Chapter 2

Tolerance-based intuitionistic fuzzy-rough set approach for attribute reduction

2.1 Introduction

There are several benefits of intuitionistic fuzzy sets over fuzzy sets are available in the literature. A vague pattern classification can be transformed into a precise and well-defined optimization problem by using intuitionistic fuzzy set approaches. Unlike fuzzy sets, intuitionistic fuzzy sets preserve a precise degree of the uncertainty.

Hereby, we propose a new type of intuitionistic fuzzy lower and upper approximations by applying a tolerance degree on the similarity between two objects and give a novel method along with a suitable algorithm to compute the reduct set [99] of an intuitionistic fuzzy decision system. Although some of the researchers have presented tolerance-based approach [41; 76] for feature selection, but none of them have considered the tolerance-based intuitionistic fuzzy rough set assisted approach. Our proposed approach can handle uncertainty, vagueness, and imprecision in a very effective manner as we propose an intuitionistic fuzzy rough set model by combining two effective tools to handle uncertainty, i.e. intuitionistic fuzzy set and rough set and further it is generalized for feature selection. Apart from all the above-mentioned advantages, we have presented the degree of dependency approach for feature selection based on an intuitionistic fuzzy rough set model. We justify our proposed method by using propositions of lower and upper approximations analogous to rough set theory. Finally, we compare our method with the tolerance-based fuzzy rough set approach for an arbitrary information system and show that our approach gives a better result.

2.2 Tolerance-based fuzzy rough set approach for attribute selection

Let FDS be a fuzzy decision system as defined in introduction chapter. Adding to these ideas, some similarity between two objects for each attribute can be defined. One of the widely used fuzzy similarity relation [42] is defined as:

$$SIM_a(x_i, x_j) = 1 - \frac{|\mu_a(x_i) - \mu_a(x_j)|}{|\mu_{a_{max}} - \mu_{a_{min}}|} \tag{2.1}$$

where, $\mu_a(x_i), \mu_a(x_j)$ are membership grades of objects x_i, x_j respectively and $\mu_{a_{max}}, \mu_{a_{min}}$ are maximum and minimum membership grades for an attribute $a \in C$ respectively.

Now, we can extend the concept of [41; 76] for tolerance-based fuzzy rough feature selection as follows:
For a subset of attributes P,

$$(x_i, x_j) \in SIM_P^\delta \text{ iff } \prod_{a \in P} SIM_a(x_i, x_j) \geq \delta \tag{2.2}$$

where, δ is a similarity threshold, which gives required level of similarity for inclusion within tolerance classes. Now, tolerance classes are generated by fuzzy similarity relation as follows:

$$SIM_P^\delta(x_i) = \{x_j \in U | (x_i, x_j) \in SIM_P^\delta\} \tag{2.3}$$

Now, lower and upper approximations of $X \subseteq U$ are defined as

$$\begin{aligned} \underline{P^\delta} X &= \{x_i | SIM_P^\delta(x_i) \subseteq X\} \\ \overline{P^\delta} X &= \{x_i | SIM_P^\delta(x_i) \cap X \neq \phi\} \end{aligned} \tag{2.4}$$

The ordered pair $(\underline{P^\delta} X, \overline{P^\delta} X)$ is called tolerance fuzzy rough set. Now, the positive region can be defined by:

$$POS_P^{\delta}(Q) = \cup_{X \in U/Q} \underline{P^{\delta}} X \tag{2.5}$$

$POS_P^{\delta}(Q)$ contains those objects that can be distinguished to classes of U/Q using information contained in set of attributes P. Using the definition of the fuzzy positive region, the dependency function of decision attribute Q over the set of conditional attributes P can be defined as

$$d_P^{\delta}(Q) = \frac{|POS_P^{\delta}(Q)|}{|U|} \tag{2.6}$$

At every step, we add one attribute in the reduct set and calculate the degree of dependency, when we obtain no increment in the degree of dependency, the algorithm terminates and hence, we get the reduct.

2.3 Tolerance-based Intuitionistic Fuzzy Rough Set approach for attribute reduction

In this approach, we find similarity between two objects with respect to an attribute or subset of attributes. Feng & Li (2013) [25] defined a similarity relation by

$$sim_a(x_i, x_j) = 1 - \sqrt{\alpha(\mu_a(x_i) - \mu_a(x_j))^2 + \beta(\nu_a(x_i) - \nu_a(x_j))^2 + \gamma(\pi_a(x_i) - \pi_a(x_j))^2} \tag{2.7}$$

where, $\mu_a(x_i), \nu_a(x_i)$ and $\pi_a(x_i)$ are membership, non-membership and hesitancy degrees of an object with respect to attribute "a" respectively and α, β and γ are weighted factors. In IFDS, the values of these parameters can be selected according to the requirement of different users along with following conditions

(i) $\alpha \geq \beta > \gamma$

(ii) $\alpha + \beta + \gamma = 1$

(iii) $0 \leq \alpha, \beta, \gamma \leq 1$

For a subset of attributes P, similarity relation between two objects is defined as follows:

$$(x_i, x_j) \in sim_P^\delta \text{ iff } \prod_{b \in P} sim_b(x_i, x_j) \geq \delta \tag{2.8}$$

where, δ is a similarity threshold. Choice of δ permits attribute values to differ to a limited extent, therefore we allow close attribute values of different objects to be considered as identical. It is user/expert dependent. We define tolerance class of an object x_i based on above similarity relation as follows:

$$sim_P^\delta(x_i) = \{x_j \in U | (x_i, x_j) \in sim_P^\delta\} \tag{2.9}$$

Now, lower and upper approximations of $X \in U$ are defined as follows:

$$\begin{aligned} \underline{approxP^\delta} X &= \{x_i | sim_P^\delta(x_i) \subseteq X\} \\ \overline{approxP^\delta} X &= \{x_i | sim_P^\delta(x_i) \cap X \neq \phi\} \end{aligned} \tag{2.10}$$

The ordered pair $(\underline{approxP^\delta} X, \overline{approxP^\delta} X)$ is called an intuitionistic fuzzy tolerance rough set. Let Q be a set of attributes generating equivalence relation over U. Therefore, the positive region and degree of dependency can be defined as:

$$POS_P^\delta(Q) = \bigcup_{X \in U/Q} \underline{approxP^\delta} X \tag{2.11}$$

$$\Gamma_P^\delta(Q) = \frac{|POS_P^\delta(Q)|}{|U|} \tag{2.12}$$

At each step, we insert one attribute in the obtained subset and calculate the degree of dependency of decision attribute over a new set of conditional attributes. In case of no increment in the degree of dependency, the process stops and we get the required reduct.

A generalization of rough sets (as proposed by Pawlak [67; 68]) is presented in an arbitrary universe of discourse by extending the concepts of crisp sets to intuitionistic fuzzy sets as follows:

Theorem 2.3.1 *Let* $(U, C \cup D, V_{IF}, IF)$ *be an IFDS. Let* $P \subseteq C$ *and* $X \subseteq U$, *then* $\underline{approxP^{\delta}}X \subseteq X \subseteq \overline{approxP^{\delta}}X$

Proof: Let $y \in \underline{approxP^{\delta}}X \Rightarrow sim_P^{\delta}(y) \subseteq X$
Since $y \in sim_P^{\delta}(y)$, hence $y \in X$. Therefore $\underline{approxP^{\delta}}X \subseteq X$.
Now, let $y \in X$, Since $y \in sim_P^{\delta}(y) \Rightarrow sim_P^{\delta}(y) \cap X \neq \phi \Rightarrow y \in \overline{approxP^{\delta}}X \Rightarrow X \subseteq \overline{approxP^{\delta}}X$.
Hence, $\underline{approxP^{\delta}}X \subseteq X \subseteq \overline{approxP^{\delta}}X$

Theorem 2.3.2 *Let* $(U, C \cup D, V_{IF}, IF)$ *be an IFDS. Let* $P_1 \subseteq P_2 \subseteq C$ *and* $X \subseteq U$, *then*

(i) $\underline{approxP_1^{\delta}}X \subseteq \underline{approxP_2^{\delta}}X$

(ii) $\overline{approxP_2^{\delta}}X \subseteq \overline{approxP_1^{\delta}}X$

Proof:

(i) Let $y \in \underline{approxP_1^{\delta}}X$, then $sim_{P_1}^{\delta}(y) \subseteq X$. Since $P_1 \subseteq P_2 \Rightarrow sim_{P_2}^{\delta}(y) \subseteq sim_{P_1}^{\delta}(y)$. Thus, $sim_{P_2}^{\delta}(y) \subseteq X \Rightarrow y \in \underline{approxP_2^{\delta}}X$. Hence, $\underline{approxP_1^{\delta}}X \subseteq \underline{approxP_2^{\delta}}X$

(ii) Let $y \in \overline{approxP_2^{\delta}}X$ then $sim_{P_2}^{\delta}(y) \cap X \neq \phi$. Since $P_1 \subseteq P_2 \Rightarrow sim_{P_2}^{\delta}(y) \subseteq sim_{P_1}^{\delta}(y)$. Thus, $sim_{P_1}^{\delta}(y) \cap X \neq \phi \Rightarrow y \in \overline{approxP_1^{\delta}}X$. Hence, $\overline{approxP_2^{\delta}}X \subseteq \overline{approxP_1^{\delta}}X$

Theorem 2.3.3 *Let* $(U, C \cup D, V_{IF}, IF)$ *be an IFDS. Let* $P \subseteq C, \delta_1 \leq \delta_2$ *and* $X \subseteq U$, *then*

(i) $\underline{approxP^{\delta_1}}X \subseteq \underline{approxP^{\delta_2}}X$

(ii) $\overline{approxP^{\delta_2}}X \subseteq \overline{approxP^{\delta_1}}X$

Proof:

(i) Let $y \in \underline{approxP^{\delta_1}}X$, then $sim_P^{\delta_1}(y) \subseteq X$. If $z \in sim_P^{\delta_2}(y)$, then $(y,z) \in sim_P^{\delta_2}$ $\iff \prod_{a \in P} sim_a(y,z) \geq \delta_2 \iff \prod_{a \in P} sim_a(y,z) \geq \delta_1$ (Since $\delta_2 \geq \delta_1$) $\iff (y,z) \in sim_P^{\delta_1} \iff z \in sim_P^{\delta_1}(y) \Rightarrow sim_P^{\delta_2}(y) \subseteq sim_P^{\delta_1}(y) \Rightarrow sim_P^{\delta_2}(y) \subseteq X$
$\Rightarrow y \in \underline{approxP^{\delta_2}}X$
Hence, $\underline{approxP^{\delta_1}}X \subseteq \underline{approxP^{\delta_2}}X$

(ii) Let $y \in \overline{approxP^{\delta_2}}X$, then $sim_P^{\delta_2}(y) \cap X \neq \phi$. Since $sim_P^{\delta_2}(y) \subseteq sim_P^{\delta_1}(y) \Rightarrow sim_P^{\delta_1}(y) \cap X \neq \phi \Rightarrow y \in \overline{approxP^{\delta_1}}X$
Hence, $\overline{approxP^{\delta_2}}X \subseteq \overline{approxP^{\delta_1}}X$

Theorem 2.3.4 $\underline{approxP^{\delta}}(X^C) = (\overline{approxP^{\delta}}(X))^C$, *where* X^C *denotes complement of set* X.

Proof: $z \in \underline{approxP^{\delta}}(X^C) \iff sim_P^{\delta}(z) \subseteq X^C \iff sim_P^{\delta}(z) \cap X = \phi \iff z \notin \overline{approxP^{\delta}}(X) \iff z \in (\overline{approxP^{\delta}}(X))^C$. Hence, $\underline{approxP^{\delta}}(X^C) = (\overline{approxP^{\delta}}(X))^C$

Theorem 2.3.5 *Let* $Y \subseteq U$ *be another set of objects, then following properties holds.*

(i) $\underline{approxP^{\delta}}(X \cap Y) = \underline{approxP^{\delta}}(X) \cap \underline{approxP^{\delta}}(Y)$

(ii) $\overline{approxP^{\delta}}(X \cup Y) = \overline{approxP^{\delta}}(X) \cup \overline{approxP^{\delta}}(Y)$

Proof:

(i) $z \in \underline{approxP^{\delta}}(X \cap Y) \iff sim_P^{\delta}(z) \subseteq X \cap Y \iff sim_P^{\delta}(z) \subseteq X$ and $sim_P^{\delta}(z) \subseteq Y \iff z \in \underline{approxP^{\delta}}(X)$ and $z \in \underline{approxP^{\delta}}(Y) \iff z \in \underline{approxP^{\delta}}(X) \cap \underline{approxP^{\delta}}(Y)$ Hence, $\underline{approxP^{\delta}}(X \cap Y) = \underline{approxP^{\delta}}(X) \cap \underline{approxP^{\delta}}(Y)$

(ii) $z \in \overline{approxP^{\delta}}(X \cup Y) \iff sim_P^{\delta}(z) \cap (X \cup Y) \neq \phi \iff (sim_P^{\delta}(z) \cap X) \cup (sim_P^{\delta}(z) \cap Y) \neq \phi$. Either $sim_P^{\delta}(z) \cap X \neq \phi$ or $sim_P^{\delta}(z) \cap Y \neq \phi$, either $z \in \overline{approxP^{\delta}}(X)$ or $z \in \overline{approxP^{\delta}}(Y)$. Therefore, $z \in \overline{approxP^{\delta}}(X) \cup \overline{approxP^{\delta}}(Y)$.
Hence, $\overline{approxP^{\delta}}(X \cup Y) = \overline{approxP^{\delta}}(X) \cup \overline{approxP^{\delta}}(Y)$

Theorem 2.3.6 $\underline{approxP^{\delta}}(U) = U = \overline{approxP^{\delta}}(U)$ *and* $\underline{approxP^{\delta}}(\phi) = \phi = \overline{approxP^{\delta}}(\phi)$

Proof: It is easy to check.

Theorem 2.3.7 $\underline{approxP^{\delta}}(sim_P^{\delta}(x)) = sim_P^{\delta}(x) = \overline{approxP^{\delta}}(sim_P^{\delta}(x))$

Proof: Since $\underline{approxP^{\delta}}(X) \subseteq X \subseteq \overline{approxP^{\delta}}(X)$

Now replacing X by $sim_P^{\delta}(x)$, we get

$\underline{approxP^{\delta}}(sim_P^{\delta}(x)) \subseteq sim_P^{\delta}(x) \subseteq \overline{approxP^{\delta}}(sim_P^{\delta}(x))$

Now, we have to show that,

$sim_P^{\delta}(x) \subseteq \underline{approxP^{\delta}}(sim_P^{\delta}(x))$ and $sim_P^{\delta}(x) \supseteq \overline{approxP^{\delta}}(sim_P^{\delta}(x))$

$$If z \in sim_P^{\delta}(x), \text{ then } \prod_{a \in P} sim_a(x,z) \geq \delta \tag{2.13}$$

$$If y \in sim_P^{\delta}(z), \text{ then } \prod_{a \in P} sim_a(z,y) \geq \delta \tag{2.14}$$

If, $sim_P^{\delta}(x)$ is a T-equivalence relation. Then,

$$min\{sim_P^{\delta}(x,z), sim_P^{\delta}(z,y)\} \leq sim_P^{\delta}(x,y), \forall x,y,z \in U \tag{2.15}$$

From, Eq.(2.13),Eq.(2.14) and Eq.(2.15),we can conclude that $\prod_{a \in P} sim_P^{\delta}(x,y) \geq \delta$ then, $y \in sim_P^{\delta}(x)$, hence, $sim_P^{\delta}(z) \subseteq sim_P^{\delta}(x)$, then $z \in \underline{approxP^{\delta}}(sim_P^{\delta}(x))$ hence, $sim_P^{\delta}(x) \subseteq \underline{approxP^{\delta}}(sim_P^{\delta}(x))$

Now, if $z \in \overline{approxP^{\delta}}(sim_P^{\delta}(x))$, then $sim_P^{\delta}(z) \cap sim_P^{\delta}(x) \neq \phi$

then $\exists y \in U$ such that $y \in sim_P^{\delta}(z) \cap sim_P^{\delta}(x)$, then $y \in sim_P^{\delta}(z)$ and $y \in sim_P^{\delta}(x)$ then $\prod_{a \in P} sim_P^{\delta}(y,z) \geq \delta$ and $\prod_{a \in P} sim_P^{\delta}(y,x) \geq \delta$. Now, using Eq.(2.15), we can conclude that $\prod_{a \in P} sim_P^{\delta}(x,z) \geq \delta$,

then $z \in sim_P^{\delta}(x)$, hence, $\overline{approxP^{\delta}}(sim_P^{\delta}(x)) \subseteq sim_P^{\delta}(x)$.

Theorem 2.3.8 *(i)* $\underline{approxP^{\delta}}(\{x\}^C) = (sim_P^{\delta}(x))^C$

(ii) $\overline{approxP^{\delta}}(\{x\}) = sim_P^{\delta}(x)$

Proof:

(i) $z \in \underline{approxP^{\delta}}(\{x\}^C)$ iff $sim_P^{\delta}(z) \subseteq \{x\}^C \iff sim_P^{\delta}(z) \not\subset \{x\} \iff z \notin sim_P^{\delta}(x) \iff z \in (sim_P^{\delta}(x))^C$.

(ii) $z \in \overline{approxP^{\delta}}(\{x\})$ iff $sim_P^{\delta}(z) \cap \{x\} \neq \phi$ iff $x \in sim_P^{\delta}(z)$ iff $z \in sim_P^{\delta}(x)$

Theorem 2.3.9 *(i)* $\underline{approxP^{\delta}}(\underline{approxP^{\delta}}(X)) = \underline{approxP^{\delta}}(X)$

(ii) $\overline{approxP^{\delta}}(\overline{approxP^{\delta}}(X)) = \overline{approxP^{\delta}}(X)$

Proof:

(i) Since, $\underline{approxP^{\delta}}(X) \subseteq X$, now, replacing X by $\underline{approxP^{\delta}}(X)$, we get $\underline{approxP^{\delta}}(\underline{approxP^{\delta}}(X)) \subseteq \underline{approxP^{\delta}}(X)$
Now, let $y \in \underline{approxP^{\delta}}(X)$, we have to show that $y \in \underline{approxP^{\delta}}(\underline{approxP^{\delta}}(X))$.

$$Ify \in \underline{approxP^{\delta}}(X), \text{ then } sim_P^{\delta}(y) \subseteq X \tag{2.16}$$

$$Letz \in sim_P^{\delta}(y), \text{ then } \prod_{a\in P} sim_a(z, y) \geq \delta \tag{2.17}$$

$$Ifu \in sim_P^{\delta}(z), \text{ this implies that } \prod_{a\in P} sim_a(u, z) \geq \delta \tag{2.18}$$

If sim_P^{δ} is an equivalence relation. Then from Eq.(2.17),Eq.(2.18) and T-transitivity property of sim_P^{δ}, we get $\prod_{a\in P} sim_P(u, y) \geq \delta$, it implies that $u \in sim_P^{\delta}(y)$. From Eq.(2.16), we get $u \in X$.
Since $u \in sim_P^{\delta}(z)$ and $u \in X$, this implies that $sim_P^{\delta}(z) \subseteq X$, then $z \in \underline{approxP^{\delta}}(X)$.
Since $z \in sim_P^{\delta}(y)$ and $z \in \underline{approxP^{\delta}}(X)$. This gives that $sim_P^{\delta}(y) \subseteq \underline{approxP^{\delta}}(X)$, this implies that, $y \in \underline{approxP^{\delta}}(\underline{approxP^{\delta}}(X))$,
hence, $\underline{approxP^{\delta}}(X) \subseteq \underline{approxP^{\delta}}(\underline{approxP^{\delta}}(X))$
Hence, we get the required result.

(ii) Since $X \subset \overline{approxP^\delta}(X)$, then replacing X by $\overline{approxP^\delta}(X)$, we get

$\overline{approxP^\delta}(X) \subseteq \overline{approxP^\delta}(\overline{approxP^\delta}(X))$

Now, let $y \in \overline{approxP^\delta}(\overline{approxP^\delta}(X))$, this implies that

$sim_P^\delta(y) \cap \overline{approxP^\delta}(X) \neq \phi$.

Let us consider $z \in sim_P^\delta(y) \cap \overline{approxP^\delta}(X)$, then $z \in sim_P^\delta(y)$ and $z \in \overline{approxP^\delta}(X)$, this gives that $\prod_{a\in P} sim_a(z, y) \geq \delta$ and $sim_P^\delta(z) \cap X \neq \phi$, this implies that $\exists u$, such that $u \in sim_P^\delta(z)$ and $u \in X$, then $\prod_{a\in P} sim_a(u, z) \geq \delta$ and $u \in X$. Since, $\prod_{a\in P} sim_a(z, y) \geq \delta$ and $\prod_{a\in P} sim_a(u, z) \geq \delta$,

if sim_P^δ is T-transitive, then we can conclude that $\prod_{a\in P} sim_a(u, y) \geq \delta$, then $u \in sim_P^\delta(y)$. Since, $u \in sim_P^\delta(y)$ and $u \in X$, this provides that $sim_P^\delta(y) \cap X \neq \phi$, this implies that $y \in \overline{approxP^\delta}(X)$. Hence $\overline{approxP^\delta}(\overline{approxP^\delta}(X)) \subseteq \overline{approxP^\delta}(X)$.

Hence, we get the result.

Theorem 2.3.10 *(i)* $\underline{approxP^\delta}(X) \subseteq \overline{approxP^\delta}(\underline{approxP^\delta}(X)) \subseteq \overline{approxP^\delta}(X)$

(ii) $\underline{approxP^\delta}(X) \subseteq \underline{approxP^\delta}(\overline{approxP^\delta}(X)) \subseteq \overline{approxP^\delta}(X)$

Proof:

(i) Since, $X \subset \overline{approxP^\delta}(X)$, then, replacing X by $\underline{approxP^\delta}(X)$, we get

$\underline{approxP^\delta}(X) \subseteq \overline{approxP^\delta}(\underline{approxP^\delta}(X))$

Now, let $y \in \overline{approxP^\delta}(\underline{approxP^\delta}(X))$, it results in, $sim_P^\delta(y) \cap \underline{approxP^\delta}(X) \neq \phi$.

it implies that $\exists z \in U$, such that $z \in sim_P^\delta(y)$ and $z \in \underline{approxP^\delta}(X)$,

then $z \in sim_P^\delta(y)$ and $sim_P^\delta(z) \subseteq X$, this gives that

$z \in sim_P^\delta(y)$ and $z \in sim_P^\delta(z) \subseteq X$, this implies that $z \in sim_P^\delta(y)$ and $z \in X$

this provides that $sim_P^\delta(y) \cap X \neq \phi$, then $y \in \overline{approxP^\delta}(X)$.

Hence $\overline{approxP^\delta}(\underline{approxP^\delta}(X)) \subseteq \overline{approxP^\delta}(X)$

(ii) Since, $\underline{approxP^\delta}(X) \subseteq X$, then replacing X by $\overline{approxP^\delta}(X)$, we can conclude that $\underline{approxP^\delta}(\overline{approxP^\delta}(X)) \subseteq \overline{approxP^\delta}(X)$.

Now, let $y \in \underline{approxP^\delta}(X)$, this implies that $sim_P^\delta(y) \subseteq X$.

Since $X \subseteq \overline{approxP^\delta}(X)$, hence, $sim_P^\delta(y) \subseteq \overline{approxP^\delta}(X)$, this results in, $y \in$

$\underline{approxP^{\delta}}(\overline{approxP^{\delta}}(X))$

Hence, $\underline{approxP^{\delta}}(X) \subseteq \underline{approxP^{\delta}}(\overline{approxP^{\delta}}(X))$

2.4 Algorithm for Tolerance-based Intuitionistic Fuzzy Rough Reduction

In this section, we establish a quick reduct algorithm for feature selection by using a tolerance-based degree of dependency, i.e. $\Gamma_P^{\delta}(Q)$. The algorithm starts with a null set and adds those attributes one by one, which provide a maximum increase in the degree of dependency of decision attribute over a subset of conditional attributes until it achieves highest potential value for any dataset (it will be 1 in case of a consistent system). This algorithm produces a close-to-minimal reduct of a decision system without exhaustively checking all possible subsets of conditional attributes, which is the key advantage of our proposed algorithm. The resulting algorithm can be given as follows:

Tolerance Based Intuitionistic Fuzzy Rough Reduct(C, D, δ)

C, Collection of all conditional attributes;

D, Collection of all decision attributes;

δ, the similarity threshold

$S \Rightarrow \{\}; \Gamma_{best}^{\delta} = 0$

do

$L \leftarrow S$

$\Gamma_{prev}^{\delta} = \Gamma_{best}^{\delta}$

$\forall x \in (C \setminus S)$

if $\Gamma_{S \cup \{x\}}^{\delta}(D) > \Gamma_L^{\delta}(D)$

$L \leftarrow S \cup \{x\}$

$\Gamma_{best}^{\delta} = \Gamma_L^{\delta}$

$S \leftarrow L$

until $\Gamma_{best}^{\delta} == \Gamma_{prev}^{\delta}$

return S

In the next section, we apply the above algorithm on an example dataset to clearly demonstrate the proposed concept.

2.5 Illustrative Example

In order to illustrate our proposed approach, an arbitrary example of fuzzy decision system is given in Table 2.1. Here, the table consists of six conditional attributes $\{a, b, c, d, e, f\}$, one decision attribute $\{Q\}$ and six objects $\{x_1, x_2, x_3, x_4, x_5, x_6\}$.

Now we apply tolerance-based fuzzy rough set based attribute selection as follows:

Since x_1, x_3 and x_6 have decision class value as 1, while x_2, x_4 and x_5 have 0, hence decision classes can be given by:

$$U/Q = \{\{x_1, x_3, x_6\}, \{x_2, x_4, x_5\}\}$$

Setting $A = \{a\}$ and $\delta = 0.8$, a tolerance class based on a similarity measure defined as Eq. (2.1) for attribute set A is given by:

$U/SIM_A^\delta = \{x_2, x_6\}$

Similarly, for $B = \{b\}, C = \{c\}, D = \{d\}, E = \{e\}, F = \{f\}$, we get tolerance classes as follows:

$U/SIM_B^\delta = \{\{x_1, x_3\}, \{x_2, x_5\}, \{x_4, x_6\}\}$

$U/SIM_C^\delta = \{x_5, x_6\}$

$U/SIM_D^\delta = \{x_1, x_2\}$

$U/SIM_E^\delta = \{x_1, x_5\}$

$U/SIM_F^\delta = \{x_1, x_3\}$

On the basis of A, lower approximations of decision classes can be obtained by

$\underline{A^\delta}\{1, 3, 6\} = \{x_i | SIM_A^\delta(x_i) \subseteq \{1, 3, 6\}\} = \phi$

$\underline{A^\delta}\{2, 4, 5\} = \{x_i | SIM_A^\delta(x_i) \subseteq \{2, 4, 5\}\} = \phi$

So, positive region can be calculated by:

$POS_A^\delta(Q) = \underline{A^\delta}\{1, 3, 6\} \cup \underline{A^\delta}\{2, 4, 5\} = \phi \cup \phi = \phi$

So, the degree of dependency of decision attribute Q over conditional attribute set A is calculated as:

$d^{\delta}_{A}(Q) = \frac{|POS^{\delta}_{A}(Q)|}{|U|} = \frac{0}{6}$

Similarly, for $B = \{b\}, C = \{c\}, D = \{d\}, E = \{e\}, F = \{f\}$,

$d^{\delta}_{B}(Q) = \frac{4}{6}, d^{\delta}_{C}(Q) = \frac{0}{6}, d^{\delta}_{D}(Q) = \frac{0}{6}, d^{\delta}_{E}(Q) = \frac{0}{6}, d^{\delta}_{F}(Q) = \frac{2}{6}$

Now, B is added to the reduct set. On adding other features with reduct set B one by one, other degree of dependencies are:

Table 2.1: Fuzzy Decision Table

Attributes / Objects	a	b	c	d	e	f	Q
x_1	0.4	0.4	1	0.8	0.4	0.2	1
x_2	0.6	1	0.6	0.8	0.2	0.2	0
x_3	0.8	0.4	0.4	0.6	1	1	1
x_4	1	0.8	0.2	1	0.6	0.4	0
x_5	0.2	1	0.8	0.4	0.4	0.6	0
x_6	0.6	0.6	0.8	0.2	0.8	0.8	1

$d^{\delta}_{\{a,b\}}(Q)=\frac{6}{6}, d^{\delta}_{\{b,c\}}(Q) = \frac{6}{6}, d^{\delta}_{\{b,d\}}(Q) = \frac{6}{6}, d^{\delta}_{\{b,e\}}(Q) = \frac{6}{6}, d^{\delta}_{\{b,f\}}(Q) = \frac{6}{6}$

As degree of dependency cannot exceed 1, therefore $\{\{a, b\}\vee\{b, c\} \vee \{b, d\} \vee \{b, e\} \vee \{b, f\}\}$ is the reduct set of given information system.

The intersection of all the reduct sets is defined as Core. Hence $\{b\}$ is the core of given dataset.

Now we convert the above FDS into IFDS by using Jurio et.al. [46] concept with hesitancy degree as 0.2. The transformed decision system is given in Table 2.2.

Table 2.2: Intuitionistic Fuzzy Decision Table

Features / Objects	a	b	c	d	e	f	Q
x_1	$\langle 0.32, 0.48 \rangle$	$\langle 0.32, 0.48 \rangle$	$\langle 0.80, 0.00 \rangle$	$\langle 0.64, 0.16 \rangle$	$\langle 0.32, 0.48 \rangle$	$\langle 0.16, 0.64 \rangle$	1
x_2	$\langle 0.48, 0.32 \rangle$	$\langle 0.80, 0.00 \rangle$	$\langle 0.48, 0.32 \rangle$	$\langle 0.64, 0.16 \rangle$	$\langle 0.16, 0.64 \rangle$	$\langle 0.16, 0.64 \rangle$	0
x_3	$\langle 0.64, 0.16 \rangle$	$\langle 0.32, 0.48 \rangle$	$\langle 0.32, 0.48 \rangle$	$\langle 0.48, 0.32 \rangle$	$\langle 0.80, 0.00 \rangle$	$\langle 0.80, 0.00 \rangle$	1
x_4	$\langle 0.80, 0.00 \rangle$	$\langle 0.64, 0.16 \rangle$	$\langle 0.16, 0.64 \rangle$	$\langle 0.80, 0.00 \rangle$	$\langle 0.48, 0.32 \rangle$	$\langle 0.32, 0.48 \rangle$	0
x_5	$\langle 0.16, 0.64 \rangle$	$\langle 0.80, 0.00 \rangle$	$\langle 0.64, 0.16 \rangle$	$\langle 0.32, 0.48 \rangle$	$\langle 0.32, 0.48 \rangle$	$\langle 0.48, 0.32 \rangle$	0
x_6	$\langle 0.48, 0.32 \rangle$	$\langle 0.48, 0.32 \rangle$	$\langle 0.64, 0.16 \rangle$	$\langle 0.16, 0.64 \rangle$	$\langle 0.64, 0.16 \rangle$	$\langle 0.64, 0.16 \rangle$	1

The decision classes of above information system are $U/Q = \{x_1, x_3, x_6\}, \{x_2, x_4, x_5\}$ Setting $A = \{a\}, B = \{b\}, C = \{c\}, D = \{d\}, E = \{e\}, F = \{f\}$ and taking $\alpha = 0.4, \beta = 0.4, \gamma = 0.2$ and $\delta = 0.8$, we obtain tolerance classes as:

$U/SIM_A^\delta = \{\{x_1, x_2, x_6\}, \{x_1, x_5\}, \{x_2, x_3, x_6\}, \{x_3, x_4\}\}$
$U/SIM_B^\delta = \{\{x_1, x_3, x_6\}, \{x_2, x_4, x_5\}\}$
$U/SIM_C^\delta = \{\{x_1, x_5, x_6\}, \{x_2, x_3\}, \{x_2, x_5, x_6\}, \{x_3, x_4\}\}$
$U/SIM_D^\delta = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_3, x_5\}, \{x_5, x_6\}\}$
$U/SIM_E^\delta = \{\{x_1, x_2\}, \{x_1, x_4, x_5\}, \{x_3, x_6\}, \{x_4, x_6\}\}$
$U/SIM_F^\delta = \{\{x_1, x_2, x_4\}, \{x_3, x_6\}, \{x_4, x_5\}, \{x_5, x_6\}\}$

Now, lower approximations of decision classes for attribute set A can be given by:
$\underline{approxA^\delta}\{1, 3, 6\} = \{x_i | SIM_A^\delta(x_i) \subseteq \{1, 3, 6\}\} = \phi$
$\underline{approxA^\delta}\{2, 4, 5\} = \{x_i | SIM_A^\delta(x_i) \subseteq \{2, 4, 5\}\} = \phi$
So, the positive region is calculated by:
$POS_A^\delta(Q) = \phi \cup \phi = \phi$
Therefore, degree of dependency can be obtained as:
$\Gamma_A^\delta(Q) = \frac{0}{6}$
Similarly, degrees of dependencies for other attributes are:

$\Gamma^{\delta}_{B}(Q) = \frac{6}{6}, \Gamma^{\delta}_{C}(Q) = \frac{0}{6}, \Gamma^{\delta}_{D}(Q) = \frac{0}{6}, \Gamma^{\delta}_{E}(Q) = \frac{2}{6}, \Gamma^{\delta}_{F}(Q) = \frac{4}{6}$

Since the degree of dependency can never exceed 1. Hence, algorithm terminates and we get reduct set of the given decision system as $\{b\}$.

2.6 Conclusion

In this chapter, we have given a novel approach for attribute reduction by using tolerance-based intuitionistic fuzzy rough set concept. We have defined lower and upper approximations against a threshold value and presented a method to calculate degree of dependency of decision attribute over a subset of conditional attributes by using the tolerance-based intuitionistic fuzzy rough set for attribute reduction. Moreover, we have validated supporting theorems based on lower and upper approximations. Furthermore, we have applied our proposed algorithm to an example data set and a comparison has been presented with the tolerance-based fuzzy rough set method. We observed that with the previous algorithm the obtained reduct set was $\{\{a,b\} \vee \{b,c\} \vee \{b,d\} \vee \{b,e\} \vee \{b,f\}\}$ and after applying our proposed method, the reduct was $\{b\}$. This clearly indicates the superiority of our proposed work. It is obvious from the given example that our model works fine when trying to discover the smallest reduct from a decision system. Moreover, reduct of the decision system can be improved by adjusting the parameter so that the ability of the model to handle tolerance for fault or noise may increase. It is also observed that the proposed algorithm is capable to handle uncertainty, vagueness as well as the noise of the information system.

Our approach would be useful for selecting the most predictive and non-redundant features for machine learning tasks and enhance the interpretability of datasets for the applications in the field of expert and intelligent systems. Nowadays, high-dimensional intuitionistic fuzzy information systems are generated in many areas related to expert and intelligent systems, which could be directly handled in an effective way by using our proposed approach. Performances of different classifiers could be enhanced by using our approach as it selects the most relevant and non-redundant features. However, the proposed approach, which considers the intuitionistic fuzzy set concept, may cause a relatively high computational complexity. We have converted FDS into IFDS with a fixed value of degree of hesitancy, which may not be always possible for real-world datasets.

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