

Chapter 6

On fuzzy topology generated by fuzzy relations

6.1 Introduction

In this chapter, we have introduced a fuzzy topology generated by a fuzzy relation, as a generalization of the corresponding concept given by Smithson[102]. We have then obtained sufficient conditions under which this generated fuzzy topology will satisfy separation axioms, fuzzy T_0 , fuzzy T_1 and fuzzy T_2 . Further, we have introduced ‘finite intersection property(F.I.P.)’ in a fuzzy topological space and then obtained a characterization of (Lowen’s) fuzzy compactness in terms of F.I.P. Using this result, we have obtained a sufficient condition under which a fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.

Now we give some definitions and results which will be used in this chapter.

Definition 6.1. Let X be a non empty set, $A \subseteq X$ and \mathcal{R} be a fuzzy relation on X . Then A is called an \mathcal{R} -*antiset* of X if no two distinct elements of A are \mathcal{R} -related, where two elements x, y are said to be \mathcal{R} -related if $\mathcal{R}(x, y) > 0$.

Definition 6.2. Let X be a non empty set, $A \subseteq X$ and \mathcal{R} be a fuzzy relation on X . Then fuzzy sets L_A and R_A on X , are defined as follows:

$$L_A(y) = \sup_{a \in A} \mathcal{R}(y, a),$$
$$R_A(y) = \sup_{a \in A} \mathcal{R}(a, y),$$

for each $y \in X$.

Smithson[102] introduced and studied a topology on a set X , induced by a relation \mathcal{R} . Here we generalize this concept in the fuzzy setting.

Definition 6.3. Let \mathcal{R} be a fuzzy relation on X and \mathcal{A} be a collection of \mathcal{R} -antisets of X . Then $\tau_{\mathcal{R},\mathcal{A}}$ is the fuzzy topology on X generated by taking

$$\mathcal{S} = \{L_A\}_{A \in \mathcal{A}} \cup \{R_A\}_{A \in \mathcal{A}} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

as a subbase for the fuzzy closed sets in X i.e., every fuzzy closed set in X can be written as an intersection of finite unions of members of \mathcal{S} .

We mention here that in [77], for a given fuzzy relation \mathcal{R} on X , the authors have introduced and studied the fuzzy topology on X generated by $\{L_{\{x\}}, R_{\{x\}}\}_{x \in X}$ considered as a subbase for the fuzzy open sets in X . This is a generalization of the corresponding concept given in [62].

Example 6.1. Let \mathcal{R} be a fuzzy relation on $X = \{a, b\}$, which is given as follows:

\mathcal{R}	a	b
a	0.1	0
b	0	0.3

and $\mathcal{A} = \{\{a\}, \{a, b\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase \mathcal{S} for the fuzzy closed sets in X :

$$\mathcal{S} = \{L_{\{a\}}, L_{\{a,b\}}, R_{\{a\}}, R_{\{a,b\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{a,b\}}, R_{\{a\}}, R_{\{a,b\}}$ are given by:

$$L_{\{a\}} = \frac{0.1}{a} + \frac{0}{b}, \quad L_{\{a,b\}} = \frac{0.1}{a} + \frac{0.3}{b}, \quad R_{\{a\}} = \frac{0.1}{a} + \frac{0}{b}, \quad R_{\{a,b\}} = \frac{0.1}{a} + \frac{0.3}{b}.$$

So, the collection \mathcal{H} of fuzzy closed sets in X consists of arbitrary intersections of finite unions of members of \mathcal{S} and hence $\tau_{\mathcal{R},\mathcal{A}}$ is the set obtained by taking complements of members of \mathcal{H} .

Definition 6.4. Let (X, τ) be a fuzzy topological space. Then (X, τ) is said to be

1. *fuzzy T_0* if for each $x, y \in X$ such that $x \neq y$, there exists a fuzzy closed set U in X such that $U(x) \neq U(y)$;
2. *fuzzy T_1* if for each $x, y \in X$ such that $x \neq y$, there exist two fuzzy closed sets U, V such that $U(x) = 1, U(y) = 0, V(x) = 0, V(y) = 1$;
3. *fuzzy T_2* if for each pair of distinct fuzzy points x_r, y_s in X , there exist two fuzzy closed sets U, V such that $r > U(x), s > V(y)$ and $U \cup V = X$.

We remark here that (1), (2), (3) given above are equivalent to the definitions of fuzzy T_0 , fuzzy T_1 and fuzzy T_2 given in [71], [107], [106], respectively.

Proposition 6.5. [110] *Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

1. (X, τ) is fuzzy T_1 .
2. $\{x\}$ is fuzzy closed, $\forall x \in X$.

6.2 Separation axioms

In this section, we prove some results in the fuzzy setting, which are counterparts of the corresponding results given in [102]. In our discussion here, we shall assume that \mathcal{R} is a fuzzy relation on X and \mathcal{A} is a collection of \mathcal{R} -antisets.

Proposition 6.6. *Let for each $x, y \in X$ such that $x \neq y$, there exist $z \in X$ such that $\mathcal{R}(x, z) \neq \mathcal{R}(y, z)$ (or $\mathcal{R}(z, x) \neq \mathcal{R}(z, y)$) and \mathcal{A} contains singletons. Then $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_0 .*

Proof. Let $x, y \in X$ such that $x \neq y$. Then by our assumption, there exists $z \in X$ such that $\mathcal{R}(x, z) \neq \mathcal{R}(y, z)$ which implies that $L_{\{z\}}(x) \neq L_{\{z\}}(y)$. Since $L_{\{z\}}$ is fuzzy closed in $\tau_{\mathcal{R}, \mathcal{A}}$ and is such that $L_{\{z\}}(x) \neq L_{\{z\}}(y)$, so $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_0 .

Similarly, we can proceed for the case when $\mathcal{R}(z, x) \neq \mathcal{R}(z, y)$. □

Example 6.2. *Let \mathcal{R} be a fuzzy relation on $X = \{a, b\}$, which is given as follows:*

\mathcal{R}	a	b
a	0.7	0.3
b	0.6	0.5

and $\mathcal{A} = \{\{a\}, \{b\}\}$. Then the fuzzy topology $\tau_{\mathcal{R}, \mathcal{A}}$ is generated by the following subbase \mathcal{S} for the fuzzy closed sets in X :

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, R_{\{a\}}, R_{\{b\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{b\}}, R_{\{a\}}, R_{\{b\}}$ are given by:

$$L_{\{a\}} = \frac{0.7}{a} + \frac{0.6}{b}, \quad L_{\{b\}} = \frac{0.3}{a} + \frac{0.5}{b}, \quad R_{\{a\}} = \frac{0.7}{a} + \frac{0.3}{b}, \quad R_{\{b\}} = \frac{0.6}{a} + \frac{0.5}{b}.$$

Note that $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_0 since for $a, b \in X$, $a \neq b$, there exists a fuzzy closed set $U = L_{\{a\}}$ in X such that $U(a) \neq U(b)$.

Proposition 6.7. *Let for each $x \in X$ and $y \in X \setminus \{x\}$, there exist $z \in X$ such that $\mathcal{R}(x, z) = 1$ (or $\mathcal{R}(z, x) = 1$) and $\mathcal{R}(y, z) = 0$ (or $\mathcal{R}(z, y) = 0$). Then \mathcal{A} contains singletons implies that $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_1 .*

Proof. In view of Proposition 6.5, we show that $\{x\}$ is fuzzy closed. Let $y_r \in X \setminus \{x\}$. Then $x \neq y$. So according to our assumption, there exists $z \in X$ such that $\mathcal{R}(x, z) = 1$ and $\mathcal{R}(y, z) = 0$, which implies that $L_{\{z\}}(x) = 1$ and $L_{\{z\}}(y) = 0$. Therefore, there exists $L_{\{z\}}^c \in \tau_{\mathcal{R}, \mathcal{A}}$ such that $y_r \in L_{\{z\}}^c \subseteq X \setminus \{x\}$, which implies that $\{x\}$ is fuzzy closed.

The other case can be handled similarly. □

Corollary 6.8. *Let \mathcal{R} be a fuzzy relation on X which is reflexive and antisymmetric. Then \mathcal{A} contains singletons implies that $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_1 .*

Proof. Let $x, y \in X$ such that $x \neq y$. Then by the antisymmetry of \mathcal{R} , either $\mathcal{R}(x, y) = 0$ or $\mathcal{R}(y, x) = 0$. Also by the reflexivity of \mathcal{R} , $\mathcal{R}(x, x) = 1$, for each $x \in X$. In both the cases, if we set $z = x$, then by Proposition 6.7, $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_1 . □

Definition 6.9. A collection \mathcal{A} of \mathcal{R} -antisets is called *separating* if for each $x \in X$ and $y \in X \setminus \{x\}$, there exists $A \in \mathcal{A}$ such that $L_A(x) = 1$ and $L_A(y) = 0$ or $R_A(x) = 1$ and $R_A(y) = 0$.

Proposition 6.10. *If \mathcal{A} is separating, then $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_1 .*

Proof. Let $x, y \in X$ such that $x \neq y$. Then $y \in X \setminus \{x\}$ and so by our assumption, there exists $A \in \mathcal{A}$ such that $L_A(x) = 1$ and $L_A(y) = 0$ or $R_A(x) = 1$ and

$R_A(y) = 0$. Similarly, for $x \in X \setminus \{y\}$, there exists $A' \in \mathcal{A}$ such that $L_{A'}(y) = 1$ and $L_{A'}(x) = 0$ or $R_{A'}(y) = 1$ and $R_{A'}(x) = 0$. Since $L_A, L_{A'}, R_A, R_{A'}$ are fuzzy closed sets in X , therefore $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_1 . \square

Example 6.3. Let \mathcal{R} be a fuzzy relation on $X = \{a, b\}$, which is given as follows:

\mathcal{R}	a	b	c
a	1	0.3	0.4
b	0	1	0
c	0	0.5	1

and $\mathcal{A} = \{\{a\}, \{b\}, \{c\}\}$. Then the fuzzy topology $\tau_{\mathcal{R}, \mathcal{A}}$ is generated by the following subbase \mathcal{S} for the fuzzy closed sets in X :

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}$ are given by:

$$\begin{aligned} L_{\{a\}} &= \frac{1}{a} + \frac{0}{b} + \frac{0}{c}, & L_{\{b\}} &= \frac{0.3}{a} + \frac{1}{b} + \frac{0.5}{c}, & L_{\{c\}} &= \frac{0.4}{a} + \frac{0}{b} + \frac{1}{c}, \\ R_{\{a\}} &= \frac{1}{a} + \frac{0.3}{b} + \frac{0.4}{c}, & R_{\{b\}} &= \frac{0}{a} + \frac{1}{b} + \frac{0}{c}, & R_{\{c\}} &= \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c}. \end{aligned}$$

Since, $L_{\{a\}} = \{a\}$, $R_{\{b\}} = \{b\}$ and $L_{\{c\}} \cap R_{\{c\}} = \{c\}$ are fuzzy closed set in X , so in view of the Proposition 6.5, $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_1 .

Definition 6.11. A collection \mathcal{A} of \mathcal{R} -antisets completely separates points of X if for each $x, y \in X$ such that $x \neq y$, there exist B_1, B_2, \dots, B_k , where for each i , $B_i = L_A$ or R_A , $A \in \mathcal{A}$ such that $X = \bigcup_{i=1}^k B_i$ and $B_i(x) > 0$ implies that $B_i(y) = 0$.

Theorem 6.12. If \mathcal{A} completely separates points of X , then $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_2 .

Proof. Let x_r and y_s be two distinct fuzzy points in X . Then $x \neq y$. Since \mathcal{A} completely separates points of X , so there exist B_1, B_2, \dots, B_k , where for each i , $B_i = L_A$ or R_A , $A \in \mathcal{A}$ such that

$$X = \bigcup_{i=1}^k B_i \tag{6.1}$$

and $B_i(x) > 0$ implies that $B_i(y) = 0$. Since $X = \bigcup_{i=1}^k B_i$, so for $x, y \in X$, there exist i_1 and i_2 such that $B_{i_1}(x) = 1$ and $B_{i_2}(y) = 1$. Next, since $B_{i_1}(x) = 1 > 0$,

so $B_{i_1}(y) = 0$. Let $X_1 = \cup\{B_i|B_i(y) = 0\}$ and $X_2 = \cup\{B_i|B_i(y) > 0\}$. Here $X_1(y) = 0, X_2(x) = 0$ and in view of (6.1), we have $X_1 \cup X_2 = X$. Note that X_1 and X_2 are fuzzy closed sets in X such that $r > X_2(x), s > X_1(y)$ and $X_1 \cup X_2 = X$. Therefore, $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_2 . \square

Example 6.4. Let \mathcal{R} be a fuzzy relation on $X = \{a, b, c\}$, which is given as follows:

\mathcal{R}	a	b	c
a	1	0	0.3
b	0.7	1	0
c	0	0.8	1

and $\mathcal{A} = \{\{a\}, \{b\}, \{c\}\}$. Then the fuzzy topology $\tau_{\mathcal{R}, \mathcal{A}}$ is generated by the following subbase \mathcal{S} for the fuzzy closed sets in X :

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}$ are given by:

$$\begin{aligned} L_{\{a\}} &= \frac{1}{a} + \frac{0.7}{b} + \frac{0}{c}, & L_{\{b\}} &= \frac{0}{a} + \frac{1}{b} + \frac{0.8}{c}, & L_{\{c\}} &= \frac{0.3}{a} + \frac{0}{b} + \frac{1}{c}, \\ R_{\{a\}} &= \frac{1}{a} + \frac{0}{b} + \frac{0.3}{c}, & R_{\{b\}} &= \frac{0.7}{a} + \frac{1}{b} + \frac{0}{c}, & R_{\{c\}} &= \frac{0}{a} + \frac{0.8}{b} + \frac{1}{c}. \end{aligned}$$

Since for the fuzzy points $x_r, y_s \in X$, there exist two fuzzy closed sets $U = L_{\{b\}} \cup R_{\{c\}}$ and $V = L_{\{c\}} \cup R_{\{a\}}$ in X such that $r > U(x), s > V(y)$ and $U \cup V = X$, for the fuzzy points $y_r, z_s \in X$, there exist two fuzzy closed sets $U = L_{\{c\}} \cup R_{\{a\}}$ and $V = L_{\{a\}} \cup R_{\{b\}}$ in X such that $r > U(y), s > V(z)$ and $U \cup V = X$ and for the fuzzy points $z_r, x_s \in X$, there exist two fuzzy closed sets $U = L_{\{a\}} \cup R_{\{b\}}$ and $V = L_{\{b\}} \cup R_{\{c\}}$ in X such that $r > U(z), s > V(x)$ and $U \cup V = X$, so $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy T_2 .

6.3 Finite intersection property and fuzzy compactness

In this section, we introduce ‘finite intersection property’ in fuzzy topological spaces and find a sufficient condition under which a fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.

Definition 6.13. A family \mathcal{F} of fuzzy sets is said to satisfy the *finite intersection property (F.I.P.)* if for every $\alpha \in (0, 1]$, there exists ϵ , $0 < \epsilon < \alpha$ such that for every finite subfamily F_1, F_2, \dots, F_n of \mathcal{F} , there exists $x \in X$ such that $(\bigcap_{i=1}^n F_i)(x) > 1 - \alpha + \epsilon$.

Note that the fuzzy set 0_X cannot be a member of a family satisfying finite intersection property.

Theorem 6.14. Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:

1. If \mathcal{F} is a family of fuzzy closed sets satisfying finite intersection property (F.I.P.), then for each $\alpha \in (0, 1]$, there exists $y \in X$ such that $(\bigcap_{F \in \mathcal{F}} F)(y) > 1 - \alpha$.
2. (X, τ) is fuzzy compact.

Proof. (1) \Rightarrow (2) Let \mathcal{G} be a family of fuzzy open sets in X such that

$$\begin{aligned} \alpha_X &\subseteq \bigcup_{G_i \in \mathcal{G}} G_i \text{ and } \alpha \in (0, 1] \\ \Rightarrow \bigcap_{G_i \in \mathcal{G}} G_i^c &\subseteq (1 - \alpha)_X \\ \Rightarrow \nexists y \in X \text{ such that } &(\bigcap_{G_i \in \mathcal{G}} G_i^c)(y) > 1 - \alpha. \end{aligned}$$

Therefore in view of (1), $\mathcal{F} = \{G_i^c : G_i \in \mathcal{G}\}$ does not satisfy F.I.P., so for each ϵ such that $0 < \epsilon < \alpha$, there exist $G_1^c, G_2^c, \dots, G_n^c \in \mathcal{F}$ such that

$$\begin{aligned} (\bigcap_{i=1}^n G_i^c)(x) &\leq 1 - \alpha + \epsilon, \quad \text{for each } x \in X \\ \Rightarrow \bigcap_{i=1}^n G_i^c &\subseteq (1 - \alpha + \epsilon)_X \\ \Rightarrow (\alpha - \epsilon)_X &\subseteq \bigcup_{i=1}^n G_i \\ \Rightarrow \alpha_X &\text{ is fuzzy compact.} \\ \Rightarrow (X, \tau) &\text{ is fuzzy compact.} \end{aligned}$$

(2) \Rightarrow (1) Conversely, assume that (X, τ) is fuzzy compact i.e., each $\alpha_X, \alpha \in [0, 1]$, is fuzzy compact. Let \mathcal{F} be a family of fuzzy closed sets satisfying F.I.P. We have to show that for each $\alpha \in (0, 1]$, there exists $x \in X$ such that

$$\left(\bigcap_{F \in \mathcal{F}} F\right)(x) > 1 - \alpha.$$

Assume the contrary, i.e, for $\alpha \in (0, 1]$,

$$\begin{aligned} & \left(\bigcap_{F \in \mathcal{F}} F\right)(x) \leq 1 - \alpha, \quad \forall x \in X \\ \Rightarrow & \bigcap_{F \in \mathcal{F}} F \subseteq (1 - \alpha)_X \\ \Rightarrow & \alpha_X \subseteq \bigcup_{F \in \mathcal{F}} F^c \end{aligned}$$

This implies that $\{F^c : F \in \mathcal{F}\}$ is an open cover of α_X . Since α_X is fuzzy compact, so for each ϵ such that $0 < \epsilon < \alpha$, there exist $F_1^c, F_2^c, \dots, F_n^c \in \mathcal{F}$ such that

$$\begin{aligned} & (\alpha - \epsilon)_X \subseteq \bigcup_{i=1}^n F_i^c \\ \Rightarrow & \bigcap_{i=1}^n F_i \subseteq (1 - \alpha + \epsilon)_X \\ \Rightarrow & \nexists \text{ any } x \text{ such that } \left(\bigcap_{i=1}^n F_i\right)(x) > 1 - \alpha + \epsilon, \end{aligned}$$

implying that \mathcal{F} does not satisfy F.I.P., which is a contradiction. \square

Definition 6.15. [61] Let \mathcal{R} be a fuzzy partial ordering on X and $A \subseteq X$. Then the *fuzzy upper bound* for A is the fuzzy set denoted by $U(\mathcal{R}, A)$ and defined by

$$U(\mathcal{R}, A) = \bigcap_{x \in A} R_x, \text{ where } R_x(y) = \mathcal{R}(x, y), \text{ for each } y \in X.$$

Definition 6.16. A subset A of X is said to be α -level bounded above, $\alpha \in (0, 1]$, if for every $\epsilon, 0 < \epsilon < \alpha$, there exists $y \in X$ such that

$$U(\mathcal{R}, A)(y) > 1 - \alpha + \epsilon.$$

We say that A is *fuzzy bounded above* if it is α -level bounded above, for each $\alpha \in (0, 1]$.

Definition 6.17. A subset A of X is said to have α -level least upper bound, $\alpha \in (0, 1]$, if for every ϵ , $0 < \epsilon < \alpha$, there exists $z \in X$ such that $U(\mathcal{R}, A)(z) > 1 - \alpha + \epsilon$ and $\mathcal{R}(z, y) > 1 - \alpha + \epsilon$, $\forall y$ such that $\mathcal{R}(x, y) > 1 - \alpha$, for each $x \in A$.

Definition 6.18. A set X is said to be α -level complete if every $A \subseteq X$ which is α -level bounded above, has an α -level least upper bound.

We say that X is *fuzzy complete* if it is α -level complete for every $\alpha \in (0, 1]$.

Proposition 6.19. If X is α -level bounded above and $A \subseteq X$, then A is also α -level bounded above.

Proof. Since X is α -level bounded above, so for every ϵ , $0 < \epsilon < \alpha$, there exists $y \in X$ such that

$$U(\mathcal{R}, X)(y) > 1 - \alpha + \epsilon.$$

We have to show that for every ϵ , $0 < \epsilon < \alpha$, there exists $y \in X$ such that $U(\mathcal{R}, A)(y) > 1 - \alpha + \epsilon$. Assume the contrary that for each $y \in X$,

$$\begin{aligned} U(\mathcal{R}, A)(y) &\leq 1 - \alpha + \epsilon \\ \Rightarrow \inf_{a \in A} \mathcal{R}(a, y) &\leq 1 - \alpha + \epsilon \\ \Rightarrow U(\mathcal{R}, X)(y) = \inf_{x \in X} \mathcal{R}(x, y) &\leq \inf_{a \in A} \mathcal{R}(a, y) \leq 1 - \alpha + \epsilon, \text{ for each } y \in Y \end{aligned}$$

which is a contradiction to the fact that X is α -level bounded above.

Theorem 6.20. Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:

1. If \mathcal{F} is a family of subbasic fuzzy closed sets satisfying finite intersection property (F.I.P.), then for each $\alpha \in (0, 1]$, there exists $y \in X$ such that $(\bigcap_{F \in \mathcal{F}} F)(y) > 1 - \alpha$.
2. For any subbase \mathcal{S} of τ , if $\mathcal{G} \subseteq \mathcal{S}$ such that $\alpha_X \subseteq \bigcup_{f \in \mathcal{G}} f$, then for each ϵ , $0 < \epsilon < \alpha$, there exists a finite subset \mathcal{G}_0 of \mathcal{G} such that $(\alpha - \epsilon)_X \subseteq \bigcup_{f \in \mathcal{G}_0} f$.
3. (X, τ) is fuzzy compact.

Proof. The equivalence of (2) and (3) has already been proved in [69]. Here we prove (1) \Leftrightarrow (2). For (1) \Rightarrow (2), let \mathcal{S} be a subbase for τ and $\mathcal{G} \subseteq \mathcal{S}$ such that

$$\begin{aligned} \alpha_X &\subseteq \bigcup_{f \in \mathcal{G}} f \\ \Rightarrow \bigcap_{f \in \mathcal{G}} f^c &\subseteq (1 - \alpha)_X \\ \Rightarrow \nexists y \in X \text{ such that } & \left(\bigcap_{f \in \mathcal{G}} f^c \right)(y) > 1 - \alpha. \end{aligned}$$

Therefore in view of (1), $\mathcal{F} = \{f^c : f \in \mathcal{G}\}$ does not satisfy F.I.P., so for each ϵ such that $0 < \epsilon < \alpha$, there exist $f_1^c, f_2^c, \dots, f_n^c \in \mathcal{G}$ such that

$$\begin{aligned} \left(\bigcap_{i=1}^n f_i^c \right)(x) &\leq 1 - \alpha + \epsilon, \quad \text{for each } x \in X \\ \Rightarrow \bigcap_{i=1}^n f_i^c &\subseteq (1 - \alpha + \epsilon)_X \\ \Rightarrow (\alpha - \epsilon)_X &\subseteq \bigcup_{i=1}^n f_i. \end{aligned}$$

Conversely, assume that \mathcal{F} be a family of subbasic fuzzy closed sets satisfying F.I.P. We have to show that for each $\alpha \in (0, 1]$, there exists $x \in X$ such that

$$\left(\bigcap_{F \in \mathcal{F}} F \right)(x) > 1 - \alpha.$$

Assume the contrary i.e, for $\alpha \in (0, 1]$,

$$\begin{aligned} \left(\bigcap_{F \in \mathcal{F}} F \right)(x) &\leq 1 - \alpha, \quad \forall x \in X \\ \Rightarrow \bigcap_{F \in \mathcal{F}} F &\subseteq (1 - \alpha)_X \\ \Rightarrow \alpha_X &\subseteq \bigcup_{F \in \mathcal{F}} F^c. \end{aligned}$$

So according to our assumption, for each ϵ such that $0 < \epsilon < \alpha$, there exist $F_1^c, F_2^c, \dots, F_n^c \in \mathcal{F}$ such that

$$\begin{aligned} (\alpha - \epsilon)_X &\subseteq \bigcup_{i=1}^n F_i^c \\ \Rightarrow \bigcap_{i=1}^n F_i &\subseteq (1 - \alpha + \epsilon)_X \\ \Rightarrow \nexists \text{ any } x \text{ such that } &\left(\bigcap_{i=1}^n F_i\right)(x) > 1 - \alpha + \epsilon, \end{aligned}$$

implying that \mathcal{F} does not satisfy F.I.P., which is a contradiction. \square

Theorem 6.21. *Let \mathcal{R} be a fuzzy partial order. If X is fuzzy complete, fuzzy bounded above and \mathcal{A} consists of singletons, then $(X, \tau_{\mathcal{R}, \mathcal{A}})$ is fuzzy compact.*

Proof. Let \mathcal{F} be a family of subbasic fuzzy closed sets of X satisfying F.I.P. First, we show that there does not exist any δ_X , where $\delta \in (0, 1)$, belonging to \mathcal{F} . Since if we assume that \mathcal{F} contains some δ_X , where $\delta \in (0, 1)$, then

$$\bigcap_{F \in \mathcal{F}_1} F \subseteq \delta_X,$$

where \mathcal{F}_1 is a finite subfamily of \mathcal{F} containing δ_X , implies that

$$\bigcap_{F \in \mathcal{F}_1} F \subseteq (1 - \alpha_1)_X \subseteq (1 - \alpha_1 + \epsilon)_X, \quad \text{for } \alpha_1 \in (0, 1) \text{ such that } \delta = (1 - \alpha_1)$$

and for each ϵ , $0 < \epsilon < \alpha_1$

$\Rightarrow \mathcal{F}$ does not satisfy F.I.P., which is a contradiction.

Next, if \mathcal{F} contains only 1_X , then the proof of the theorem is trivial and if there exists some member of the form $L_{\{x\}}$ or $R_{\{x\}}$ other than 1_X in \mathcal{F} , then 1_X plays no role in the intersection $\bigcap_{F \in \mathcal{F}} F$, so it is sufficient to prove the theorem for \mathcal{F} of the form $\{L_{\{x_\beta\}} \mid \beta \in \Omega_1\} \cup \{R_{\{x_\beta\}} \mid \beta \in \Omega_2\}$. Now, we have to show that for each $\alpha \in (0, 1]$, there exists $z \in X$ such that $(\bigcap_{F \in \mathcal{F}} F)(z) > 1 - \alpha$.

Let $A = \{x_\beta \mid \beta \in \Omega_2\} \subseteq X$. Since for $\alpha \in (0, 1]$, X is α -level bounded above, so by Proposition 6.19, A is also α -level bounded above. Therefore, by the fuzzy completeness of X , there exists an α -level least upper bound of A . So, for each ϵ

such that $0 < \epsilon < \alpha$, there exists an element $x_0 \in X$ such that

$$U(\mathcal{R}, A)(x_0) > 1 - \alpha + \epsilon \text{ and } \mathcal{R}(x_0, y) > 1 - \alpha + \epsilon, \forall y \in X$$

$$\text{such that } \mathcal{R}(x, y) > 1 - \alpha, \text{ for each } x \in A \quad (6.2)$$

$$\begin{aligned} \Rightarrow \inf_{x \in A} \mathcal{R}(x, x_0) &> 1 - \alpha + \epsilon \\ \Rightarrow \mathcal{R}(x_\beta, x_0) &> 1 - \alpha + \epsilon, \text{ for each } \beta \in \Omega_2 \\ \Rightarrow R_{\{x_\beta\}}(x_0) &> 1 - \alpha + \epsilon, \text{ for each } \beta \in \Omega_2. \end{aligned} \quad (6.3)$$

Since for $\beta \in \Omega_2$, $\mathcal{B} = \{R_{\{x_\beta\}}, L_{\{x_\gamma\}}\}$, where $\gamma \in \Omega_1$, is a finite subfamily of \mathcal{F} , so there exists ϵ_1 , $0 < \epsilon_1 < \alpha$ and $y \in X$ such that

$$\begin{aligned} (R_{\{x_\beta\}} \cap L_{\{x_\gamma\}})(y) &> 1 - \alpha + \epsilon_1 \\ \Rightarrow \min\{R_{\{x_\beta\}}(y), L_{\{x_\gamma\}}(y)\} &> 1 - \alpha + \epsilon_1 \\ \Rightarrow \min\{\mathcal{R}(x_\beta, y), \mathcal{R}(y, x_\gamma)\} &> 1 - \alpha + \epsilon_1 \\ \Rightarrow \mathcal{R}(x_\beta, x_\gamma) &> 1 - \alpha + \epsilon_1 \text{ (Using transitivity of } \mathcal{R}\text{)} \\ \Rightarrow \mathcal{R}(x_\beta, x_\gamma) &> 1 - \alpha. \end{aligned}$$

Since the above inequality holds for each $\beta \in \Omega_2$, so we have

$$\begin{aligned} \Rightarrow \mathcal{R}(x_0, x_\gamma) &> 1 - \alpha + \epsilon \text{ (Using 6.2 and putting } y = x_\gamma\text{)} \\ \Rightarrow L_{\{x_\gamma\}}(x_0) &> 1 - \alpha + \epsilon. \end{aligned}$$

The above inequality holds for every $\gamma \in \Omega_1$. So

$$L_{\{x_\beta\}}(x_0) > 1 - \alpha + \epsilon, \text{ for each } \beta \in \Omega_1. \quad (6.4)$$

From (6.3) and (6.4), we get

$$\begin{aligned} \inf_{F \in \mathcal{F}} F(x_0) &\geq 1 - \alpha + \epsilon > 1 - \alpha \\ \Rightarrow \left(\bigcap_{F \in \mathcal{F}} F \right)(x_0) &> 1 - \alpha. \end{aligned} \quad (6.5)$$

□

6.4 Conclusion

In this chapter, we have introduced fuzzy topologies generated by fuzzy relations, as a generalization of the corresponding concept given by Smithson[102]. We have then obtained sufficient conditions under which this generated fuzzy topology satisfies separation axioms, fuzzy T_0 , fuzzy T_1 and fuzzy T_2 . Further, we have introduced ‘finite intersection property’ in fuzzy topological spaces and obtained a characterization of Lowen’s fuzzy compactness in terms of this property. Using this result, we have obtained a sufficient condition under which a fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.

