Chapter 7

Representability of fuzzy biorders and fuzzy weak orders

7.1 Introduction

The idea of providing a representation for a binary relation $\mathcal{R}$ between two non empty sets A and X was formulated by Guttman[46] in 1944, by proposing two functions $f : A \to \mathbb{R}$ and $g : X \to \mathbb{R}$ such that

$$a\mathcal{R}x \Leftrightarrow f(a) > g(x),$$

for each $a \in A$ and $x \in X$. In 1969, Ducamp and Falmagne[36] showed that if A and X both are finite, then the existence of such type of functions f and g for a binary relation $\mathcal{R}$ between A and X is equivalent to the following condition:

$$a\mathcal{R}x \text{ and } b\mathcal{R}y \Rightarrow a\mathcal{R}y \text{ or } b\mathcal{R}x, \tag{7.1}$$

for each $a, b \in A$ and $x, y \in X$.

A relation satisfying the condition (7.1) (called as Ferrers condition[79, 91]) is said to be a biorder. It has been proved by Doignon et al.[34] that the condition (7.1) on $\mathcal{R}$ is equivalent to the condition

$$\mathcal{R}\mathcal{R}^d\mathcal{R} \subseteq \mathcal{R}, \tag{7.2}$$

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where $\mathcal{R}^d$ denotes the dual of $\mathcal{R}$. They also studied representability of biorders and interval orders. Representations of different types of ordering have been studied by several authors(see e.g., [19, 82, 86]).

The fuzzy analogues of the Ferrers conditions (7.1) and (7.2), which are given as follows:

$$\min\{\mathcal{R}(a,x),\mathcal{R}(b,y)\} \leq \max\{\mathcal{R}(a,y),\mathcal{R}(b,x)\}, \quad \text{for each } a,b \in A \text{ and } x,y \in X \tag{7.3}$$

and

$$\mathcal{R} o_T \mathcal{R}^d o_T \mathcal{R} \subseteq \mathcal{R} \tag{7.4}$$

are no more equivalent in general. A condition under which (7.3) and (7.4) are equivalent has been provided by Fodor[41]. Keeping this in view, in literature, there are two different definitions of fuzzy biorders, corresponding to these two non-equivalent Ferrers conditions(also called $T-$Ferrers conditions)[41, 97]. Comparative studies of $T-$Ferrers relations, fuzzy biorders and fuzzy interval orders have been carried out by several authors(see e.g., [28, 30]). Baets and Walle[10] had introduced the notion of $T-$fuzzy interval orders and studied two particular types of $T-$fuzzy interval orders: Weak and Strong fuzzy interval orders.

The study of fuzzy weak orders with respect to a left continuous t-norm T and their representability by the residual implication operator associated with T (called $T-$representable fuzzy weak orders) has been done by several authors(see e.g., [11, 98]). Characterizations for a T_M-representable(also called Gödel representable) fuzzy weak orders and for the fuzzy relation which can be written as the union or intersection of a finite family of T_M-representable(or Gödel representable) fuzzy weak orders have been obtained by Baets et al.[11]. Characterizations for a T_P-representable fuzzy weak order and for finite intersections of fuzzy weak orders with respect to any left continuous t-norm T have been obtained by Sali et al.[98].

The representability of a fuzzy total preorder additive fuzzy preference structure without incomparability and a compatible fuzzy semiorder, in terms of the $\alpha-$cuts of their corresponding fuzzy weak preference relation have been respectively studied by Agud et al.[1] and Induráin et al.[52].

In this chapter, we have studied representability of both fuzzy biorders and fuzzy weak orders. It is observed that union of a finite family of fuzzy weak orders with respect to T is a fuzzy quasi-transitive relation with respect to T. In the last theorem, we have obtained a characterization for a T_L-representable fuzzy weak order.

Now we recall some definitions and results which will be used throughout the chapter.

Definition 7.1. [123] Let $\mathcal{R}$ be a fuzzy relation between A and X. Then for $\alpha \in [0, 1]$, the *α-cut* $\mathcal{R}_\alpha$ is given by

$$\mathcal{R}_\alpha = \{(a, x) \in A \times X : \mathcal{R}(a, x) \geq \alpha\}.$$

Note that each $\mathcal{R}_\alpha$ is a binary relation between A and X.

Definition 7.2. [84] Let $\mathcal{R}$ be a fuzzy relation on A. Then its *strict part* $P_\mathcal{R}$ is the fuzzy relation on A given by:

$$P_\mathcal{R}(a, b) = \begin{cases} \mathcal{R}(a, b), & \text{if } \mathcal{R}(a, b) > \mathcal{R}(b, a) \\ 0, & \text{otherwise.} \end{cases}$$

Definition 7.3. [11] Let T be a t-norm. Then a fuzzy relation $\mathcal{R}$ on A is called a *fuzzy weak order* with respect to T if it is:

1. strongly S_M-complete;
2. $T-$ transitive.

Definition 7.4. [11] Let T be a t-norm. Then a fuzzy relation $\mathcal{R}$ on A is called a *fuzzy quasi order* with respect to T if it is:

1. reflexive;
2. $T-$transitive.

Definition 7.5. [11] Let T be a t-norm and S be its dual t-conorm. Then a fuzzy relation $\mathcal{R}$ on A is called a *fuzzy quasi-transitive* relation with respect to T if it is:

1. strongly S_M- complete;
2. negatively $S-$transitive.

Definition 7.6. [34] Let $\mathcal{R}$ and $\mathcal{Q}$ be two binary relations between A and X, X and Y respectively. Then the *composition* $\mathcal{R}\mathcal{Q}$ is the binary relation between A and Y given by

$$\mathcal{R}\mathcal{Q} = \{(a, y) : \text{there exists}\, x \in X \,\text{such that}\, (a, x) \in \mathcal{R} \text{ and } (x, y) \in \mathcal{Q}\}.$$

Definition 7.7. [61] Let $\mathcal{R}$ and $\mathcal{Q}$ be two fuzzy relations between A and X, X and Y respectively and T be a t-norm. Then the *fuzzy composition* $\mathcal{R}o_T\mathcal{Q}$ with respect to T is the fuzzy relation between A and Y given by

$$(\mathcal{R}o_T\mathcal{Q})(a, y) = \sup_{x \in X} T\{\mathcal{R}(a, x), \mathcal{Q}(x, y)\}, \text{for each}\, (a, y) \in A \times Y.$$

Definition 7.8. [34, 79, 91] A binary relation $\mathcal{R}$ between A and X is said to satisfy the *Ferrers property* if

$$(a, x) \in \mathcal{R} \,\text{and}\, (b, y) \in \mathcal{R} \Rightarrow (a, y) \in \mathcal{R} \,\text{or}\, (b, x) \in \mathcal{R}, \tag{7.5}$$

for each $a, b \in A$ and $x, y \in X$. Equivalently,

$$\mathcal{R}\mathcal{R}^d\mathcal{R} \subseteq \mathcal{R}. \tag{7.6}$$

Definition 7.9. [34] A binary relation $\mathcal{R}$ between A and X is said to be a *biorder* if $\mathcal{R}$ satisfies the Ferrers property.

7.2 Fuzzy biorder and its representability

Corresponding to (7.5) and (7.6), in case of fuzzy relations on A we have the following:

Definition 7.10. [42] Let T be a t-norm and S be a t-conorm. Then a fuzzy relation $\mathcal{R}$ on A is said to satisfy *type1* $T - S$ *Ferrers property* if $T\{\mathcal{R}(a, b), \mathcal{R}(c, d)\} \leq S\{\mathcal{R}(a, d), \mathcal{R}(c, b)\}$, for each $a, b, c, d \in A$. In case T is a t-norm and S is the corresponding t-conorm(or dual t-conorm), then the type1 $T - S$ Ferrers property is simply called type1 T Ferrers property.

Definition 7.11. [31] A fuzzy relation $\mathcal{R}$ on A is said to satisfy *type2* T *Ferrers property* if $\mathcal{R}o_T\mathcal{R}^d o_T\mathcal{R} \subseteq \mathcal{R}$.

Now we state the following important result in this context, given by Fodor[42], which shows under which condition, the two types of Ferrers properties turn out to be equivalent.

Proposition 7.12. *[41] The following statements are equivalent for a fuzzy relation on A:*

1. *A type1 T Ferrers relation is also type2 T Ferrers.*

2. *A type2 T Ferrers relation is also type1 T Ferrers.*

3. *T is rotational invariant(i.e., $T(x, y) \leq z \Leftrightarrow T(x, 1-z) \leq (1-y)$, for each $x, y, z \in [0, 1]$).*

From the above proposition, it is clear that type1 T Ferrers property and type2 T Ferrers property are not equivalent in general. Keeping this in view, in literature, there are two different definitions of fuzzy biorders, one corresponding to type1 T_M Ferrers property of a fuzzy relation between A and X and another corresponding to type2 T Ferrers property of a fuzzy relation on A(cf.[41, 97]), which we have called here as type1 T_M biorder and type2 T biorder respectively.

Definition 7.13. [97] A fuzzy relation $\mathcal{R}$ between A and X is said to be *type1 T_M biorder* if $\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\}$, for every $a, b \in A$ and $x, y \in X$.

Definition 7.14. [41] A fuzzy relation $\mathcal{R}$ on A is said to satisfy *type2 T biorders* if $\mathcal{R} o_T \mathcal{R}^d o_T \mathcal{R} \subseteq \mathcal{R}$.

Proposition 7.15. *Let $\mathcal{R}$ be a fuzzy relation between A and X. Then $\mathcal{R}$ is a type1 T_M biorder iff each $\mathcal{R}_\alpha$ is a biorder between A and X.*

Proof. Let $\mathcal{R}$ be a type1 T_M biorder between A and X. Then

$$\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\},$$

for each $a, b \in A$ and $x, y \in X$.

We have to show that $\mathcal{R}_\alpha$ is a biorder between A and X, for each $\alpha \in [0, 1]$. Assume the contrary. Let for some $\alpha \in [0, 1]$, there exist $a, b \in A$ and $x, y \in X$

such that

$$\begin{aligned} & a\mathcal{R}_\alpha x, b\mathcal{R}_\alpha y, a(\mathcal{R}_\alpha)^c y, b(\mathcal{R}_\alpha)^c x \\ \Rightarrow\ & \mathcal{R}(a,x) \geq \alpha, \mathcal{R}(b,y) \geq \alpha, \mathcal{R}(a,y) < \alpha, \mathcal{R}(b,x) < \alpha \\ \Rightarrow\ & \min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \geq \alpha > \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\}, \end{aligned}$$

which is a contradiction.

Conversely, assume that each $\mathcal{R}_\alpha$ is a biorder between A and X, for $\alpha \in [0,1]$. We have to show that $\mathcal{R}$ is a type1 T_M biorder between A and X i.e,

$$\min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \leq \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\}$$

for each $a, b \in A$ and $x, y \in X$. For this, we need to consider the following cases:
Case 1: If $\mathcal{R}(a,x) = 0$ or $\mathcal{R}(b,y) = 0$. Then the above inequality is obviously satisfied.
Case 2: If $\mathcal{R}(a,x) = \beta \neq 0$ and $\mathcal{R}(b,y) = \gamma \neq 0$. Set $\delta = \min\{\beta, \gamma\}$. Then

$$\begin{aligned} & \mathcal{R}(a,x) \geq \delta \,\text{and}\, \mathcal{R}(b,y) \geq \delta \\ \Rightarrow\ & a\mathcal{R}_\delta x \text{ and } b\mathcal{R}_\delta y \\ \Rightarrow\ & a\mathcal{R}_\delta y \text{ or } b\mathcal{R}_\delta x \quad (\text{Since } \mathcal{R}_\delta \text{ is a biorder}) \\ \Rightarrow\ & \mathcal{R}(a,y) \geq \delta \text{ or } \mathcal{R}(b,x) \geq \delta \\ \Rightarrow\ & \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\} \geq \delta = \min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\}. \end{aligned}$$

Thus $\mathcal{R}$ is a type1 T_M biorder. □

The following Example 7.1 shows that Proposition 7.15 is not true if we replace type1 T_M biorder with type2 T_M biorder or type2 T_{nM} biorder(which is equivalent to type1 T_{nM} biorder using Proposition 7.12, as T_{nM} is the largest rotational invariant t-norm[28]).

Example 7.1. *Consider the following fuzzy relation $\mathcal{R}$ between A and X, where $A = X = \{a, b\}$ as follows:*

$\mathcal{R}$	a	b
a	*1*	*0.5*
b	*0.5*	*0.6*

1. *It can be checked easily that $\mathcal{R}$ is a type2 T_M biorder.*

2. *It can also be verified that $\mathcal{R}$ is a type1 T_{nM} biorder. Further, since T_{nM} is the largest rotational invariant t-norm(cf.[28]) and hence in view of Proposition 7.12, it is also a type2 T_{nM} biorder.*

Now if we take $\alpha = 0.6$, then $(a, a) \in \mathcal{R}_\alpha$ and $(b, b) \in \mathcal{R}_\alpha$, but (a, b) and (b, a) both do not belong to $\mathcal{R}_\alpha$, which implies that $\mathcal{R}_{0.6}$ is not a biorder.

The representability of a fuzzy total preorder additive fuzzy preference structure without incomparability and a compatible fuzzy semiorder, in terms of the $\alpha-$cuts of their corresponding fuzzy weak preference relation have been respectively studied by Agud et al.[1] and Induráin et al.[52]. Motivated by these facts and keeping in view the Proposition 7.15, **from now onwards we mean fuzzy biorders in the sense of Definition 7.13** i.e., a fuzzy relation $\mathcal{R}$ between A and X will be called a fuzzy biorder relation if $\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\}$, for each $a, b \in A$ and $x, y \in X$.

In [41], It has been shown that if $\mathcal{R}$ is a fuzzy relation on A, then $\mathcal{R}$ is a fuzzy biorder if and only if $\mathcal{R}^d$ is a fuzzy biorder. In the following proposition, we show that this result also holds good if we take a fuzzy relation between A and X.

Proposition 7.16. *Let $\mathcal{R}$ be a fuzzy relation between A and X. Then $\mathcal{R}$ is a fuzzy biorder between A and X iff $\mathcal{R}^d$ is a fuzzy biorder between X and A.*

Proof. Let $\mathcal{R}$ be a fuzzy biorder between A and X. So, we have

$$\min\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \leq \max\{\mathcal{R}(a, y), \mathcal{R}(b, x)\}, \tag{7.7}$$

for each $a, b \in A$ and $x, y \in X$. Now,

$$\begin{aligned}
\min\{\mathcal{R}^d(x, a), \mathcal{R}^d(y, b)\} &= \min\{1 - \mathcal{R}(a, x), 1 - \mathcal{R}(b, y)\} \\
&= 1 - \max\{\mathcal{R}(a, x), \mathcal{R}(b, y)\} \\
&\leq 1 - \min\{\mathcal{R}(a, y), \mathcal{R}(b, x)\} \quad \text{(Since } \mathcal{R} \text{ is a fuzzy biorder)} \\
&= \max\{1 - \mathcal{R}(a, y), 1 - \mathcal{R}(b, x)\} \\
&= \max\{\mathcal{R}^d(y, a), \mathcal{R}^d(x, b)\},
\end{aligned}$$

for each $a, b \in A$ and $x, y \in X$. Therefore $\mathcal{R}^d$ is a fuzzy biorder between X and A.

Conversely, assume that $\mathcal{R}^d$ is a fuzzy biorder between X and A. So, we have

$$\min\{\mathcal{R}^d(x,a),\mathcal{R}^d(y,b)\} \leq \max\{\mathcal{R}^d(x,b),\mathcal{R}^d(y,a)\}, \tag{7.8}$$

for each $a, b \in A$ and $x, y \in X$.

Now,

$$\begin{aligned}
\min\{\mathcal{R}(a,x),\mathcal{R}(b,y)\} &= \min\{1-\mathcal{R}^d(x,a), 1-\mathcal{R}^d(y,b)\} \\
&= 1-\max\{\mathcal{R}^d(x,a),\mathcal{R}^d(y,b)\} \\
&\leq 1-\min\{\mathcal{R}^d(x,b),\mathcal{R}^d(y,a)\} \quad \text{(Since } \mathcal{R}^d \text{ is a fuzzy biorder)} \\
&= \max\{1-\mathcal{R}^d(x,b), 1-\mathcal{R}^d(y,a)\} \\
&= \max\{\mathcal{R}(b,x),\mathcal{R}(a,y)\},
\end{aligned}$$

for each $a, b \in A$ and $x, y \in X$. Therefore $\mathcal{R}$ is a fuzzy biorder between A and X. □

Union and intersection of fuzzy biorders need not be a fuzzy biorder. This is exhibited through the following examples.

Example 7.2. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations between A and X, where $A = X = \{a, b\}$ and are given as follows:*

$\mathcal{R}_1$	a	b
a	*0.7*	*0.7*
b	*0.3*	*0.8*

and

$\mathcal{R}_2$	a	b
a	*0.6*	*0.3*
b	*0.6*	*0.8*

.

It is easy to verify that $\mathcal{R}_1$ and $\mathcal{R}_2$ both are fuzzy biorder between A and X, but $\mathcal{R}_1 \cap \mathcal{R}_2$ which is given as follows:

$\mathcal{R}_1 \cap \mathcal{R}_2$	a	b
a	*0.6*	*0.3*
b	*0.3*	*0.8*

is not a fuzzy biorder between A and X as

$$\min\{\mathcal{R}(a,a),\mathcal{R}(b,b)\} = 0.6 > \max\{\mathcal{R}(a,b),\mathcal{R}(b,a)\} = 0.3,$$

where $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$.

Example 7.3. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations between A and X, where $A = X = \{a, b\}$ and are given as follows:*

$\mathcal{R}_1$	a	b
a	0.7	0.5
b	0.4	0.4

and

$\mathcal{R}_2$	a	b
a	0.6	0.6
b	0.3	0.8

.

It is easy to verify that $\mathcal{R}_1$ and $\mathcal{R}_2$ both are fuzzy biorder between A and X, but $\mathcal{R}_1 \cup \mathcal{R}_2$ which is given as follows:

$\mathcal{R}_1 \cup \mathcal{R}_2$	a	b
a	0.7	0.6
b	0.4	0.8

is not a fuzzy biorder between A and X as

$$\min\{\mathcal{R}(a,a),\mathcal{R}(b,b)\} = 0.7 > \max\{\mathcal{R}(a,b),\mathcal{R}(b,a)\} = 0.6,$$

where $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$.

Definition 7.17. [61, 97] Let T be a left continuous t-norm. Then the *residual implication operator* I_T associated with T is defined as $I_T(x,y) = \sup\{z \in [0,1] : T(x,z) \leq y\}$, for each $(x,y) \in [0,1] \times [0,1]$. For example,

1. If $T = T_P$, then the residual implication operator I_{T_P} associated with T_P is given by

$$I_{T_P}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise,} \end{cases}$$

for each $(x, y) \in [0, 1] \times [0, 1]$.

2. If $T = T_L$, then the residual implication operator I_{T_L} associated with T_L is given by

$$I_{T_L}(x, y) = \min\{1, 1 - x + y\} = \begin{cases} 1, & \text{if } x \leq y \\ 1 - x + y, & \text{otherwise,} \end{cases}$$

for each $(x, y) \in [0, 1] \times [0, 1]$.

Definition 7.18. [34] Let $\mathcal{R}$ be a binary relation between A and X and for any $a \in A$ and $x \in X$, $a\mathcal{R} = \{y \in X : a\mathcal{R}y\}$ and $\mathcal{R}x = \{c \in A : c\mathcal{R}x\}$. Then the binary relations $\mathcal{R}_A$ on A and $\mathcal{R}_X$ on X are defined as follows:

$$a\mathcal{R}_A b \text{ if } b\mathcal{R} \subseteq a\mathcal{R}$$
$$x\mathcal{R}_X y \text{ if } \mathcal{R}x \subseteq \mathcal{R}y.$$

Now, we prove the following:

Proposition 7.19. *Let $\mathcal{R}$ be a fuzzy biorder on A. Then*

1. *If $\mathcal{R}$ is reflexive on A, then $\mathcal{R}$ is strongly S_M- complete and negatively S_M- transitive on A.*

2. *If $\mathcal{R}$ is irreflexive on A, then $\mathcal{R}$ is T_M-asymmetric and T_M-transitive on A.*

Proof. 1. Since T_M has no zero divisors, it follows from Theorem 3 in [32] that $\mathcal{R}$ is strongly S_M- complete on A. Now we prove that $\mathcal{R}$ is negatively S_M- transitive as follows:

Since $\min\{\mathcal{R}(a, b), \mathcal{R}(c, c)\} \leq \max\{\mathcal{R}(a, c), \mathcal{R}(c, b)\}$, for each $a, b, c \in A$ and $\mathcal{R}(a', a') = 1$, for each $a' \in A$, so $\mathcal{R}(a, b) \leq \max\{\mathcal{R}(a, c), \mathcal{R}(c, b)\}$, for each $a, b, c \in A$.

2. We show that $\min\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} = 0$, for each $a, b \in A$. Since

$$\min\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} \leq \max\{\mathcal{R}(b, b), \mathcal{R}(a, a)\},$$

for each $a, b \in A$ and by the irreflexiviy of $\mathcal{R}$, $\mathcal{R}(a', a') = 0$, for each $a' \in A$, so we get $\min\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} \leq 0$ and hence $\min\{\mathcal{R}(a, b), \mathcal{R}(b, a)\} = 0$, for each $a, b \in A$. Now it follows from Lemma 5.1(i) in [33] that $\mathcal{R}$ is T_M- transitive on A.

□

Definition 7.20. [34] A binary relation $\mathcal{R}$ between A and X is said to be *representable* with respect to $\leq$(resp.,$<$) if there exist two mappings $u : A \to \mathbb{R}$ and $v : X \to \mathbb{R}$ such that $(a, x) \in \mathcal{R}$ iff $u(a) \leq v(x)$(resp., $u(a) < v(x)$), for each $a \in A$ and $x \in X$.

In the case of biorders, we have the following result related to its representability.

Proposition 7.21. *[34] Let $\mathcal{R}$ be a binary relation between A and X, where A and X both are countable. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a biorder.*
2. *$\mathcal{R}$ is representable with respect to $\leq$.*
3. *$\mathcal{R}$ is representable with respect to $<$.*

Now we prove the fuzzy analogue of this proposition.

Proposition 7.22. *Let $\mathcal{R}$ be a fuzzy relation between A and X, where A and X both are countable. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a fuzzy biorder.*
2. *Each $\mathcal{R}_\alpha$, $\alpha \in [0, 1]$ is representable with respect to $\leq$.*
3. *Each $\mathcal{R}_\alpha$, $\alpha \in [0, 1]$ is representable with respect to $<$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{R}$ be a fuzzy biorder between A and X, where A and X both are countable. Then in view of Proposition 7.15, each $\mathcal{R}_\alpha$, $\alpha \in [0, 1]$ is a biorder between A and X, where A and X both are countable. Therefore from the Proposition 7.21, we get a representation of $\mathcal{R}_\alpha$ with respect to $\leq$, for each $\alpha \in [0, 1]$.
(2) $\Rightarrow$ (3) Let each $\mathcal{R}_\alpha$, $\alpha \in [0, 1]$ be representable with respect to $\leq$. Then from

the Proposition 7.21, each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ is representable with respect to $<$.

(3) $\Rightarrow$ (1) Let each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ be representable with respect to $<$. Then from the Proposition 7.21, each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ is a biorder between A and X. Finally from the Proposition 7.15, we get $\mathcal{R}$ is a fuzzy biorder. □

Definition 7.23. [34] Let $\mathcal{R}$ be a binary relation between A and X. A subset $M^* \subseteq A \cup X$ is said to be *widely dense* if for each $a \in A$ and $x \in X$, $a\mathcal{R}^c x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in A$, $m^*\mathcal{R}^c x$ and $m^*\mathcal{R}_A a$ or $m^* \in X$, $x\mathcal{R}_X m^*$ and $a\mathcal{R}^c m^*$.

Proposition 7.24. *[34] Let $\mathcal{R}$ be a binary relation between A and X. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a biorder with a countable widely dense subset M^*.*
2. *$\mathcal{R}$ is representable with respect to $\leq$.*

Now we give the fuzzy analogue of the this proposition.

Proposition 7.25. *Let $\mathcal{R}$ be a fuzzy relation between A and X. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a fuzzy biorder and for each $\alpha \in [0,1]$, there exists a countable subset $M^*_\alpha \subseteq A \cup X$ such that for each $a \in A$ and $x \in X$, $a(\mathcal{R}_\alpha)^c x$ implies that there exists an element $m^* \in M^*_\alpha$ such that either $m^* \in A$, $m^*(\mathcal{R}_\alpha)^c x$ and $m^*(\mathcal{R}_\alpha)_A a$ or $m^* \in X$, $x(\mathcal{R}_\alpha)_X m^*$ and $a(\mathcal{R}_\alpha)^c m^*$.*
2. *Each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ is representable with respect to $\leq$.*

The proof follows from Proposition 7.15 and 7.24.

Definition 7.26. [34] Let $\mathcal{R}$ be a binary relation between A and X. A subset M^* of $A \cup X$ is said to be *strictly dense* if for all $a \in A$ and $x \in X$, $a\mathcal{R}x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in X$, $a\mathcal{R}m^*$ and $m^*\mathcal{R}_X x$ or $m^* \in A$, $a\mathcal{R}_A m^*$ and $m^*\mathcal{R}x$.

Proposition 7.27. *[34] Let $\mathcal{R}$ be a binary relation between A and X. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a biorder with a countably strictly dense subset M^*.*

2. *$\mathcal{R}$ is representable with respect to $<$.*

Now we give the fuzzy analogue of this proposition.

Proposition 7.28. *Let $\mathcal{R}$ be a fuzzy relation between A and X. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a fuzzy biorder and for each $\alpha \in [0,1]$, there exists a countable subset $M^*_\alpha \subseteq A \cup X$ such that for each $a \in A$ and $x \in X$, $a\mathcal{R}_\alpha x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in X$, $a\mathcal{R}_\alpha m^*$ and $m^*(\mathcal{R}_\alpha)_X x$ or $m^* \in A$, $a(\mathcal{R}_\alpha)_A m^*$ and $m^*\mathcal{R}_\alpha x$.*
2. *Each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ is representable with respect to $<$.*

The proof follows from Proposition 7.15 and 7.27.

Definition 7.29. [14] A binary relation $\mathcal{R}$ on A is said to be an *interval order* if:

1. it is asymmetric i.e, if $(a,b) \in \mathcal{R}$, then $(b,a) \notin \mathcal{R}$;
2. it satisfies the Ferrers property i.e, $(a,b) \in \mathcal{R}$ and $(c,d) \in \mathcal{R} \Rightarrow (a,d) \in \mathcal{R}$ or $(c,b) \in \mathcal{R}$.

Now we prove the following result which is a corollary of Proposition 7.15:

Corollary 7.30. *Let $\mathcal{R}$ be a fuzzy relation on A. Then $\mathcal{R}$ is an irreflexive fuzzy biorder on A iff for each $\alpha \in (0,1]$, $\mathcal{R}_\alpha$ is an interval order on A.*

The proof follows from Proposition 7.15, Proposition 7.19 and the facts that a fuzzy relation is T_M-asymmetric if and only if its $\alpha-$cuts are asymmetric for every $\alpha \in (0,1]$ and every T_M-asymmetric fuzzy relation is irreflexive.

Proposition 7.31. *[34] Let $\mathcal{R}$ be a binary relation on A. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is an interval order with a countable widely dense subset M^*.*
2. *There exist two mappings $u : A \to \mathbb{R}$ and $r : A \to \mathbb{R}^+_0$ such that for $a,b \in A$, $a\mathcal{R}b \Leftrightarrow u(a) + r(a) \leq u(b)$.*

Proposition 7.32. *[34] Let $\mathcal{R}$ be a binary relation on A. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is an interval order with a countable strictly dense subset M^*.*
2. *There exist two mappings $u : A \to \mathbb{R}$ and $r : A \to \mathbb{R}^+$ such that for all $a, b \in A$, $a\mathcal{R}b$ iff $u(a) + r(a) < u(b)$.*

We now give the following two propositions:

Proposition 7.33. *Let $\mathcal{R}$ be a fuzzy relation on A. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is an irreflexive fuzzy biorder such that for each $\alpha \in (0, 1]$, there exists a countable widely dense subset M^*_α.*
2. *For each $\alpha \in (0, 1]$, there exist two mappings $u_\alpha : A \to \mathbb{R}$ and $r_\alpha : A \to \mathbb{R}^+_0$ such that for $a, b \in A$, $a\mathcal{R}_\alpha b \Leftrightarrow u_\alpha(a) + r_\alpha(a) \leq u_\alpha(b)$.*

The proof follows from Corollary 7.30 and Proposition 7.31.

Proposition 7.34. *Let $\mathcal{R}$ be a fuzzy relation on A. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is an irreflexive fuzzy biorder such that for each $\alpha \in (0, 1]$, there exists a countable strictly dense subset M^*_α.*
2. *For each $\alpha \in (0, 1]$, there exist two mappings $u_\alpha : A \to \mathbb{R}$ and $r_\alpha : A \to \mathbb{R}^+$ such that for $a, b \in A$, $a\mathcal{R}_\alpha b \Leftrightarrow u_\alpha(a) + r_\alpha(a) < u_\alpha(b)$.*

The proof follows from Corollary 7.30 and Proposition 7.32.

7.3 Representability of fuzzy weak orders using the residual implication operator

Definition 7.35. [98] Let T be a left continuous t-norm. A fuzzy relation $\mathcal{R}$ on a finite set A is called $T-$*representable*, if there exists a mapping $f : A \to [0, 1]$ such that $\mathcal{R}(a,b) = \mathcal{R}_{T,f}(a,b)$, for each $(a,b) \in A \times A$, where $\mathcal{R}_{T,f}$ is the fuzzy relation defined by $\mathcal{R}_{T,f}(a,b) = I_T(f(b), f(a))$, for each $(a,b) \in A \times A$ and I_T denotes the residual implication operator associated with T .

Theorem 7.36. *[97] A $T-$representable fuzzy relation on a finite set A is a fuzzy weak order on A with respect to T.*

Theorem 7.37. *[98] Let $\mathcal{R}$ be a fuzzy relation on a finite set A and T be a left continuous t-norm. Then $\mathcal{R}$ is a fuzzy quasi order with respect to T iff it is intersection of a finite family of fuzzy weak orders with respect to T.*

Remark 7.38. Let $\mathcal{R}$ be a $T-$transitive fuzzy relation on A. Then the following property holds for every $a, b, c \in A$:

$$(\mathcal{R}(a,b) = 1 \text{ and } \mathcal{R}(b,c) = 1) \Rightarrow (\mathcal{R}(a,c) = 1 \text{ and } \mathcal{R}(c,a) \leq \mathcal{R}(c,b)).$$

The proof is trivial.

Baets et al.[11] had proved the following result in case of T_M. These authors have also given an example to show that the converse of the following result does not hold good in that case.

Proposition 7.39. *[11] If a fuzzy relation $\mathcal{R}$ on A is negatively S_M-transitive, then its strict part $P_{\mathcal{R}}$ is T_M-transitive.*

Here we show that none of the implications hold good in case of T_P, through counter examples.

Example 7.4. *Let $\mathcal{R}$ be a fuzzy relation on $A = \{a, b, c\}$ whose matrix representation is as follows:*

$\mathcal{R}$	a	b	c
a	0	0.84	0.6
b	0.8	0	0.7
c	0.7	0.6	0

Then its strict part $P_{\mathcal{R}}$ is as follows:

$P_{\mathcal{R}}$	a	b	c
a	0	0.84	0
b	0	0	0.7
c	0.7	0	0

Now it is easy to verify that $\mathcal{R}$ is negatively S_P-transitive but $P_\mathcal{R}$ is not T_P-transitive as $P_\mathcal{R}(a,b).P_\mathcal{R}(b,c) = 0.84 \times 0.7 = 0.58 > P_\mathcal{R}(a,c) = 0$.

Example 7.5. *Let $\mathcal{R}$ be a fuzzy relation on $A = \{a, b, c\}$ whose matrix representation is as follows:*

$\mathcal{R}$	a	b	c
a	1	0.5	0.9
b	1	1	0.6
c	1	1	1

Then its strict part $P_\mathcal{R}$ is as follows:

$P_\mathcal{R}$	a	b	c
a	0	0	0
b	1	0	0
c	1	1	0

Now it is easy to verify that $P_\mathcal{R}$ is T_P-transitive but $\mathcal{R}$ is not negatively S_P-transitive as $\mathcal{R}(a,b) + \mathcal{R}(b,c) - \mathcal{R}(a,b).\mathcal{R}(b,c) = 0.5 + 0.6 - 0.3 = 0.8 < \mathcal{R}(a,c) = 0.9$.

Proposition 7.40. *If $\mathcal{R}$ is a fuzzy relation on A which is strongly S_M-complete. It satisfies, for each $a, b, c \in A$*

$$P_\mathcal{R}(a,b) = 1 \text{ and } P_\mathcal{R}(b,c) = 1 \Rightarrow \mathcal{R}(c,a) \leq S(\mathcal{R}(c,b), \mathcal{R}(b,a)),$$

if and only if $\mathcal{R}$ is negatively $S-$transitive.

Proof. To show that $\mathcal{R}$ is negatively $S-$transitive, we have to show that

$$S(\mathcal{R}(a,b), \mathcal{R}(b,c)) \geq \mathcal{R}(a,c) \tag{7.9}$$

for each $a, b, c \in A$. Assume the contrary. Let

$$\mathcal{R}(a,c) > S(\mathcal{R}(a,b), \mathcal{R}(b,c)) \tag{7.10}$$

for some $a, b, c \in A$. Then from (7.10) and since $\mathcal{R}$ is strongly S_M-complete, we get $\mathcal{R}(b,a) = 1 > \mathcal{R}(a,b)$ and $\mathcal{R}(c,b) = 1 > \mathcal{R}(b,c)$. This implies that $P_\mathcal{R}(b,a) = 1$

and $P_{\mathcal{R}}(c, b) = 1$. So by our assumption, we must have

$$\mathcal{R}(a, c) \leq S(\mathcal{R}(a, b), \mathcal{R}(b, c)),$$

which contradicts (7.10).

Conversely, if $\mathcal{R}$ is negatively $S-$transitive, then by its definition itself: $\mathcal{R}(c, a) \leq S(\mathcal{R}(c, b), \mathcal{R}(b, a)$, for each $a, b, c \in A$. □

Proposition 7.41. *The union $\mathcal{R}$ of any finite family $\{\mathcal{R}_i\}_{i=1}^n$ of fuzzy weak orders on A with respect to T is a fuzzy quasi-transitive relation on A with respect to T.*

Proof. Obviously, $\mathcal{R}$ is strongly S_M-complete. Let $a, b, c \in A$ be such that $P_{\mathcal{R}}(a, b) = 1$ and $P_{\mathcal{R}}(b, c) = 1$. Now $P_{\mathcal{R}}(a, b) = 1$ implies that $1 > \mathcal{R}(b, a)$ and $P_{\mathcal{R}}(b, c) = 1$ implies that $1 > \mathcal{R}(c, b)$. Then for each $i \in \{1, 2, ..., n\}$, $\mathcal{R}_i(b, a) < 1$ and since $\mathcal{R}_i$ is strongly S_M-complete so $\mathcal{R}_i(a, b) = 1$. Similarly, $\mathcal{R}(c, b) < 1$ implies that $\mathcal{R}_i(b, c) = 1$. Since each $\mathcal{R}_i$ is a fuzzy weak order with respect to T, so by Remark 7.38, we have

$$\mathcal{R}_i(c, a) \leq \mathcal{R}_i(c, b), \quad \text{for each } i \in \{1, 2, ..., n\} \tag{7.11}$$

Now, $\mathcal{R}(c, a) = \max_i \mathcal{R}_i(c, a) = \mathcal{R}_t(c, a)$, for some $t \in \{1, 2, ..., n\}$, then

$$\begin{aligned} S(\mathcal{R}(c, b), \mathcal{R}(b, a)) &\geq S_M(\mathcal{R}(c, b), \mathcal{R}(b, a)) \\ &\geq \mathcal{R}(c, b) \\ &\geq \mathcal{R}_t(c, b) \\ &\geq \mathcal{R}_t(c, a) = \mathcal{R}(c, a) \end{aligned}$$

Therefore, by Proposition 7.40, $\mathcal{R}$ is negatively $S-$ transitive. Thus, $\mathcal{R} = \bigcup_i \mathcal{R}_i$ is strongly S_M-complete as well as negatively $S-$transitive and hence it is a fuzzy quasi-transitive relation with respect to T. □

Characterizations for fuzzy weak orders with respect to T_M and T_P on a finite set which are T_M-representable and T_P-representable have been respectively obtained by Baets et al.[11] and Sali et al.[98]. In the following theorem, we have obtained a characterization for fuzzy weak orders with respect to T_L on a finite set which are T_L representable.

Theorem 7.42. *A fuzzy weak order $\mathcal{R}$ with respect to T_L on a finite set A is T_L-representable if and only if*

$$\mathcal{R}(a,b) < 1 \text{ and } \mathcal{R}(b,c) < 1 \Rightarrow \mathcal{R}(a,c) = \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1 \tag{7.12}$$

holds for each $a, b, c \in A$.

To prove the above theorem, we need to prove the following lemma, the proof of which is on the similar lines as that of Lemma 7 in [98].

Lemma 7.43. *Let $\mathcal{R}$ be a reflexive fuzzy relation on a finite set A satisfying (7.12). Then there exists $c \in A$ such that $\mathcal{R}(c,a) = 1$ for each $a \in A$.*

Proof. Assume the contrary, i.e, for each $c \in A$, there exists $a_c \in A$ such that $\mathcal{R}(c, a_c) < 1$. Now define an oriented graph $\vec{G} = (V, E)$, where $V = A$ and there is an arc from a to b iff $\mathcal{R}(a,b) < 1$. By our assumption the out-degree of each node is atleast one, so there is a directed cycle C in $\vec{G}$, which is obtained by taking connected nodes in the cyclic order. Let the nodes of C be $\{a_1, a_2, ..., a_n\}$. Then by using (7.12), we have

$$\begin{aligned} 1 = n - (n-1) &> \mathcal{R}(a_1,a_2) + \mathcal{R}(a_2,a_3) + ... + \mathcal{R}(a_{n-1},a_n) + \mathcal{R}(a_n,a_1) - (n-1) \\ &= \mathcal{R}(a_1,a_1) \end{aligned}$$

which contradicts the reflexivity of $\mathcal{R}$. □

Proof of the theorem. Let $\mathcal{R}$ be a fuzzy weak order with respect to T_L which is T_L-representable. Then for each $a, b \in A$, $\mathcal{R}(a,b) = \mathcal{R}_{T_L,f}(a,b) = I_{T_L}(f(b), f(a))$ for some mapping $f : A \to [0,1]$. If $\mathcal{R}(a,b) < 1$ and $\mathcal{R}(b,c) < 1$, for some $a, b, c \in A$, then $\mathcal{R}(a,b) = 1 - f(b) + f(a)$ and $\mathcal{R}(b,c) = 1 - f(c) + f(b)$ such that $f(b) > f(a)$ and $f(c) > f(b)$. This implies that $f(c) > f(a)$ and hence $\mathcal{R}(a,c) = 1 - f(c) + f(a) = \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1$.

Conversely, let $\mathcal{R}$ be a fuzzy weak order with respect to T_L satisfying (7.12). Then $\mathcal{R}$ is reflexive and so by the previous lemma, there exists $c \in A$ such that $\mathcal{R}(c,a) = 1$, for each $a \in A$. Now define the mapping $f : A \to [0,1]$ by $f(a) =$

$\mathcal{R}(a,c)$, for each $a \in A$. We show that

$$\mathcal{R}(a,b) = \mathcal{R}_{T_L,f}(a,b) = I_{T_L}(f(b),f(a)) = \begin{cases} 1, & \text{if } f(b) \leq f(a) \\ 1 - f(b) + f(a), & \text{otherwise,} \end{cases}$$

To prove this we need to consider the following cases:
Case 1: If $f(a) \geq f(b)$ (i.e, $\mathcal{R}(a,c) \geq \mathcal{R}(b,c)$). In this case $I_{T_L}(f(b),f(a)) = 1$, so we have to show that $\mathcal{R}(a,b) = 1$. Assume the contrary that $\mathcal{R}(a,b) < 1$. If $\mathcal{R}(b,c) < 1$, then by (7.12),

$$\begin{aligned} \mathcal{R}(a,c) &= \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1 \\ &\leq \mathcal{R}(a,b) + \mathcal{R}(a,c) - 1 \\ &< \mathcal{R}(a,c) \quad (\text{since } \mathcal{R}(a,b) < 1) \end{aligned}$$

which is a contradiction. Next, if $\mathcal{R}(b,c) = 1$, then $\mathcal{R}(a,c) = 1$ and in view of the previous lemma, $\mathcal{R}(c,b) = 1$. So by the T_L-transitivity of $\mathcal{R}$, we have

$$\begin{aligned} T_L(\mathcal{R}(a,c),\mathcal{R}(c,b)) \leq \mathcal{R}(a,b) \\ 1 = \max\{0, \mathcal{R}(a,c) + \mathcal{R}(c,b) - 1\} \leq \mathcal{R}(a,b) < 1 \end{aligned}$$

which is again a contradiction. Hence in this case we are done.
Case 2: If $f(a) < f(b)$ (i.e.,$\mathcal{R}(a,c) < \mathcal{R}(b,c)$). In this case $I_{T_L}(f(b),f(a)) = 1 - f(b) + f(a)$, so we have to show that $\mathcal{R}(a,b) = 1 - f(b) + f(a)$. Let $\mathcal{R}(b,c) < 1$. Now by the T_L-transitivity of $\mathcal{R}$, we have $T_L(\mathcal{R}(a,b),\mathcal{R}(b,c)) = \max\{0, \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1\} \leq \mathcal{R}(a,c)$. If $\max\{0, \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1\} = \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1$, then $\mathcal{R}(a,b) + \mathcal{R}(b,c) - 1 \leq \mathcal{R}(a,c) < \mathcal{R}(b,c)$ which implies that $\mathcal{R}(a,b) < 1$. If $\max\{0, \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1\} = 0$, then $\mathcal{R}(a,b) \leq 1 - \mathcal{R}(b,c)$ and hence $0 < \mathcal{R}(a,b) < 1$(since $0 < \mathcal{R}(b,c) < 1$). Now by using (7.12), we have $\mathcal{R}(a,c) = \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1$ which implies that $\mathcal{R}(a,b) = 1 - \mathcal{R}(b,c) + \mathcal{R}(a,c) = 1 - f(b) + f(a)$. Next, if $\mathcal{R}(b,c) = 1$, then by the T_L-transitivity of $\mathcal{R}$, we have $T_L(\mathcal{R}(a,b),\mathcal{R}(b,c)) = \max\{0, \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1\} \leq \mathcal{R}(a,c)$ which implies that $\mathcal{R}(a,b) \leq \mathcal{R}(a,c)$. Again by using T_L-transitivity of $\mathcal{R}$, we have $T_L(\mathcal{R}(a,c),\mathcal{R}(c,b)) = \max\{0, \mathcal{R}(a,c) + \mathcal{R}(c,b) - 1\} \leq \mathcal{R}(a,b)$ which implies that $\mathcal{R}(a,c) \leq \mathcal{R}(a,b)$, since $\mathcal{R}(c,b) = 1$ using the previous lemma. Hence $\mathcal{R}(a,b) = \mathcal{R}(a,c) = 1 - \mathcal{R}(b,c) + \mathcal{R}(a,c) = 1 - f(b) + f(a)$. This proves the theorem.

7.4 Conclusion

In this chapter, representability of fuzzy biorders in terms of their $\alpha-$cuts and fuzzy weak orders using residual implication operators, have been studied. Further, we have shown that union of a finite family of fuzzy weak orders with respect to a t-norm T is fuzzy quasi-transitive with respect to T and counter examples have been produced to show that unions and intersections of fuzzy biorders need not be fuzzy biorder. In the last theorem, we have also obtained a characterization for a T_L-representable fuzzy weak orders.