JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **202,** 53–71 (1996)
ARTICLE NO. 0302

Generalized Proper Efficiency and Duality for a Class of Nondifferentiable Multiobjective Variational Problems with *V*-Invexity

S. K. Mishra

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 221005, India

Submitted by Firdaus E. Udwadia

Received November 7, 1994

A Mond–Weir type dual for a class of nondifferentiable multiobjective variational problems in which every component of the objective function contains a term involving the square root of a certain positive semidefinite quadratic form, is considered and various duality results, viz. weak, strong, and converse duality theorems, are developed for conditionally properly efficient solutions. These results are obtained under *V*-invexity assumptions and its generalizations on objective and constraint functions. This work extends many results on variational problems established earlier.

1. INTRODUCTION

Hanson [8] extended the duality results of mathematical programming to a class of functions subsequently called invex. Since that time, it has been shown [10, 11] that many results in mathematical programming previously established for convex functions actually hold for the wider class of invex functions. Mond *et al*. [11] extended the concept of invexity to the continuous case and used it to generalize earlier duality results for a class of variational problems. Mond and Smart [15] extended the duality theorems for a class of static nondifferentiable problems with Wolfe type and Mond–Weir type duals, and further extended these for the continuous analogues. Mishra and Mukherjee [10] extended the work of Mond *et al*. [11] for multiobjective variational problems which in particular extended an earlier work of Bector and Husain [1] for invex functions.

Jeyakumar and Mond [9] introduced a wider class than that of invex functions subsequently known as *V*-invex functions, which preserves the

0022-247X/96 $18.00

sufficient optimality and duality results in the scalar case, and avoids the major difficulty of verifying that the inequality holds for the same function $\eta(.,.)$. Mukherjee and Mishra [16] extended the work of [9] to variational problems with the concept of weak minima.

Singh and Hanson [17] generalized the concept of proper efficiency, introduced by Geoffrion [7], to cover practical situations where Geoffrion's definition does not apply and they applied it to the relationship between multiobjective programming and scalarized multiobjective programming, and to the duality theory of mathematical programming.

The aim of this paper is to extend an earlier work of Mond *et al*. [13] to the continuous case, under the assumption of V-invexity and its generalizations on the objective and constraint functions. Further, duality results are developed for conditionally properly efficient solutions. A close relationship between these variational problems and nonlinear multiobjective programming problems is also indicated. The results generalize various well known results in variational problems with differentiable functions and also give a dynamic analogue of certain nondifferentiable programming problems.

2. NOTATION AND PRELIMINARIES

Let $I = [a, b]$ be a real interval and $f\colon I \times R^n \times R^n \to R$ and $g\colon I \times R^n \times R^n \to R^m$ be continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$, where $x\colon I \to R^n$ with derivative $\dot{x}$ denote the partial derivative of f with respect to t, x, and $\dot{x}$, respectively, by f_t, f_x, and $f_{\dot{x}}$, such that

$$f_x = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right), \qquad f_{\dot{x}} = \left(\frac{\partial f}{\partial \dot{x}_1}, \frac{\partial f}{\partial \dot{x}_2}, \ldots, \frac{\partial f}{\partial \dot{x}_n} \right).$$

We write the partial derivatives of the vector function g using matrices with m rows instead of one.

Let $C(I, R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|$ where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\, ds,$$

where α is a given boundary value. Therefore $D = d/dt$ except at discontinuities.

Consider the following vector minimum problem

$$(\mathrm{P}^0)\ \text{Minimize} \int_a^b f(t, x(t), \dot{x}(t))\,dt$$
$$= \left(\int_a^b f^1\left(t, x(t), \dot{x}(t)\,dt, \int_a^b f^2(t, x(t)\dot{x}(t))\,dt, \ldots,\right.\right.$$
$$\left.\int_a^b f^p(t, x(t), \dot{x}(t))\,dt\right)$$

$$\text{subject to } x(a) = \alpha, \qquad x(b) = \beta \tag{1}$$

$$g(t, x(t), \dot{x}(t)) \leq 0, \qquad t \in I. \tag{2}$$

Craven [6] obtained Kuhn−Tucker type necessary conditions for the above problem and proved that the necessary conditions are also sufficient if objective functions are pseudoconvex and constraints are quasiconvex.

Let K the set of feasible solutions for (P^0) be given by

$$K = \{x \in C(I, R^n): x(a) = \alpha, x(b) = \beta$$
$$g(t, x(t), \dot{x}(t)) \leq 0, t \in I\}.$$

The following definitions will be needed in the sequel.

DEFINITION 1. A point x^* in K is said to be an efficient solution of (P) if for all x in K

$$\int_a^b f^i(t, x^*(t), \dot{x}(t))\,dt \geq \int_a^b f^i(t, x(t), \dot{x}(t))\,dt \qquad \forall\ i \in \{1, 2, \ldots, p\}$$
$$\Rightarrow \int_a^b f^i(t, x^*(t), \dot{x}^*(t))\,dt = \int_a^b f^i(t, x(t), \dot{x}(t))\,dt,$$
$$\forall\ i \in \{1, 2, \ldots, p\}.$$

DEFINITION 2 (Borwein [2]). *A point x^* in K is said to be a weak minimum for* (P) *if there exists no other x in K for which*

$$\int_a^b f(t, x^*(t), \dot{x}^*(t))\,dt > \int_a^b f(t, x(t), \dot{x}(t))\,dt.$$

From this it follows that if an x^* in K is efficient for (P^0) *then it is also a weak minimum for* (P^0).

DEFINITION 3 (Geoffrion [7]). A point x^* in K is said to be properly efficient solution of (P^0) if there exists scalar $M > 0$ such that $\forall\ i \in \{1, 2, \ldots, p\}$

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt - \int_a^b f^i(t, x(t), \dot{x}(t))\, dt \leq M\left(\int_a^b f^j(t, x(t), \dot{x}(t))\, dt - \int_a^b f^j(t, x^*(t), \dot{x}^*(t))\, dt\right)$$

for some j such that

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt > \int_a^b f^j(t, x^*(t), \dot{x}^*(t))\, dt$$

whenever x is in K and

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt < \int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt.$$

An efficient solution that is not properly efficient is said to be improperly efficient. Thus for x^* to be improperly efficient means that to every sufficiently large $M > 0$ there is an x in K and an $i \in \{1, 2, \ldots, p\}$ such that

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt < \int_a^b f^i(t, x^*(t), x^*(t))\, dt$$

and

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt - \int_a^b f^i(t, x(t), \dot{x}(t))\, dt > M\left(\int_a^b f^j(t, x(t), \dot{x}(t))\, dt - \int_a^b f^j(t, x^*(t) \dot{x}^*(t)\, dt)\right) \quad \forall\ j \in \{1, 2, \ldots, p\},$$

such that

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt > \int_a^b f^j(t, x^*(t), \dot{x}^*(t))\, dt.$$

DEFINITION 4 (Singh and Hanson [17]). A point x^* in K is said to be conditionally properly efficient for (P^0) if x^* is efficient for (P^0) and there exists a positive function $M(x)$ such that for each $i \in \{1, 2, \ldots, p\}$, we have

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt - \int_a^b f^i(t, x(t), \dot{x}(t))\, dt$$

$$\leq \int_a^b M(x) f^j(t, x(t), \dot{x}(t))\, dt$$

$$- \int_a^b M(x) f^j(t, x^*(t), \dot{x}^*(t))\, dt,$$

for some j such that

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt < \int_a^b f^j(t, x^*(t), \dot{x}^*(t))\, dt$$

whenever x is in K and

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt < \int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt.$$

Now we consider the following Singh and Hanson [17] type parametric variational problem for predetermined positive functions $\tau_i(x)$ such that $a_i < \tau_i k(x) < b_i$, $i \in \{1, \ldots, p\}$ where a_1 and b_1 are specified constants

$$(\mathrm{P}_\tau^0) \text{ Minimize } \sum_{i=1}^{p} \int_a^b \tau_i(x) f^i(t, x(t), \dot{x}(t))\, dt$$

$$\text{subject to (1) and (2).}$$

Problems (P^0) and (P_τ^0) are equivalent in the sense of Singh and Hanson [17]. Theorems 1 and 2 are valid when R^n is replaced by some normed space of functions, as the proofs of these theorems do not depend on the dimensionality of the space in which the feasible set of (P) lies. For our variational problems the feasible set K lies in the normed space $C(I, R^n)$. For completeness we shall merely state these theorems characterizing conditional proper vector minima of (P^0) in terms of solutions of (P_τ^0).

THEOREM 1. *If x^* is an optimal solution of (P_τ^0) then x^* is conditionally properly efficient for (P^0).*

THEOREM 2. *If x^* is conditionally properly efficient for (P^0) then x^* is optimal for (P_τ^0) for some $\tau_i(x^*) > 0$, $i = 1, 2, \ldots, p$.*

Consider the following nondifferentiable multiobjective variational primal problem as

$$\text{(P)}\quad \text{Minimize} \int_a^b \psi(t, x(t), \dot{x}(t)\, dt$$

$$= \left(\int_a^b \left\{ f^1(t, x(t), \dot{x}(t)) + \left(x^T(t)\beta x(t)\right)^{1/2} \right\} dt, \ldots, \right.$$

$$\left. \int_a^b \left\{ f^p(t, x(t), \dot{x}(t)) + \left(x(t)^T B^p(t) x(t)\right)^{1/2} \right\} dt \right)$$

subject to (1) and (2),

where $B_i(t)$, $i \in \{1, 2, \ldots, p\}$ is a positive semi-definite (symmetric) matrix, with $E_i(\cdot)$, $i \in \{1, 2, \ldots, p\}$ continuous on I.

The aim of this paper is to show that the requirements of objective and constraint functions to be invex, pseudo-invex, or quadi-invex can be weakened to requiring V-invexity, V-pseudo-invexity, and V-quasi-invexity, respectively.

3. VECTOR INVEXITY

Vector invexity was introduced into mathematical programming by Jeyakumar and Mond [9]. Mukherjee and Mishra [16] have extended the concept of V-invexity to the continuous case and obtained sufficient optimality criteria and duality results for a class of multiobjective variational problems under V-invexity assumptions.

DEFINITION. A vector function $F = (F_1, \ldots, F_p)$, $F_i(x) = \int_a^b f^i(t, x(t), \dot{x}(t))\, dt$ is said to be V-invex if there exist different vector functions $\eta(t, x, \bar{x}) \in R^p$ with $\eta(t, x, x) = 0$ and α_i: $I \times X_0 \times X_0 \to R_+ \setminus \{0\}$ such that for each $x, \bar{x} \in X_0$ and $i \in \{1, 2, \ldots, p\}$

$$F_i(x) - F_i(x) \geq \int_a^b \left\{ \alpha_i(t, x(t), \bar{x}(t)) f_x^i(t, \bar{x}(t), \dot{\bar{x}}(t)) \eta(t, x(t), \bar{x}(t)) \right.$$

$$+ \frac{d}{dt} \eta(t, x(t), \bar{x}(t))\, \alpha_i(t, x(t), \bar{x}(t))$$

$$\left. \times f_{\dot{\bar{x}}}^i(t, \bar{x}(t), \dot{\bar{x}}(t)) \right\} dt.$$

If we take $\alpha_i(t, x(t), \bar{x}(t)) = 1$, $\forall\ i = 1, \dots, p$ this definition reduces to invex functions, see Mishra and Mukherjee [10], and if we take $p = 1$, $\alpha(t, x(t), \bar{x}(t)) = 1$, the above definition reduces to invex in x and $\bar{x}$ on $[a,b]$ with respect to η, see Mond and Smart [15] and Mond, Chandra, and Husain [11]. V-pseudo and V-quasi-invexity are simply defined; the vector function F is V-pseudo-invex with respect to η and β_i: $I \times X_0 \times X_0 \to R_+ \setminus\{0\}$ is for each $x, \bar{x} \in X_0$,

$$\int_a^b \sum_{i=1}^p \left\{ \eta(t, x, \bar{x}) f_x^i(t, \bar{x}, \dot{\bar{x}}) + \frac{d}{dt}\eta(t, x, \bar{x}) f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}}) \right\} dt \geq 0$$

$$\Rightarrow \int_a^b \sum_{i=1}^p \beta_i(t, x, \bar{x}) f_i(t, x, \dot{x})\, dt \geq \int_a^b \sum_{i=1}^p \beta_i(t, x, \bar{x}) f^i(t, \bar{x}, \dot{\bar{x}})\, dt;$$

or equivalently,

$$\int_a^b \sum_{i=1}^p \beta_i(t, x, \bar{x}) f_i(t, \bar{x}, \dot{\bar{x}})\, dt < \int_a^b \sum_{i=1}^p \beta_i(t, x, \bar{x}) f_i(t, \bar{x}, \dot{\bar{x}})\, dt$$

$$\Rightarrow \int_a^b \sum_{i=1}^p \left\{ \eta(t, x, \bar{x}) f_x^i(t, \bar{x}, \dot{\bar{x}}) + \frac{d}{dt}\eta(t, x, \bar{x}) f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}}) \right\} dt < 0.$$

The vector function F is said to be V-quasi-invex if there exist functions η: $I \times X_0 \times X_0 \to R^p$ with $\eta(t, \bar{x}, \bar{x}) = 0$ and γ_i: $I \times X_0 \times X_0 \to R_+ \setminus\{0\}$ such that for each $x, \bar{x} \in X_0$,

$$\int_a^b \sum_{i=1}^p \gamma_i(t, x, \bar{x}) f_i(t, x, \dot{x})\, dt \leq \int_a^b \sum_{i=1}^p \gamma_i(t, x, \bar{x}) f_i(t, \bar{x}, \dot{\bar{x}})\, dt$$

$$\Rightarrow \int_a^b \sum_{i=1}^p \left\{ \eta(t, x, \bar{x}) f_x^i(t, \bar{x}, \dot{\bar{x}}) + \frac{d}{dt}\eta(t, x, \bar{x}) f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}}) \right\} dt \leq 0;$$

or equivalently,

$$\int_a^b \sum_{i=1}^p \left\{ \eta(t, x, \bar{x}) f_x^i(t, \bar{x}, \dot{\bar{x}}) + \frac{d}{dt}\eta(t, x, \bar{x}) f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}}) \right\} dt > 0$$

$$\Rightarrow \int_a^b \sum_{i=1}^p \gamma_i(t, x, \bar{x}) f_i(t, x, \dot{x})\, dt > \int_a^b \sum_{i=1}^p \gamma_i(t, x, \bar{x}) f_i(t, \bar{x}, \dot{\bar{x}})\, dt.$$

Again all V-pseudo-invex functionals are also V-invex. It is to be noted here that if f is independent of t, then V-invexity, V-pseudo-invexity, and V-quasi-invexity defined above reduce to the definitions of V-invexity, V-pseudo-invexity, and V-quasi-invexity of Jeyakumar and Mond [9], respectively. Note also that linear functional multiobjective variational problems are V-invex variational problems. Moreover, invex multiobjective variational problems are necessarily V-invex variational problems, but not conversely, which is clear from the following example of Mukherjee and Mishra [16].

EXAMPLE. Consider

$$\min_{x_1, x_2 \in R} \left(\int_a^b \frac{x_1^2(t)}{x_2(t)} \, dt, \int_a^b \frac{x_2(t)}{x_1(t)} \, dt \right)$$

subject to

$$1 - x_1(t) \leq 0, \qquad 1 - x_2(t) \leq 0.$$

Then, for

$$\alpha_1(t, x(t), u(t)) = \frac{u_2(t)}{x_2(t),}, \qquad \alpha_2(t, x(t), u(t)) = \frac{u_1(t)}{x_1(t)}.$$

$$\beta_i(t, x(t), u(t)) = 1 \text{ for } i = 1, 2, \qquad \eta(t, x(t), u(t)) = x(t) - u(t),$$

we shall show that

$$\int_a^b \left\{ f_i(t, x, \dot{x}) - f_i(t, u, \dot{u}) - \alpha_i(t, x, u) f_x^i(t, u, \dot{u}) \eta(t, x, u) \right\} dt \geq 0$$

for $i = 1, 2$.

Now,

$$\int_a^b \frac{x_1^2(t)}{x_2(t)} \, dt - \int_a^b \frac{u_1^2(t)}{u_2(t)} \, dt - \int_a^b \frac{u_2(t)}{x_2(t)} \left(\frac{2u_1(t)}{u_2(t)} - \frac{u_1^2(t)}{u_2^2(t)} \right)$$
$$\times (x_1(t) - 1)(x_2(t) - 1) \, dt$$
$$= \int_a^b \frac{(x_1(t) - 1)^2}{x_2(t)} \, dt \geq 0.$$

Thus, V-invexity does not necessarily imply invexity.

It has been shown in [9] that V-invex functions can be formed from certain nonconvex functions. In the subsequent analysis, we shall frequently use the following generalized Schwarz inequality

$$x^T B z \leq (x^T B x)^{1/2} (Z^T B_z)^{1/2},$$

where B is an $n \times n$ positive semidefinite Matrix.

PROPOSITION. *If f_i, $i = 1, \ldots, p$, in V-invex with respect to α_i and η, $i = 1, 2, \ldots, p$, with $\eta(x, u) = x - u + y(x, u)$, where $B_i(x, u) = 0$ then $f_i + \cdot^T B_i$ is also V-invex with respect to η.*

Proof. The proof follows easily from the proof of Proposition 2 of Mond and Smart [15].

4. DUALITY

In view of Proposition 1 of [14], the Mond–Weir type dual for (P_τ) is

$$(\mathrm{D}_\tau) \text{ Maximize } \int_a^b \sum_{i=1}^p \tau_i\big(f_i(u) + u^T B_i z_i\big)\, dt$$

subject to

$$u(a) = \alpha, \qquad u(b) = \beta \tag{3}$$

$$\sum_{i=1}^p \tau_i f_x^i(t, u, \dot{u}) + B_i(t) z_i(t) + \sum_{j=1}^m y_j g_x^j(t, u, \dot{u})$$

$$= \frac{d}{dt}\left\{\sum_{i=1}^p \tau_i f_{\dot{x}}^i(t, u, \dot{u}) + \sum_{j=1}^m y_j(t) g_{\dot{x}}^j(t, u, \dot{u})\right\} \tag{4}$$

$$z_i^T B_i z_i \leq 1, \qquad i = 1, 2, \ldots, p \tag{5}$$

$$\int_a^b y_j(t) g_j(t, u, \dot{u})\, dt \geq 0, \qquad j = 1, 2, \ldots, m \tag{6}$$

$$y(t) \geq 0, \qquad t \in I, \tau e = 1, \tau \geq 0, \tag{7}$$

where

$$e = (1, 1, \ldots, 1) \in R^p. \qquad (7)'$$

Now Theorems 1 and 2 motivated us to define the following vector maximization variational problem:

$$\text{Dual (D) Maximize } \left(\int_a^b \left\{f_1(t, u(t), \dot{u}(t)) + u(t)^T B_1(t) z_1(t)\right\} dt, \ldots,\right.$$

$$\left.\int_a^b \left\{f_p(t, u(t), \dot{u}(t)) + u(t)^T B_p(t) Z_p(t)\right\} dt\right)$$

subject to (3)–(7).

In problems (P_τ) and (D_τ), the vector $0 < \tau \in R^p$ is predetermined. Note that if $p = 1$, then (P) and (D) become the pair of nondifferentiable nonsymmetric dual problems treated by Mond and Smart [15]. In case

$B_i = 0$, $\forall\ i \in \{1, 2, \ldots, p\}$ then (P) and (D) reduce to the problems studied by Bector and Husain [1], and Mukherjee and Mishra [15]. If we take $p = 1$, $B_i = 90$, $\forall\ i \in \{1, 2, \ldots, p\}$, then (P) and (D) reduce to the variational problem dealt by Mond *et al.* [11].

Let K and H denote the sets of feasible solutions of (P) and (D), respectively.

THEOREM 3 (Weak Duality). *Let* $x \in K$ *and* $(u, \tau, y, z_1, \ldots, z_p) \in H$. *Let*

$$\left(\int_a^b \left\{\tau_i\left(f_1(t,0,0) + \cdot^T B_1(t) z_1(t)\right)\right\} dt, \ldots,\right.$$

$$\left.\int_a^b \left\{\tau_p\left(f_p(t,0,0) + 0^T B_p(t) z_p(t)\right\} dt\right)$$

by V-pseudo-invex and

$$\left(\int_a^b y_1(t) g_1(t,0,0)\, dt, \ldots, \int_a^b y_m(t) g_m(t,0,0)\, dt\right)$$

be V-quasi-invex both with respect to η *for all piecewise smooth* $z_i\colon I \to R^n$. *Then*

$$\left(\int_a^b \left(f_1(t, x(t), \dot{x}(t)) + \left(x^T(t) B_1(t) x(t)\right)^{1/2}\right) dt, \ldots,\right.$$

$$\left.\int_a^b \left(f_p(t, x(t), \dot{x}(t)) + \left(x(t)^T B_p(t) x(t)\right)^{1/2} dt\right)\right.$$

$$-\left(\int_a^b \left(f_1(t, u(t), \dot{u}(t)) + u(t)^T B_1(t) z_1(t)\right) dt, \ldots,\right.$$

$$\left.\int_a^b \left(f_p(t, u(t), \dot{u}(t)) + u(t)^T B_p(t) z_p(t)\right) dt\right)$$

$$\notin -\operatorname{int} R^p + .$$

Proof. Let $x \in K$ and $(u, \tau, y, z_1, \ldots, z_p) \in H$. Then by the feasibility condition and $\gamma_j(t, x(t), u(t)) > 0$, we have

$$\int_a^b \sum_{j=1}^m \gamma_j(t, x(t), u(t)) y_j(t) g_j(t, x(t), \dot{x}(t))\, dt$$

$$-\int_a^b \sum_{j=1}^m \gamma_j(t, x(t), u(t)) y_j(t) g_j(t, u(t), \dot{u}(t))\, dt \leq 0 \qquad \forall\ x \in K.$$

Thus, V-quasi-invexity of $\left(\int_a^b y_1 g_1(t,0,0)\,dt,\ldots,\int_a^b y_m g_m(t,0,0)\,dt\right)$ gives

$$\int_a^b \sum_{j=1}^m \left\{\eta(t,x,u)\, y_j g_x^j(t,u,\dot{u}) + \frac{d}{dt}\eta(t,x,u)\, y_j g_u^j(t,u,\dot{u})\right\} dt \le 0. \quad (8)$$

Integration by parts (4) gives

$$\begin{aligned}
&\int_a^b \eta(t,x,u)\Bigg[\sum_{i=1}^p \tau_i\left\{f_x^i(t,u(t),\dot{u}(t)) + B_i(t)z_i(t)\right\} \\
&\qquad\qquad + \sum_{j=1}^m y_j g_x^j(t,u(t),\dot{u}(t))\Bigg] dt \\
&\quad = \eta(t,x,u)\left[\sum_{i=1}^p \tau_i f_{\dot{x}}^i(t,u(t),\dot{u}(t)) + \sum_{j=1}^m y_j(t) g_{\dot{x}}^j(t,u(t)\dot{u}(t))\right] \\
&\qquad - \int_a^b \frac{d}{dt}\eta(t,x(t),u(t))\Bigg[\sum_{j=1}^p \tau_i f_{\dot{x}}^i(t,u(t),\dot{u}(t)) \\
&\qquad\qquad + \sum_{j=1}^m y_j(t) g_{\dot{x}}^j(t,u(t),\dot{u}(t))\Bigg]. \quad (9)
\end{aligned}$$

Since, $\eta(t,u(t),u(t)) = 0$, from (9) we have

$$\begin{aligned}
&\int_a^b \sum_{j=1}^m \Bigg\{\eta(t,x(t),u(t))\, y_j(t) g_x^j(t,u(t),\dot{u}(t)) \\
&\qquad + \frac{d}{dt}\eta(t,x(t),u(t))\, y_j(t) g_{\dot{x}}^j(t,u(t),\dot{u}(t))\Bigg\} dt \\
&\quad = -\int_a^b \sum_{i=1}^p \Bigg\{\eta(t,x(t),u(t))\,\tau_i\left(f_x^i(t,u(t),\dot{u}(t)) + B_i(t)z_i(t)\right) \\
&\qquad + \frac{d}{dt}\eta(t,x(t),u(t))\,\tau_i f_{\dot{x}}^i(t,u(t),\dot{u}(t)\Bigg\} dt. \quad (10)
\end{aligned}$$

From (10) and (8), we have

$$\begin{aligned}
&\int_a^b \sum_{i=1}^p \eta(t,x(t),u(t))\,\tau_i\left(f_x^i(t,u(t),u(t)) + B_i(t)z_i(t)\right) \\
&\qquad + \frac{d}{dt}\eta(t,x(t),u(t))\,\tau_i f_{\dot{x}}^i(t,u(t),\dot{u}(t))\Bigg\} dt \ge 0. \quad (11)
\end{aligned}$$

By V-pseudo-invexity of $(\int_a^b \tau_1(f_1(t,0,0) + 0^T B_1(t))\,dt, \ldots, \int_a^b \tau_p(f_p(t,0,0) + 0^T B_p(t) z_p(t))\,dt)$, Swchwarz Inequality, and (5), we have

$$\int_a^b \sum_{i=1}^p i(t, x(t), u(t))\,\tau_i\Big\{ f_i(t, x(t), x(t)) + \big(x(t)^T B_i(t) x(t)\big)^{1/2} \Big\}\,dt$$

$$\geq \int_a^b \sum_{i=1}^p \beta_i(t, x(t), u(t))\,\tau_i\Big\{ f_i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) z_i(t) \Big\}\,dt. \tag{12}$$

The conclusion now follows, since $\tau e = 1$, and $\beta_i(t, x(t), u(t)) > 0$. ∎

PROPOSITION. *Let $u \in K$, $(u, \tau, y, z_1, \ldots, z_p) \in H$. Let the V-pseudo and V-quasi-invexity conditions of Theorem* 3 *hold. If*

$$\big(u(t)^T B_i(t) u(t)\big)^{1/2} = u(t)^T B_i(t) z_i(t) \tag{13}$$

$\forall \; i = 1, 2, \ldots, p$, *then u is conditionally properly efficient for* (P) *and* $(u, \tau, y, z_1, \ldots, z_p)$ *is conditionally properly efficient for* (D).

Proof. From (12) and (13) it follows that for all $x \in K$

$$\int_a^b \sum_{i=1}^p \tau_i\Big\{ f_i(t, u(t), \dot{u}(t)) + \big(u(t)^T B_i(t) u(t)\big)^{1/2} \Big\}\,dt$$

$$= \int_a^b \sum_{i=1}^p \tau_i\Big\{ f_i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) z_i(t) \Big\}\,dt$$

$$\leq \int_a^b \sum_{i=1}^p \tau_i\Big\{ f_i(t, x(t), \dot{x}(t)) + \big(x(t)^T B_i(t) x(t)\big)^{1/2} \Big\}\,dt. \tag{14}$$

Thus $u(t)$ is an optimal solution of (P_τ). Hence, by Theorem 1, $u(t)$ is a conditionally properly efficient solution of (P).

We first show that $(u, \tau, y, z_1, \ldots, z_p)$ is an efficient solution of (D). Assume that it is not efficient, i.e., there exists $(\bar{u}, \bar{\tau}, \bar{y}, \bar{z}_1, \ldots, \bar{z}_p) \in H$ such that

$$\int_a^b \Big\{ f^i\big(t, \bar{u}(t), \dot{\bar{u}}(t)\big) + \bar{u}(t)^T B_i(t) \bar{z}_i(t) \Big\}\,dt$$

$$\geq \int_a^b \Big(f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) z_i(t) \Big)\,dt, \qquad \forall\; i$$

and

$$\int_a^b \Big(f^j\big(t, \bar{u}(t), \dot{\bar{u}}(t)\big) + \bar{u}(t)^T B_j(t) z_j(t) \Big)\, dt$$

$$> \int_a^b \Big(f^j(t, u(t)\dot{u}(t)) + u(t)^T B_j(t) z_j(t) \Big)\, dt$$

$$\text{for at least one } j \in \{1, 2, \ldots, p\}.$$

Thus, from (13),

$$\int_a^b \Big(f^i(t, u(t), \dot{u}(t)) + \big(u(t)^T B_i(t) u(t) \big)^{1/2} \Big)\, dt$$

$$\leq \int_a^b \Big(f^i\big(t, \bar{u}(t), \dot{\bar{u}}(t)\big) + u(t)^T B_i(t) z_i(t)\, dt \qquad \forall\, i \in \{1, 2, \ldots, p\}$$

and

$$\int_a^b \Big\{ f^j(t, u(t), \dot{u}(t)) + \big(u(t)^T B_j(t) u(t) \big)^{1/2} \Big\}\, dt$$

$$< \int_a^b \Big(f^j(t, u(t), u(t)) + \big(u(t)^T B_j(t) z_j(t) \big) \, dt$$

$$\text{for at least one } j \in \{1, 2, \ldots, p\},$$

contradicting weak duality. Hence $(u, \tau, y, z_1, \ldots, z_p)$ is efficient.

Now we show that $(u, \tau, y, z_1, \ldots, z_p)$ is a conditionally properly efficient solution of (D). Assume that it is not conditionally properly efficient, i.e., there exists $(\bar{u}, \bar{\tau}, \bar{y}, \bar{z}_1, \ldots, \bar{z}_p) \in H$ such that for some i and all $M(\bar{u}) > 0$

$$\int_a^b \Big\{ f^i\big(t, \bar{u}(t), \dot{\bar{u}}(t)\big) + \bar{u}(t)^T B_i(t) \bar{z}_i(t) \Big\}\, dt$$

$$- \int_a^b \Big\{ f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) z_i(t) \Big\}\, dt$$

$$> \int_a^b M(\bar{u}) \Big\{ f^j(t, u(t), \dot{u}(t)) + u(t)^T B_j(t) z_j(t) \Big\}\, dt$$

$$- \int_a^b M(\bar{u}) \Big\{ f^j\big(t, \bar{u}(t), \dot{\bar{u}}(t)\big) + \bar{u}(t)^T B_j(t) \bar{z}_j(t) \Big\}\, dt \qquad (15)$$

and $\forall\, j \in \{1, 2, \ldots, p\}$, such that

$$\int_a^b \Big\{ f^j(t, u(t), \dot{u}(t)) + uu(t)^T B_j(t) z_j(t) \Big\}\, dt$$
$$> \int_a^b \Big\{ f^j\big(t, \bar{u}(t), \dot{\bar{u}}(t)\big) + \bar{u}(t)^T B_j(t) \bar{z}_j(t) \Big\}\, dt.$$

Since $\tau \geq 0$, $\tau \neq 0$,

$$\int_a^b \sum_{i=1}^{p} \bar{\tau}_i \Big(f_i\big(t, \bar{u}(t), \dot{\bar{u}}(t)\big) + \bar{u}(t) B_i(t) \bar{z}_i(t) \Big)\, dt$$
$$> \int_a^b \sum_{i=1}^{p} \bar{\tau}_i \Big(f_i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) z_i(t)\, dt. \tag{16}$$

Now (16) and (13) lead to

$$\int_a^b \sum_{i=1}^{p} \bar{\tau}_i \Big(f_i\big(t, \bar{u}(t), \dot{\bar{u}}(t)\big) + \bar{u}(t) B_i(t) \bar{z}_i(t) \Big)\, dt$$
$$> \int_a^b \sum_{i=1}^{p} \bar{\tau}_i \bigg(f_i(t, u(t), \dot{u}(t)) + \Big(u(t)^T B_i(t) u(t) \Big)^{1/2} \bigg)\, dt$$

contradicting (12). Thus $(u, \tau, y, z_1, \ldots, z_p)$ is conditionally properly efficient.

THEOREM 4 (Strong Duality). *Let the V-pseudo-invexity and V-quasi-invexity conditions of Theorem* 3 *hold. Let* x^0 *be normal* [13] *and a conditionally properly efficient solution for* (P). *Then for some* $\bar{\tau} \in \Lambda^+$, *there exists a piecewise smooth* $y^0\colon I \to R^m$ *such that* $(u = x^0, \bar{\tau}, y^0)$ *is a conditionally properly efficient solution of* (D) *and*

$$\int_a^b \bigg\{ f^i\big(tx^0(t)\big), \dot{x}^0(t)\big) + \Big(x^0(t)^T B_i(t) x^0(t) \Big)^{1/2} \bigg\}\, dt$$
$$= \int_a^b \Big\{ f^i\big(t, u^0(t), \dot{u}^0(t)\big) + u^0(t)^T B_i(t) z_i^0(t) \Big\}\, dt$$
$$\forall\, i \in \{1, 2, \ldots, p\}.$$

Proof. Since x^0 is a conditionally properly efficient solution of (P) and generalized V-invexity conditions are satisfied, by Theorem 2, x^0 is optimal for ($P_{\bar{\tau}}$) for some $\bar{\tau} \in \Lambda^{+}$. Therefore, by Theorem 3, there exists a piecewise smooth $y^0: I \to R^m$ such that for $t \in I$,

$$\sum_{i=1}^{p} \bar{\tau}_i \left[f_x^i\left(t, u^0(t), \dot{u}^0(t)\right) + B_i(t) z_i^0(t) \right] + \sum_{j=1}^{m} y_j^0\left(t, u^0(t), \dot{u}^0(t)\right) = \frac{d}{dt}\left[\sum_{i=1}^{p} \bar{\tau}_i f_{\dot{x}}^i\left(t, u^0(t), \dot{u}^0(t)\right) + \sum_{j=1}^{m} y_j^0(t) g_{\dot{x}}^j\left(t, u^0(t), \dot{u}^0(t)\right) \right. \tag{17}$$

$$\left(x^0(t)^T B_i(t) x^0(t)\right)^{1/2} = x^0(t)^T B_i(t) z_i^0(t), \qquad i = 1, 2, \ldots, p \tag{18}$$

$$z_i(t)^T B_i(t) z_i^0(t) \leq 1, \qquad i = 1, 2, \ldots, p \tag{19}$$

$$y^0(t)^T g\left(t, x^0(t), \dot{x}^0(t)\right) = 0, \tag{20}$$

$$y^0(t) \geq 0. \tag{21}$$

From (17) and (21) it follows that $(x^0, \bar{\tau}, y^0, z_1^0, \ldots, z_p^0) \in H$. In view of (18), by Proposition 1, $(u^0, = x^0, \bar{\tau}, y^0, z_1^0, \ldots, z_p^0)$ is a conditionally properly efficient solution of (D). Using (18) we have

$$\int_a^b \left\{ f^i\left(t, x^0(t), \dot{x}^0(t)\right) + \left(x^0(t)^T B_i(t) x^0(t)\right)^{1/2} \right\} dt = \int_a^b \left\{ f^i\left(t, u^0(t), \dot{u}^0(t)\right) + u^0(t)^T B_i(t) z_i^0(t) \right\} dt$$

$$\forall\ i \in \{1, 2, \ldots, p\}. \qquad \blacksquare$$

For validating the converse duality theorem (Theorem 5) we make the assumption that X_2 denotes the space of piecewise differentiable function $x: IR^n$ for which $x(a) = 0 = x(b)$ equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty$ defining D as before. The problem (D) may be rewritten in the form

$$\text{Minimize} - \phi(u, \tau, y) = \left(-\phi^1(u, \tau, y), -\phi^2(u, \tau, y), \ldots, -\phi^2(u, \tau, y)\right)$$

subject to

$$u(a) = \alpha, \qquad u(b) = \beta$$
$$\Theta(t, u(t), \dot{u}(t), \ddot{u}(t), \tau, y(t), \dot{y}(t)) = 0$$
$$y(t) \geq 0, \qquad t \in I,$$

where

$$\phi^i(u, \tau, y) = \int_a^b \left\{ f^i(t, u(t), \dot{u}(t)) + \left(x(t)^T B_i(t) x(t) \right)^{1/2} \right\} dt,$$
$$\forall\ i = 1, 2, \ldots, p$$

and

$$\begin{aligned} \Theta &= \Theta(t, u(t), \dot{u}(t), \ddot{u}(t), \tau, y(t), \dot{y}(t)) \\ &= \sum_{i=1}^{p} \tau_i \left(f_x^i(t, u(t), \dot{u}(t)) + B_i(t) z_i(t) \right) \\ &\quad + \sum_{j=1}^{m} y_j(t)^T g_x^j(t, u(t), \dot{u}(t)) \\ &\quad - D\left[\sum_{i=1}^{p} \tau_i \left(f_{\dot{x}}^i(t, u(t), \dot{u}(t)) + \sum_{j=1}^{m} y_j(t) g_{\dot{x}}^j(t, u(t), \dot{u}(t) \right) \right] \end{aligned}$$

with $u(t) = D^2 u(t)$.

Consider $\Theta(0, u(\cdot), \dot{u}(\cdot), \ddot{u}(\cdot), \tau, y(\cdot)\dot{y}(\cdot))$ as defining a map ψ: $X_2 \times Y \times \Lambda^+ \to A$, where Y is the space of piecewise differentiable function y: $t \to R^m$ and A is a Banach space. A Fritz John Theorem [4, 5] for infinite dimensional multiobjective programming problem may be applied to problem (D) along with the analysis outlined in [11] or [3] for the derivation of optimality conditions. However, some restrictions are required as in [3] on the equality constraint $\Theta(\cdot) = 0$, since infinite dimensional space is involved here. It suffices to assume that the Frechet derivative

$$\psi = \left(\psi_x, \psi_y, \psi_z \right) \text{ has a (weak*) closed range}.$$

THEOREM 5 (Converse Duality). *Let the V-pseudo invexity and V-quasi-invexity conditions of Theorem* 3 *hold. Let* (u^0, τ^0, y^0) *with* $u^0 \in X_2$, $\tau^0 \in \Lambda^+$, *and* $y^0 \in y$ *be a conditionally properly efficient solution of* (D). *Let*

(I) ψ' *have* (*weak**) *closed range,*

(II) *f and g be twice continuously differentiable*

(III) $f_x^i - Df_{\dot{x}}^i$, $i = 1, 2, \ldots, p$ *be linearly independent, and*

(IV) $(\beta(t)^T \Theta_x - D(t)^T \Theta_{\dot{x}} + D^2 \beta(t)^T \Theta_{\ddot{x}})\beta(t) = 0 \Rightarrow \beta(t) = 0,\ t \in I.$

Then, the objective functions of (P) *and* (D) *are equal and* u^0 *is a properly efficient solution of* (P).

Proof. Since (u^0, τ^0, y^0), with $u^0 \in X_2$ and having a (weak*) closed range, is conditionally properly efficient, it is a weak maximum. Hence, there exists $\alpha \in R^P, \gamma, \mu \in R^m$, and piecewise smooth $\beta\colon I \to R^n$ and $\gamma\colon I \to R^m$ satisfying the following Fritz John conditions [4, 5], which are derived by means of the analysis of [3]

$$\sum_{i=1}^{p} \alpha_i \big[f_x^i(t, u^0(t), \dot{u}^0(t)) + B_i(t) z_i^0(t) \big]$$

$$+ \sum_{j=1}^{m} \gamma_j y_j^{0\,T}(t) g_x^j(t, u^0(t), \dot{u}^0(t))$$

$$- D\Bigg[\sum_{i=1}^{p} \alpha_i f_{\dot{x}}^i(t, u^0(t), \dot{u}^0(t)) + \sum_{j=1}^{m} \gamma_j y^j(t)^T g_{\dot{x}}^j(t, u(t), \dot{u}(t)$$

$$-\Big((\beta(t))^T \Theta_x - D(\beta(t))^T \Theta_{\dot{x}} + D^2 \beta(t)^T \Theta_{\ddot{x}} \Big) = 0, \qquad t \in I. \quad (22)$$

$$(\alpha^T e) g - \Big((\beta(t))^T \Theta_y - D\beta(t)^T \Theta_{\dot{y}} \Big) + \delta(t) = 0, \qquad t \in I, \quad (23)$$

$$\beta(t)^T \big(f_x^i(t, u^0(t), \dot{u}^0(t)) + B_i(t) z_i^0(t) \big) - \mu_i = 0 \qquad \forall\, i \in \{1, 2, \ldots, p\} \quad (24)$$

$$\delta(t)^T y(t) = 0, \qquad t \in I, \quad (25)$$

$$\mu^T \tau = 0 \quad (26)$$

$$(\alpha, \beta(t), \delta(t), \mu) > 0, \qquad t \in I, \quad (27)$$

$$(\alpha, \beta(t), \delta(t), \mu) \neq 0, \qquad t \in I, \quad (28)$$

where

$$f = f(t, u^0(t), \dot{u}^0(t)), \qquad g = g(t, u^0(t), \dot{u}^0(t)),$$
$$f_x = f_x(t, u^0(t), \dot{u}^0(t)), \qquad \text{etc.},$$

with all derivatives evaluated at $u = u^0$.

Now following the lines of the proof of Theorem 5 of [1] and Theorem 3 of this paper, we get

$$\int_a^b \left\{ f^i(t, u^0(t), \dot{u}^0(t)) + \left(u^0(t)^T B_i(t) u^0(t) \right)^{1/2} \right\} dt$$
$$= \int_a^b \left\{ f^i(t, u^0(t), \dot{u}^0(t)) + u^0(t)^T B_i(t) u^0(t) \right\} dt$$
$$\forall\, i \in \{1, 2, \ldots, p\}$$

and, by Proposition 1, $u^0(t)$ is a properly efficient solution of (P). ∎

ACKNOWLEDGMENTS

The author is grateful to his supervisor Professor R. N. Mukherjee for his guidance in preparation of this paper. The author is indebted to two unknown referees for useful comments which led to the present improved version of the paper as it stands. The contents of this paper are a part of the author's Ph.D. work under the supervision of Professor R. N. Mukherjee.

REFERENCES

1. C. R. Bector and I. Husain, Duality for multiobjective variational problems, *J. Math. Anal. Appl.* **166** (1992), 214–229.
2. J. M. Borwein, "Optimization with Respect to Partial Ordering," Ph.D. Thesis, Oxford University, 1974.
3. S. Chandra, B. D. Craven, and I. Husain, A class of nondifferentiable continuous programming problem, *J. Math. Anal. Appl.* **107** (1985), 122–131.
4. B. D. Craven, Lagrangian conditions and quasi duality, *Bull. Austral. Math. Soc.* **16** (1977), 325–339.
5. B. D.. Craven, "Lagrangian Conditions, Vector Minimization and Local Duality," Research Report 37, Department of Mathematics, University of Melbourne, Australia, 1980.
6. B. D. Craven, On continuous programming with generalized convexity, *Asia-Pacific J. Oper. Res.* **10** (1993), 219–232.
7. A. M. Geoffrion, Proper efficiency and the theory of vector maximization, *J. Math. Anal. Appl.* **22** (1968), 618–630.
8. M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *J. Math. Anal. Appl.* **80** (1981), 544–550.

9. V. Jeyakumar and B. Mond, On generalized convex mathematical programming, *J. Austral. Math. Soc. Ser. B* **34** (1992), 43–53.
10. S. K. Mishra and R. N. Mukherjee, Duality for multiobjective fractional variational problems, *J. Math. Anal. Appl.* **186** (1994), 711–725.
11. B. Mond, S. Chandra, and I. Husain, Duality for variational problems with invexity, *J. Math. Anal. Appl.* **134** (1988), 322–328.
12. B. Mond and M. A. Hanson, Duality for variational problems, *J. Math. Anal. Appl.* **18** (1967), 355–364.
13. B. Mond, I. Husain, and M. V. Durga Prasad, Duality for a class of nondifferentiable multiobjective programs, *Utilitas Math.* **39** (1991), 3–19.
14. B. Mond, I. Husain, and M. V. Durga Prasad, Duality for a class of non-differentiable multiple objective programming problems, *J. Inform. Optim. Sci.* **9** (1988), 331–341.
15. B. Mond and I. Smart, Duality with invexity for a class of nondifferentiable static and continuous programming problems, *J. Math. Anal. Appl.* **141** (1989), 373–388.
16. R. N. Mukherjee and S. K. Mishra, Sufficient optimality criteria and duality for multiobjective variational problems with V-invexity, *Indian J. Pure Appl. Math.* **25**, No. 8 (1994), 801–813.
17. C. Singh and M. A. Hanson, Generalized proper efficiency in multiobjective programming, *J. Inform. Optim. Sci.* **12** (1991), 139–149.