0) \geq 0 \qquad \forall x \in X. \tag{5.4}$$

The Kuhn−Tucker necessary conditions can be given as follows.

THEOREM 5.2. *Suppose that x_0 is an efficient solution of the primal problem* (P) *and the generalized Slater's constraint qualification is satisfied then there exists $y \in Q$ such that*

$$\left(e^t f + y^t g\right)^+ (x_0; x - x_0) \geq 0 \qquad \forall x \in X, \tag{5.5}$$

$$y^t g(x_0) = 0, \tag{5.6}$$

where $e = (1, 1, \dots, 1) \in E_+^p$.

Proof. Assume that x_0 is an efficient solution of primal problem (P). Then Fritz John conditions (5.1) and (5.2) are satisfied for $x = x_0$ for some $\tau \in E_+^p$ and $y \in Q^*$ not both zero. If $\tau = 0$, then $y \neq 0$ and $(y^t g)^+(x_0; x - x_0) \geq 0$ for all $x \in X$ and $y^t g(x_0) = 0$. Since g is semilocally convex, it follows that $y^t g(x) \geq y^t g(x_0) = 0$, $\forall x \in X$, which contradicts the generalized Slater's condition; hence $\tau \neq 0$ and we can assume that $\tau = e$, where $e = (1, 1, \dots, 1) \in E_+^p$. The following lemma scalarizes the primal problem (P).

LEMMA 5.3 [8]. *If for fixed* $\lambda > 0$ $(\lambda \in E_+^p)$, $\bar{x}$ *is an optimal solution of the parametric programming problem* (P_λ):

$$(\mathrm{P}_\lambda) \ \ \underset{x \in X}{\text{Minimize}} \, \lambda^t f(x),$$

where $\lambda > 0$ $(\lambda \in E_+^p)$ *is a preset vector then* $\bar{x}$ *is a properly efficient solution of the primal problem* (P).

Now we can state the following sufficient optimality conditions for primal problem (P).

THEOREM 5.4. *Suppose for a feasible point* $x \in X$ *for primal problem* (P) *the following conditions are satisfied*: *There exists* $\tau > 0$ $(\tau \in E_+^p)$ *and* $y \in Q^*$ *such that*

$$(\tau^t f + y^t g)^+ (x_0, x - x_0) \geq 0, \tag{5.7}$$

$$y^t g(x_0) = 0; \tag{5.8}$$

then x_0 *is a properly efficient solution of primal problem* (P).

Proof. Let x be a feasible point for (P) and (5.7), and (5.8) are satisfied. Then

$$\begin{aligned}
\tau^t f(x) - \tau^t f(x_0) &\geq (\tau f)^+ (x_0; x - x_0), \quad \text{since } \tau f \text{ is semilocally convex} \\
&\geq -(y^t g)(x_0; x - x_0), \quad \text{by (5.7)} \\
&\geq -y^t(g(x) - g(x_0)), \quad \text{since } g \text{ is semilocally convex} \\
&= -y^t g(x), \qquad \text{since } y^t g(x_0) = 0 \\
&\geq 0, \qquad \text{since } -g(x) \in Q \text{ and } y \in Q^*.
\end{aligned}$$

Hence $\tau^t f(x) \geq \tau^t f(x_0)$; therefore by Lemma 5.3, x_0 is properly efficient for primal problem (P).

A Wolfe-type dual for primal program (P) for the multiobjective case can be given as

$$\begin{aligned}
(\mathrm{DVP})_2 \quad & \text{Maximize } f(u) + y^t g(u) \\
& \text{subject to } (\tau^t f + y^t g)^+(u, x - u) \geq 0, \qquad \forall x \in X \\
& u \in X, y \in Q^*, \tau \in E_+^p, \tau^t e = 1.
\end{aligned}$$

We give below the corresponding parametric problem to $(\mathrm{DVP})_2$:

$$\begin{aligned}
(\mathrm{DVP}) \quad & \text{Maximize } \tau^t f(u) + y^t g(u) \\
& \text{subject to } (\tau^t f + y^t g)^+(u, x - u) \geq 0, \qquad \forall x \in X \\
& u \in X, y \in Q^*, \tau \in E_+^p;
\end{aligned}$$

where $\tau > 0$ $(\tau \in E_+^p)$ is a predetermined vector.

THEOREM 5.5 (Weak duality). *Let x be a feasible for program* (PVP) *and* (u, λ, y) *be feasible for dual program* $(\text{DVP})_2$ *such that* $\lambda^t f + y^t g$ *is semilocally convex at u; then*

$$\lambda^t f(x) \geq \lambda^t f(u) + y^t g(u) \qquad \forall x \in S.$$

Proof. Since $\lambda^t f + y^t g$ is semilocally convex at u, therefore

$$\lambda^t f(x) + y^t g(x) \geq \lambda^t f(u) + y^t g(u) + \left(\lambda^t f + y^t g\right)^+ (u, x - u). \quad (5.9)$$

Now since (u, λ, y) is feasible for the dual program (D_2), therefore $(\lambda f + y^t g)^+(u, x - u) \geq 0$, $\forall x \in S$, and, hence, from (5.9) and the fact that $y^t g(x) \leq_Q 0$,

$$\lambda^t f(x) \geq \lambda^t f(u) + y^t g(u). \qquad \text{G.E.D.}$$

THEOREM 5.6 (Strong duality). *Let* $\bar{x}$ *be properly efficient solution of the multiobjective programming problem* (PVP) *at which a constraint qualification is satisfied. Then there exists* $(\bar{\lambda}, \bar{y})$ *such that* $(\bar{x}, \bar{\lambda}, \bar{y})$ *is a feasible solution of the dual program* $(\text{DVP})_2$ *and* $\bar{y}^t g(\bar{x}) = 0$. *If also for each feasible point* $(\bar{u}, \bar{\lambda}, \bar{y})$ *in the dual programming problem* $(\text{DVP})_2$, $\lambda^t \underline{f} + y^t g$ *is semilocally convex and directionally differentiable at u, then* $(\bar{x}, \bar{\lambda}, \bar{y})$ *is a properly efficient solution of* $(\text{DVP})_2$ *and the objective function values are equal.*

Proof. Since a constraint qualification is satisfied at $\bar{x}$, from the Kuhn–Tucker necessary conditions there exists $(\bar{\lambda}, \bar{y})$ such that $(\bar{x}, \bar{\lambda}, \bar{y})$ is feasible in the dual programming problem $(\text{DVP})_2$ and $y^t g(\bar{x}) = 0$. Now for each feasible $(u, \bar{\lambda}, \bar{y})$ in the dual programming problem $(\text{DVP})_2$ we have

$$y^t g(x) - y^t g(u) \leq_Q 0 \qquad \forall x \in X = \{x \in X' : -g(x) \in Q, X' \subset E^m\}$$

and since $y^t g$ is semilocally convex with a directional derivative $(y^t g)^+(u, x - u)$ at u, we have

$$\left(y^t g\right)^+ (u, x - u) \leq_Q 0.$$

Using the inequality $(\bar{\lambda}^t f + \bar{y}^t g)^+(u, x - u) \geq 0$, we have that

$$\left(\bar{\lambda}^t f\right)^+ (u; x - u) \geq 0.$$

Now the semilocal convexity of $\bar{\lambda}^t f$ with a directional derivative at u implies

$$\bar{\lambda}^t f(x) \geq \lambda^t f(u) \qquad \forall x \in X. \quad (5.10)$$

By hypothesis, $\bar{x}$ is feasible for $(\text{DVP})_2$; hence equality holds in (5.10).

Therefore for all feasible $(u, \bar{\lambda}, y)$ in the dual $(\mathrm{DVP})_2$, we have

$$\bar{\lambda}^t f(u) \leq \bar{\lambda}^t f(x),$$

from which we get $\bar{\lambda}^t f(u) + \bar{y}^t g(u) \leq \bar{\lambda}^t f(x) + \bar{y}^t g(x)$ which implies that for $\bar{\lambda}, (\bar{x}, \bar{y})$ is an optimal solution of the parametric problem (DVP). Finally, because $\bar{\lambda} > 0$, applying Lemma 5.3, we get $(\bar{x}, \bar{\lambda}, \bar{y})$ as a properly efficient solution of the dual programming problem $(\mathrm{DVP})_2$. ∎

6. APPLICATION

In introducing the following multiobjective program with fractional objectives, we adopt the following notation:

$$\frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right) \qquad \text{for a function } \frac{f}{g}: X \to E^p.$$

Also $h: X \to E^m$, with $h(x) = (h_1(x), h_2(x), \ldots, h_m(x))$.

The primal problem is defined as

$$\text{(FP)} \qquad \underset{x \in X}{\text{Minimize}} \quad \frac{f(x)}{g(x)} \qquad (6.1)$$
$$\text{subject to} \quad h(x) \leq 0.$$

where $f(x) \geq 0$ and $g(x) > 0$ and in relation to the duality results, we would require $f(x)$ and $g(x)$ to have directional derivatives with semilocally convexity properties attached to them, whenever such assumptions are appropriate.

We can state the following Mond–Weir type of dual problem

$$\text{Maximize} \quad \frac{f(u)}{g(u)}$$
$$\text{(FD)} \qquad \text{Subject to} \quad \left(\lambda^t \frac{f}{g} + y^t h \right)^+ (u, x - u) \geq 0 \qquad \forall x \in X, \qquad (5.2)$$
$$y^t h(u) \geq 0,$$
$$u \in X, y \geq 0, \lambda \in E_+^p, \lambda^t e = 1,$$

where we assume that (f/g) and h have directional derivatives at u in the direction of $x - u$.

THEOREM 6.1 (Weak duality). *Let x be feasible for program* (FP) *and let* (u, λ, y) *be feasible for dual problem* (FD) *such that $\lambda^t f$ and $-\lambda^t g$ are semilocally convex at u and directionally differentiable at u, and $y^t h$ is semilocally convex at u and directionally differentiable at u; then*

$$\frac{f(x)}{g(x)} \nleq \frac{f(u)}{g(u)}.$$

Proof. Note that $(f/g)^+(u; x-u) = (1/g(u))[f^+(u; x-u) - v_0 g^+(u, x-u)]$ where, $v_0 = f(u)/g(u)$. (6.3)

Suppose, on the contrary to the conclusions, there is an x feasible for (FP) and (u, y, λ) feasible for (FD) such that

$$\frac{f(x)}{g(x)} \leq \frac{f(u)}{g(u)}. \tag{6.4}$$

Then for some $i_0 \in \{1, 2, \ldots, k\}$

$$\frac{f_{i_0}(x)}{g_{i_0}(x)} < \frac{f_{i_0}(u)}{g_{i_0}(u)} \tag{6.5}$$

and

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(u)}{g_i(u)} \qquad \text{for } i \neq i_0.$$

From the semilocal convexity of $\lambda^t f$ and $-\lambda^t g$ we have, in view of (6.3), (6.4), and (6.5),

$$\left(\lambda^t \frac{f}{g}\right)^+ (u; x-u) \leq 0,$$

which implies because of the feasibility conditions of the dual (FD),

$$y^t h^+(u; x-u) \geq 0. \tag{6.6}$$

Also, because of feasibility of the primal and dual problems,

$$y^t h(x) \leq 0, \qquad y^t h(u) \geq 0,$$

which, in view of semilocal convexity of $y^t h$ at u, gives

$$(y^t h)^+ (u; x - u) \leq 0,$$

which is a contradiction to (6.6). ∎

THEOREM 6.2 (Strong duality). *Let x_0 be a properly efficient solution of* (FP) *at which a Kuhn–Tucker constraint qualification is satisfied. Then there exists* $(\bar{\lambda}, \bar{y})$ *such that* $(x_0, \bar{\lambda}, \bar{y})$ *is a feasible solution of* (FD) *and* $y^t h(x_0) = 0$. *If, also, for each feasible point* $(\bar{u}, \bar{\lambda}, \bar{y})$ *in the* (FD) $\lambda^t f$ *and* $-\lambda^t g$ *are semilocally convex and directionally differentiable at* u, *then* $(x_0, \bar{\lambda}, \bar{y})$ *is a properly efficient solution of* (FD) *and the objective function values are equal.*

Proof. Since a Kuhn–Tucker constraint qualification is satisfied at x_0, Kuhn–Tucker necessary conditions imply that there exists $(\bar{\lambda}, \bar{y})$ such that $(x_0, \bar{\lambda}, \bar{y})$ is feasible for (FD) and $\bar{y}^t h(x_0) = 0$. Now suppose $(x_0, \bar{\lambda}, \bar{y})$ is not efficient for (FD); then there exists (x^*, λ^*, y^*) feasible for (FD) such that for some $i_0 \in \{1, 2, \dots, k\}$

$$\frac{f_{i_0}(x^*)}{g_{i_0}(x^*)} > \frac{f_{i_0}(x^0)}{g_{i_0}(x^0)} \tag{6.7}$$

and

$$\frac{f_j(x^*)}{g_j(x^*)} \geq \frac{f_j(x^0)}{g_j(x^0)}, \qquad \text{for all } j \neq i_0. \tag{6.8}$$

Now since $\lambda^t f$ and $-\lambda^t g$ are semilocally convex at x^*, because of (6.3), (6.7), and (6.8),

$$\left(\lambda^t \frac{f}{g}\right)^+ (x^*; x_0 - x^*) \leq 0. \tag{6.9}$$

Using the feasible conditions in (FD) we have

$$(y^{*t} h)^+ (x^*, x^0 - x^*) \geq 0. \tag{6.10}$$

But the primal and dual feasibility implies that $y^{*t} h(x_0) \leq 0$ and $y^{*t} h(x^*) \geq 0$. The semilocal convexity of $y^t h$ then implies that

$$(y^{*t} h)^+ (x^*; x^0 - x^*) \leq 0, \tag{6.11}$$

which is a contradiction to (6.8).

Now from the above proof it follows that $(x_0, \bar{\lambda}, \bar{y})$ is an optimal solution to the scalarized version of the dual problem (FD); hence by Lemma 5.3 it follows that $(x_0, \overline{\lambda}, \bar{y})$ is a properly efficient solution of (FD). Now it is easy to see that the objective function values of (FP) and (FD) are equal at their respective properly efficient points. ∎

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