Chapter 5

Fuzzy topologies generated by fuzzy relations

5.1 Introduction

Binary relations are fundamental concept for expressing preferences, but the two valued concept is not suitable for expressing complexity of real life preferences. Fuzzy relations are generally used to overcome this limitation of binary relations. In literature, fuzzy relations have been studied by several authors(cf.[21–23, 40, 55, 65] etc).

Topologies induced by different types of binary relations have been studied by several authors, in literature. Campión et al.[18] introduced and studied preorderable topologies. They obtained a characterization of preorderable topologies. Earlier, Dallen and Wattel[27] had obtained a characterization of orderable topologies. Smithson[102] initiated the study of topologies induced by binary relations of a general kind. Since then, many researchers have been working in this direction(cf. [6, 75, 95, 111, 112]). Knoblauch[62] introduced the notion of topologies induced by a binary relation $\mathcal{R}$ of a general kind, in a different way, where the topology on a non empty set X is generated by the set of all upper and lower contours of elements of X with respect to $\mathcal{R}$. He had also obtained a characterization for topologies induced by a binary relation of a general kind. Further, Induráin et al.[53] had studied topologies induced by binary relations in the sense of

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Knoblauch[62] and also introduced and studied bitopological spaces induced by binary relations. Motivated by these facts, in this chapter, we have introduced fuzzy topologies generated by a fuzzy relation, which is a generalization of the corresponding concept in [62] and studied related results in fuzzy setting. We have also introduced and studied fuzzy bitopological spaces generated by a fuzzy relation. In particular, we have introduced the notions of preorderable and orderable fuzzy topologies and obtained characterizations of a fuzzy topology generated by a fuzzy relation, a fuzzy topology generated by a fuzzy interval order, preorderable and orderable fuzzy topologies and a fuzzy bitopological space generated by a fuzzy relation.

5.2 Fuzzy topology generated by a fuzzy relation

Knoblauch[62] had introduced a topology generated by a binary relation. Here we extend this concept in the case of fuzzy topology.

Definition 5.1. Let $\mathcal{R}$ be a fuzzy relation on a set X. Then for $x \in X$, the fuzzy sets L_x and R_x, which are defined as

$$L_x(y) = \mathcal{R}(y,x), \text{for all } y \in X,$$
$$R_x(y) = \mathcal{R}(x,y), \text{for all } y \in X,$$

are called *lower* and *upper* contours, respectively of the element $x \in X$.

The fuzzy topology generated by the collection $\mathcal{S}_1$ of all lower contours (i.e., $\mathcal{S}_1 = \{L_x : x \in X\}$) will be denoted by τ_1 and the fuzzy topology generated by the collection $\mathcal{S}_2$ of all upper contours (i.e, $S_2 = \{R_x : x \in X\}$) will be denoted by τ_2.

Definition 5.2. The fuzzy topology which is generated by the subbase $\mathcal{S} = \{L_x\}_{x\in X} \cup \{R_x\}_{x\in X}$ is called the *fuzzy topology generated by* $\mathcal{R}$ and is denoted by $\tau_{\mathcal{R}}$.

Example 5.1. *Let* $\mathcal{R}$ *be a fuzzy relation on* $X = \{x, y\}$*, given by*

$\mathcal{R}$	x	y
x	*0.5*	*0.7*
y	*0.3*	*0.4*

Then, L_x, L_y, R_x, R_y are the fuzzy sets in X given by:

$$L_x = \frac{0.5}{x} + \frac{0.3}{y}, \quad L_y = \frac{0.7}{x} + \frac{0.4}{y}, \quad R_x = \frac{0.5}{x} + \frac{0.7}{y}, \quad R_y = \frac{0.3}{x} + \frac{0.4}{y}.$$

Therefore,

$$\begin{aligned} &\tau_1 = \{0_X, 1_X, L_x, L_y\}, \\ &\tau_2 = \{0_X, 1_X, R_x, R_y\}, \\ \text{and} \quad &\tau_{\mathcal{R}} = \{0_X, 1_X, L_x, L_y, R_x, R_y, \frac{0.5}{x} + \frac{0.4}{y}, \frac{0.3}{x} + \frac{0.3}{y}, \frac{0.7}{x} + \frac{0.7}{y}\}. \end{aligned}$$

Example 5.2. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{x, y\}$, which is given as follows:*

$\mathcal{R}$	x	y
x	*0.7*	*0.8*
y	*0.7*	*0.5*

Then the fuzzy topology $\tau_{\mathcal{R}}$ is generated by the following subbase $\mathcal{S}$:

$$\mathcal{S} = \{L_x, L_y, R_x, R_y\},$$

where L_x, L_y, R_x, R_y are given by:

$$L_x = \frac{0.7}{x} + \frac{0.7}{y}, \quad L_y = \frac{0.8}{x} + \frac{0.5}{y}, \quad R_x = \frac{0.7}{x} + \frac{0.8}{y}, \quad R_y = \frac{0.7}{x} + \frac{0.5}{y}.$$

Therefore, $\tau_{\mathcal{R}} = \{0_X, 1_X, L_x, L_y, R_x, R_y, \frac{0.8}{x} + \frac{0.7}{y}, \frac{0.8}{x} + \frac{0.8}{y}\}$ and since for $x, y \in X$ such that $x \neq y$, there exists $L_y \in \tau_{\mathcal{R}}$ such that $L_y(x) \neq L_y(y)$, so $(X, \tau_{\mathcal{R}})$ is fuzzy T_0.

Example 5.3. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{x, y, z\}$, which is given as follows:*

$\mathcal{R}$	x	y	z
x	*1*	*0.5*	*0*
y	*0*	*1*	*0.8*
z	*0.7*	*0*	*1*

Then the fuzzy topology $\tau_{\mathcal{R}}$ is generated by the following subbase $\mathcal{S}$:

$$\mathcal{S} = \{L_x, L_y, L_z, R_x, R_y, R_z\},$$

where $L_x, L_y, L_z, R_x, R_y, R_z$ *are given by:*

$$L_x = \frac{1}{x} + \frac{0}{y} + \frac{0.7}{z}, \quad L_y = \frac{0.5}{x} + \frac{1}{y} + \frac{0}{z}, \quad L_z = \frac{0}{x} + \frac{0.8}{y} + \frac{1}{z},$$
$$R_x = \frac{1}{x} + \frac{0.5}{y} + \frac{0}{z}, \quad R_y = \frac{0}{x} + \frac{1}{y} + \frac{0.8}{z}, \quad R_z = \frac{0.7}{x} + \frac{0}{y} + \frac{1}{z}.$$

Note that $(X, \tau_{\mathcal{R}})$ *is fuzzy* T_1*, since for the fuzzy points* x_r, y_s *in* X*, there exist fuzzy open sets* $U = L_x \cup R_z$ *and* $V = L_z \cup R_y$ *such that* $x_r \in U$*,* $x_r \notin V$*,* $y_s \notin U$*,* $y_s \in V$*, for the fuzzy points* y_r, z_s *in* X*, there exist fuzzy open sets* $U = R_x \cup L_y$ *and* $V = L_x \cup R_z$ *such that* $y_r \in U$*,* $y_r \notin V$*,* $z_s \notin U$*,* $z_s \in V$ *and for fuzzy points* x_r, z_s *in* X*, there exist fuzzy open sets* $U = R_x \cup L_y$ *and* $V = L_z \cup R_y$ *such that* $x_r \in U$*,* $x_r \notin V$*,* $z_s \notin U$*,* $z_s \in V$*.*

Example 5.4. *Let* $\mathcal{R}$ *be a fuzzy relation on* $X = \{x, y, z\}$*, which is given as follows:*

$\mathcal{R}$	x	y	z
x	*1*	*0.3*	*0.5*
y	*0*	*1*	*0*
z	*0*	*0.9*	*1*

Then the fuzzy topology $\tau_{\mathcal{R}}$ *is generated by the following subbase* $\mathcal{S}$*:*

$$\mathcal{S} = \{L_x, L_y, L_z, R_x, R_y, R_z\},$$

where $L_x, L_y, L_z, R_x, R_y, R_z$ *are given by:*

$$L_x = \frac{1}{x} + \frac{0}{y} + \frac{0}{z}, \quad L_y = \frac{0.3}{x} + \frac{1}{y} + \frac{0.9}{z}, \quad L_z = \frac{0.5}{x} + \frac{0}{y} + \frac{1}{z},$$
$$R_x = \frac{1}{x} + \frac{0.3}{y} + \frac{0.5}{z}, \quad R_y = \frac{0}{x} + \frac{1}{y} + \frac{0}{z}, \quad R_z = \frac{0}{x} + \frac{0.9}{y} + \frac{1}{z}.$$

Note that $(X, \tau_{\mathcal{R}})$ *is fuzzy* T_2*, since for the fuzzy points* x_r, y_s *in* X*, there exist fuzzy open sets* $U = L_x$ *and* $V = R_y$ *such that* $x_r \in U$*,* $y_s \in V$*,* $U \cap V = 0_X$*, for the fuzzy points* y_r, z_s *in* X*, there exist fuzzy open sets* $U = R_y$ *and* $V = L_z \cap R_z$ *such that* $y_r \in U$*,* $z_s \in V, U \cap V = 0_X$ *and for fuzzy points* x_r, z_s *in* X*, there exist fuzzy open sets* $U = L_x$ *and* $V = L_z \cap R_z$ *such that* $x_r \in U$*,* $z_s \in V, U \cap V = 0_X$*.*

Definition 5.3. ([61]) A fuzzy relation $\mathcal{R}$ on a set X is called

1. *reflexive* if $\mathcal{R}(x, x) = 1,$ for each $x \in X$;

2. *irreflexive* if $\mathcal{R}(x,x) \neq 1$, for some $x \in X$;

3. *antireflexive* if $\mathcal{R}(x,x) = 0$, for each $x \in X$;

4. *symmetric* if $\mathcal{R}(x,y) = \mathcal{R}(y,x)$, for each $(x,y) \in X \times X$.
   The following definitions are from [40]:

5. *transitive* if $\mathcal{R}(x,z) \geq \min\{\mathcal{R}(x,y), \mathcal{R}(y,z)\}$, for each $x, y, z \in X$;

6. *asymmetric* if $\min\{\mathcal{R}(x,y), \mathcal{R}(y,x)\} = 0$, for each $(x,y) \in X \times X$;

7. *antisymmetric* if $\min\{\mathcal{R}(x,y), \mathcal{R}(y,x)\} = 0$, for each $(x,y) \in X \times X$ such that $x \neq y$;

8. *negatively transitive* if $\max\{\mathcal{R}(x,y), \mathcal{R}(y,z)\} \geq \mathcal{R}(x,z)$, for each x, y, $z \in X$.

9. *total* if $\max\{\mathcal{R}(x,y), \mathcal{R}(y,x)\} = 1$, for each $x, y \in X$.

10. *connecting* if $\max\{\mathcal{R}(x,y), \mathcal{R}(y,x)\} = 1$, for each $(x,y) \in X \times X$, such that $x \neq y$.

We mention here that the definitions 9 and 10 have been called as '*strongly complete*' and '*complete*' respectively in [40] and the definition 3 has been called '*irreflexive*' in [41].

Definition 5.4. [123] A fuzzy relation is called a *fuzzy preorder* relation if it is reflexive and transitive.

Definition 5.5. [123] A fuzzy relation is called a *fuzzy partial order* relation if it is reflexive, transitive and antisymmetric.

Definition 5.6. [123] A fuzzy relation is called a *similarity* relation if it is reflexive, symmetric and transitive.

Now we prove:

Proposition 5.7. *If $\mathcal{R}$ is a symmetric fuzzy relation, then $\tau_1 = \tau_2$.*

Proof. Since $\mathcal{R}$ is a symmetric fuzzy relation, so $\mathcal{R}(x,y) = \mathcal{R}(y,x)$, for each $x, y \in X$. This implies that $R_x(y) = L_x(y)$, for each $x, y \in X$ and hence $R_x = L_x$, for each $x \in X$. Thus the topologies τ_1 and τ_2, which are generated by $\{L_x : x \in X\}$ and $\{R_x : x \in X\}$ respectively, are same.

Proposition 5.8. *If $\mathcal{R}$ is a fuzzy preorder relation, then*

1. *If $A \in \tau_1$, then $A \supseteq \bigcup_{x:A(x)=1} R_x$.*
2. *If $A \in \tau_2$, then $A \supseteq \bigcup_{x:A(x)=1} L_x$.*

Proof. 1. To show that $A \supseteq \bigcup_{x:A(x)=1} L_x$, let $y_r \in \bigcup_{x:A(x)=1} L_x$. This implies that there exists some x such that $A(x) = 1$ and $y_r \in L_x$. So $r < \mathcal{R}(y, x)$. Now since A is open and $A(x) = 1$, so $x_r \in A$ and there exists a basic fuzzy open set $\bigcap_{i=1}^{n} L_{x_i}$ such that

$$
\begin{aligned}
& x_r \in \bigcap_{i=1}^{n} L_{x_i} \subseteq A \\
\Rightarrow\ & r < \mathcal{R}(x, x_i), \quad \text{for each } i = 1, 2, ..., n \\
\Rightarrow\ & r < \min\{\mathcal{R}(y, x), \mathcal{R}(x, x_i)\} \leq \mathcal{R}(y, x_i), \quad \text{for each } i = 1, 2, ..., n \\
\Rightarrow\ & y_r \in L_{x_i}, \quad \text{for each } i = 1, 2, ..., n \\
\Rightarrow\ & y_r \in \bigcap_{i=1}^{n} L_{x_i} \subseteq A \\
\Rightarrow\ & y_r \in A \\
\Rightarrow\ & A \supseteq \bigcup_{x:A(x)=1} L_x
\end{aligned}
$$

2. The proof is similar to that of part 1.

□

Theorem 5.9. *Let (X, τ) be a fuzzy topological space. Then the fuzzy topology τ is generated by a fuzzy relation $\mathcal{R}$ if and only if it has a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$.*

Proof. First assume that τ is generated by some fuzzy relation $\mathcal{R}$, then obviously it has a subbase $\{U_x, V_x : x \in X\}$, where $U_x = L_x$ and $V_x = R_x$, for each $x \in X$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$.

Conversely, assume that τ has a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$. Now to show that τ is generated by some fuzzy relation $\mathcal{R}$, define a fuzzy relation $\mathcal{R} : X \times X \to I$ by $\mathcal{R}(x, y) = U_y(x) = V_x(y)$, for each

$(x, y) \in X \times X$. Then for $x \in X, L_x(y) = \mathcal{R}(y, x) = U_x(y)$ and $R_x(y) = \mathcal{R}(x, y) = V_x(y)$, for each $y \in X$ which implies that $L_x = U_x$ and $R_x = V_x$, for each $x \in X$. So from the hypothesis of the theorem, we have that the family $\{L_x, R_x : x \in X\}$ is a subbase for τ. Hence τ is generated by the fuzzy relation $\mathcal{R}$. □

Theorem 5.10. *Let (X, τ) be a fuzzy topological space. Suppose that τ has a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$. Let $a, b \in X$ such that $a_r \in O \Rightarrow b_r \in O$, for each $O \in \tau$ and $r \in (0, 1)$. Then $U_a \subseteq U_b$ and $V_a \subseteq V_b$.*

Proof. Let $z_r \in U_a$, for some $r \in (0, 1)$.

$$\begin{aligned}
&\Rightarrow \quad r < U_a(z) = V_z(a)\\
&\Rightarrow \quad a_r \in V_z\\
&\Rightarrow \quad b_r \in V_z \quad (\text{Since}\, V_z \in \tau)\\
&\Rightarrow \quad r < V_z(b) = U_b(z)\\
&\Rightarrow \quad z_r \in U_b\\
&\Rightarrow \quad U_a \subseteq U_b
\end{aligned}$$

Similarly we can prove that $V_a \subseteq V_b$. □

So far we have obtained the result that τ is generated by some fuzzy relation $\mathcal{R}$ if τ exhibits a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$. From this result, for the given fuzzy topology τ with the above mentioned subbase, the fuzzy relation $\mathcal{R}$ that generates τ(i.e, $\tau = \tau_{\mathcal{R}}$) can directly be obtained by defining $\mathcal{R} : X \times X \to I$ by $\mathcal{R}(x, y) = U_y(x) = V_x(y)$, for each $(x, y) \in X \times X$, and in this case the lower contour L_x and the upper contour R_x of the element $x \in X$ are same as U_x and V_x, respectively.

In the following theorem we have obtained that the fuzzy relation $\mathcal{R}$ that generates τ will satisfy some additional properties if we impose some conditions on the subbase elements U_x and V_x.

Theorem 5.11. *Let (X, τ) be a fuzzy topological space where τ has a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$. Consider the fuzzy relation $\mathcal{R} : X \times X \to I$ defined by $\mathcal{R}(x, y) = U_y(x) = V_x(y)$, for each $(x, y) \in X \times X$. Then the following properties hold good:*

1. *$\mathcal{R}$ is reflexive if and only if $U_x(x) = 1$, for each $x \in X$.*

2. *$\mathcal{R}$ is irreflexive if and only if $U_x(x) \neq 1$, for some $x \in X$.*

3. *$\mathcal{R}$ is antireflexive if and only if $U_x(x) = 0$, for each $x \in X$*

4. *$\mathcal{R}$ is symmetric if and only if $U_x = V_x$, for each $x \in X$.*

5. *$\mathcal{R}$ is asymmetric if and only if $U_x \cap V_x = 0_X$, for each $x \in X$.*

6. *$\mathcal{R}$ is antisymmetric if and only if $(U_x \cap V_x)(y) = 0$, for each $x, y \in X$ such that $x \neq y$.*

7. *$\mathcal{R}$ is transitive if and only if $U_z(x) \geq (V_x \cap U_z)(y)$ holds for each $x, y, z \in X$.*

8. *$\mathcal{R}$ is negatively transitive if and only if $(V_x \cup U_z)(y) \geq U_z(x)$, for each $x, y, z \in X$.*

9. *$\mathcal{R}$ is total if and only if $U_x \cup V_x = 1_X$, for each $x \in X$.*

10. *$\mathcal{R}$ is connecting if and only if $(U_x \cup V_x)(y) = 1$, for each $x, y \in X$ such that $x \neq y$.*

The proof is straightforward.

Definition 5.12. [13] Let $\mathcal{R}$ be a fuzzy preorder relation (resp., fuzzy partial order relation). Then the *associated asymmetric* fuzzy relation $\mathcal{R}_1$ is given by:

$$\mathcal{R}_1(x, y) = \max\{\mathcal{R}(x, y) - \mathcal{R}(y, x), 0\}, \quad \text{for each } (x, y) \in X \times X.$$

$\mathcal{R}_1$ is called the asymmetric part of the fuzzy preorder (resp., fuzzy partial order) $\mathcal{R}$.

Now we define a preorderable fuzzy topology on similar lines as in [18].

Definition 5.13. Let (X, τ) be a fuzzy topological space. Then the fuzzy topology τ is said to be a *preorderable*(resp., *orderable*) on X if it is generated by the asymmetric part of some total fuzzy preorder(resp., total fuzzy partial order) relation.

Example 5.5. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{x, y\}$, given by*

$\mathcal{R}$	x	y
x	*1*	*0.7*
y	*0.6*	*1*

It is easy to verify that $\mathcal{R}$ *is a fuzzy preorder relation and its associated asymmetric part* $\mathcal{R}_1$ *is the fuzzy relation on* $X = \{x, y\}$*, given by*

$\mathcal{R}_1$	x	y
x	*0*	*0.1*
y	*0*	*0*

Now, $L_x^{\mathcal{R}_1}, L_y^{\mathcal{R}_1}, R_x^{\mathcal{R}_1}, R_y^{\mathcal{R}_1}$ *are the fuzzy sets in* X*, given by*

$$L_x^{\mathcal{R}_1} = \frac{0}{x} + \frac{0}{y}, \quad L_y^{\mathcal{R}_1} = \frac{0.1}{x} + \frac{0}{y}, \quad R_x^{\mathcal{R}_1} = \frac{0}{x} + \frac{0.1}{y}, \quad R_y^{\mathcal{R}_1} = \frac{0}{x} + \frac{0}{y}.$$

Therefore, the preorderable fuzzy topology $\tau_{\mathcal{R}_1}$ *on* X *is given by*

$$\tau_{\mathcal{R}_1} = \{0_X, 1_X, L_y^{\mathcal{R}_1}, R_x^{\mathcal{R}_1}, \frac{0.1}{x} + \frac{0.1}{y}\}$$

Example 5.6. *Let* $\mathcal{R}$ *be a fuzzy relation on* $X = \{x, y\}$*, given by*

$\mathcal{R}$	x	y
x	*1*	*0.3*
y	*0*	*1*

It is easy to verify that $\mathcal{R}$ *is a fuzzy partial order relation and its associated asymmetric part* $\mathcal{R}_1$ *is the fuzzy relation on* $X = \{x, y\}$*, given by*

$\mathcal{R}_1$	x	y
x	*0*	*0.3*
y	*0*	*0*

Now, $L_x^{\mathcal{R}_1}, L_y^{\mathcal{R}_1}, R_x^{\mathcal{R}_1}, R_y^{\mathcal{R}_1}$ *are the fuzzy sets in* X*, given by*

$$L_x^{\mathcal{R}_1} = \frac{0}{x} + \frac{0}{y}, \quad L_y^{\mathcal{R}_1} = \frac{0.3}{x} + \frac{0}{y}, \quad R_x^{\mathcal{R}_1} = \frac{0}{x} + \frac{0.3}{y}, \quad R_y^{\mathcal{R}_1} = \frac{0}{x} + \frac{0}{y}.$$

Therefore, the orderable fuzzy topology $\tau_{\mathcal{R}_1}$ *on* X *is given by*

$$\tau_{\mathcal{R}_1} = \{0_X, 1_X, L_y^{\mathcal{R}_1}, R_x^{\mathcal{R}_1}, \frac{0.3}{x} + \frac{0.3}{y}\}$$

Theorem 5.14. *A fuzzy relation is asymmetric part of a total fuzzy preorder relation if and only if it is asymmetric and negatively transitive.*

Proof. Let $\mathcal{R}$ be a total fuzzy preorder relation and $\mathcal{R}_1$ be its associated asymmetric part. Since $\mathcal{R}$ is a total fuzzy preorder relation so the following conditions are satisfied for each $x, y, z \in X$:

$$\max\{\mathcal{R}(x,y), \mathcal{R}(y,x)\} = 1, \tag{5.1}$$

$$\mathcal{R}(x,x) = 1, \tag{5.2}$$

$$\mathcal{R}(x,z) \geq \min\{\mathcal{R}(x,y), \mathcal{R}(y,z)\}. \tag{5.3}$$

It has been already shown in [29] that $\mathcal{R}_1$ is asymmetric. So we only need to show that $\mathcal{R}_1$ is negatively transitive i.e.,

$$\max\{\mathcal{R}_1(x,y), \mathcal{R}_1(y,z)\} \geq \mathcal{R}_1(x,z), \text{ for each } x, y, z \in X.$$

In view of 5.1, we have to consider the following cases:
Case 1: If $\mathcal{R}(x,y) = 1$, $\mathcal{R}(y,z) = 1$, then by the transitivity of $\mathcal{R}$, $\mathcal{R}(x,z) = 1$. Now $\max\{\mathcal{R}_1(x,y), \mathcal{R}_1(y,z)\} = \max\{1-\mathcal{R}(y,x), 1-\mathcal{R}(z,y)\} = 1-\min\{\mathcal{R}(y,x), \mathcal{R}(z,y)\} \geq 1 - \mathcal{R}(z,x)) = \max\{\mathcal{R}(x,z) - \mathcal{R}(z,x), 0\} = \mathcal{R}_1(x,z)$.
Case 2: If $\mathcal{R}(x,y) = 1$, $\mathcal{R}(z,y) = 1$, then $\max\{\mathcal{R}_1(x,y), \mathcal{R}_1(y,z)\} = \max\{1 - \mathcal{R}(y,x), 0\} = 1-\mathcal{R}(y,x) \geq 1-\mathcal{R}(z,x) \geq \max\{\mathcal{R}(x,z)-\mathcal{R}(z,x), 0\} = \mathcal{R}_1(x,z)$ as by the transitivity of $\mathcal{R}$, we have $\mathcal{R}(z,x) \geq \min\{\mathcal{R}(z,y), \mathcal{R}(y,x)\} = \min\{1, \mathcal{R}(y,x)\} = \mathcal{R}(y,x)$ which implies that $1 - \mathcal{R}(z,x) \leq 1 - \mathcal{R}(y,x)$.
Case 3: If $\mathcal{R}(y,x) = 1$, $\mathcal{R}(y,z) = 1$, then $\max\{\mathcal{R}_1(x,y), \mathcal{R}_1(y,z)\} = \max\{0, 1 - \mathcal{R}(z,y)\} = 1-\mathcal{R}(z,y) \geq 1-\mathcal{R}(z,x) \geq \max\{\mathcal{R}(x,z)-\mathcal{R}(z,x), 0\} = \mathcal{R}_1(x,z)$ as by the transitivity of $\mathcal{R}$, we have $\mathcal{R}(z,x) \geq \min\{\mathcal{R}(z,y), \mathcal{R}(y,x)\} = \min\{\mathcal{R}(z,y), 1\} = \mathcal{R}(z,y)$ which implies that $1 - \mathcal{R}(z,x) \leq 1 - \mathcal{R}(z,y)$.
Case 4: If $\mathcal{R}(y,x) = 1$, $\mathcal{R}(z,y) = 1$. Then by the transitivity of $\mathcal{R}$, $\mathcal{R}(z,x) = 1$. Now $\max\{\mathcal{R}_1(x,y), \mathcal{R}_1(y,z)\} = 0$ and $\mathcal{R}_1(x,z) = \max\{\mathcal{R}(x,z) - \mathcal{R}(z,x), 0\} = 0$.

Conversely, assume that $\mathcal{R}'$ is an asymmetric and negatively transitive fuzzy relation. Then to show that it is the asymmetric part of some total fuzzy preorder relation, define a fuzzy relation given by $\mathcal{R}(x,y) = \max\{\mathcal{R}'(x,y), 1-\mathcal{R}'(y,x)\}$, for each $(x,y) \in X \times X$. Since $\mathcal{R}'$ is asymmetric so $\min\{\mathcal{R}'(x,x), \mathcal{R}'(x,x)\} = 0$ which implies that $\mathcal{R}'(x,x) = 0$. Hence $\mathcal{R}(x,x) = \max\{\mathcal{R}'(x,x), 1 - \mathcal{R}'(x,x)\} = 1$. Thus $\mathcal{R}$ is reflexive. Since $\mathcal{R}'$ is asymmetric so $\min\{\mathcal{R}'(x,y), \mathcal{R}'(y,x)\} = 0$

which implies that $\mathcal{R}'(x,y)=0$ or $\mathcal{R}'(y,x)=0$. Hence $\max\{\mathcal{R}(x,y),\mathcal{R}(y,x)\} = \max\{\max\{\mathcal{R}'(x,y),1-\mathcal{R}'(y,x)\},\max\{\mathcal{R}'(y,x),1-\mathcal{R}'(x,y)\}\}=1$. Next to show that $\mathcal{R}$ is transitive i.e., $\mathcal{R}(x,z)\geq \min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\}$, for each $x,y,z\in X$. Since $\mathcal{R}'$ is asymmetric(i.e, $\min\{\mathcal{R}'(x,y),\mathcal{R}'(y,x)\}=0$, for each $x,y\in X$) so we have to consider the following cases:

Case 1: Let $\mathcal{R}'(x,y)=0$ and $\mathcal{R}'(y,z)=0$. Since $\mathcal{R}'$ is negatively transitive so $\mathcal{R}'(x,z)\leq \max\{\mathcal{R}'(x,y),\mathcal{R}'(y,z)\}=0$ which implies that $\mathcal{R}'(x,z)=0$. Now $\mathcal{R}(x,z)=\max\{\mathcal{R}'(x,z),1-\mathcal{R}'(z,x)\}=1-\mathcal{R}'(z,x)\geq 1-\max\{\mathcal{R}'(y,x),\mathcal{R}'(z,y)\} = \min\{1-\mathcal{R}'(y,x),1-\mathcal{R}'(z,y)\}=\min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\}$.

Case 2: Let $\mathcal{R}'(x,y)=0$ and $\mathcal{R}'(z,y)=0$. Now $\min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\}=\min\{1-\mathcal{R}'(y,x),1\}=1-\mathcal{R}'(y,x)\leq 1-\mathcal{R}'(z,x)\leq \max\{\mathcal{R}'(x,z),1-\mathcal{R}'(z,x)\}=\mathcal{R}(x,z)$ as $\mathcal{R}'$ is negatively transitive so $\mathcal{R}'(z,x)\leq \max\{\mathcal{R}'(z,y),\mathcal{R}'(y,x)\}=\mathcal{R}'(y,x)$ which implies that $1-\mathcal{R}'(z,x)\geq 1-\mathcal{R}'(y,x)$.

Case 3: Let $\mathcal{R}'(y,x)=0$ and $\mathcal{R}'(y,z)=0$, then $\min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\} = \min\{1,1-\mathcal{R}'(z,y)\}=1-\mathcal{R}'(z,y)\leq 1-\mathcal{R}'(z,x)\leq \max\{\mathcal{R}'(x,z),1-\mathcal{R}'(z,x)\} = \mathcal{R}(x,z)$ as $\mathcal{R}'$ is negatively transitive so $\mathcal{R}'(z,x)\leq \max\{\mathcal{R}'(z,y),\mathcal{R}'(y,x)\} = \mathcal{R}'(z,y)$ which implies that $1-\mathcal{R}'(z,x)\geq 1-\mathcal{R}'(z,y)$.

Case 4: Let $\mathcal{R}'(y,x)=0$ and $\mathcal{R}'(z,y)=0$, then $\min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\} = \min\{1,1\}=1$ and $\mathcal{R}(x,z)=\max\{\mathcal{R}'(x,z),1-\mathcal{R}'(z,x)\}=1$ as $\mathcal{R}'$ is negatively transitive so $\mathcal{R}'(z,x)\leq \max\{\mathcal{R}'(z,y),\mathcal{R}'(y,x)\}=0$ which implies that $\mathcal{R}'(z,x)=0$.

Now it remains to prove that $\mathcal{R}'$ is asymmetric part of $\mathcal{R}$. Let $\mathcal{R}_1$ be the asymmetric part of $\mathcal{R}$. Then by definition, $\mathcal{R}_1$ is given by $\mathcal{R}_1(x,y)=\max\{\mathcal{R}(x,y)-\mathcal{R}(y,x),0\}$. We show that $\mathcal{R}_1=\mathcal{R}'$. Since $\mathcal{R}'$ is asymmetric, therefore $\min\{\mathcal{R}'(x,y),\mathcal{R}'(y,x)\}=0$ which implies that $\mathcal{R}'(x,y)=0$ or $\mathcal{R}'(y,x)=0$. Let $\mathcal{R}'(x,y)=0$. Then $\mathcal{R}_1(x,y)=\max\{\max\{\mathcal{R}'(x,y),1-\mathcal{R}'(y,x)\}-\max\{\mathcal{R}'(y,x),1-\mathcal{R}'(x,y)\},0\}=\max\{1-\mathcal{R}'(y,x)-1,0\}=0=\mathcal{R}'(x,y)$. Next, if $\mathcal{R}'(y,x)=0$. Then $\mathcal{R}_1(x,y)=\max\{\max\{\mathcal{R}'(x,y),1-\mathcal{R}'(y,x)\}-\max\{\mathcal{R}'(y,x),1-\mathcal{R}'(x,y)\},0\} = \max\{1-(1-\mathcal{R}'(x,y)),0\}=\mathcal{R}'(x,y)$. This completes the proof.

□

Proposition 5.15. *A fuzzy relation is asymmetric part of a total fuzzy partial order relation if and only if it is asymmetric, negatively transitive and connecting.*

Proof. Since it has already been shown in Theorem 5.14 that a fuzzy relation is asymmetric part of a total preorder if and only if it is asymmetric and negatively transitive, so we need to show that the asymmetric part of a total fuzzy partial order is connecting also and conversely we have to show that a fuzzy relation which is asymmetric, transitive and connecting, is the asymmetric part of some total fuzzy partial order.

Let $\mathcal{R}$ be a total fuzzy partial order and $\mathcal{R}_1$ be its asymmetric part. Then the following conditions are satisfied:

$$\max\{\mathcal{R}(x,y),\mathcal{R}(y,x)\} = 1, \text{ for each } x,y \in X,$$
$$\mathcal{R}(x,x) = 1, \text{ for each } x \in X$$
$$\mathcal{R}(x,z) \geq \min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\} \text{ for each } x,y,z \in X,$$
$$\min\{\mathcal{R}(x,y),\mathcal{R}(y,x)\} = 0, \text{ for each } x,y \in X, x \neq y.$$

Therefore, for $x,y \in X$, $x \neq y$ either $\mathcal{R}(x,y) = 0$ and $\mathcal{R}(y,x) = 1$ or $\mathcal{R}(x,y) = 1$ and $\mathcal{R}(y,x) = 0$. Thus, for $x,y \in X, x \neq y$ either $\mathcal{R}_1(x,y) = 1$ or $\mathcal{R}_1(y,x) = 1$ and hence $\max\{\mathcal{R}_1(x,y),\mathcal{R}_1(y,x)\} = 1$, for each $x,y \in X, x \neq y$, which implies $\mathcal{R}_1$ is connecting.

Conversely suppose that $\mathcal{R}'$ is a asymmetric, negatively transitive and connecting fuzzy relation. Then to show that it is the asymmetric part of some total fuzzy partial order, define a fuzzy relation $\mathcal{R}$ by $\mathcal{R}(x,y) = \max\{\mathcal{R}'(x,y), 1-\mathcal{R}'(y,x)\}$. Since it has been already shown in Theorem 5.14 that this fuzzy relation $\mathcal{R}$ is total preorder, so we only need to show that $\mathcal{R}$ is antisymmetric also i.e, $\min\{\mathcal{R}(x,y),\mathcal{R}(y,x)\} = 0$, for each $x,y \in X, x \neq y$. Since $\mathcal{R}'$ is asymmetric as well as connecting, so for each $x,y \in X, x \neq y$, either $\mathcal{R}'(x,y) = 0$ and $\mathcal{R}'(y,x) = 1$ or $\mathcal{R}'(x,y) = 1$ and $\mathcal{R}'(y,x) = 0$. Therefore for $x,y \in X, x \neq y$, either $\mathcal{R}(x,y) = 0$ or $\mathcal{R}(y,x) = 0$ and hence $\min\{\mathcal{R}(x,y),\mathcal{R}(y,x)\} = 0$ for each $x,y \in X, x \neq y$. Finally $\mathcal{R}'$ is asymmetric part of the total fuzzy partial order $\mathcal{R}$ has already been proved in Theorem 5.14. □

Proposition 5.16. *A fuzzy topology on a non empty set X is preorderable if and only if it has a subbase $\{U_x, V_x : x \in X\}$ such that the following conditions are satisfied:*

1. $U_y(x) = V_x(y)$, *for each* $x,y \in X$.
2. $U_x \cap V_x = 0_X$, *for each* $x \in X$.
3. $(V_x \cup U_z)(y) \geq U_z(x)$, *for each* $x,y,z \in X$.

Proof. A fuzzy topology is preorderable if it is generated by asymmetric part of some total fuzzy preorder relation. Since it has already been shown in the Theorem 5.14 that a fuzzy relation is asymmetric part of a total fuzzy preorder relation if and only if it is asymmetric and negatively transitive so the required conditions follow from Theorem 5.11. □

Proposition 5.17. *A fuzzy topology on a non empty set X is orderable if and only if it has a subbase $\{U_x, V_x : x \in X\}$ such that the following conditions are satisfied:*

1. $U_y(x) = V_x(y)$*, for each* $x, y \in X$*.*
2. $U_x \cap V_x = 0_X$*, for each* $x \in X$*.*
3. $(V_x \cup U_z)(y) \geq U_z(x)$*, for each* $x, y, z \in X$*.*
4. $(U_x \cup V_x)(y) = 1$*, for each* $x, y \in X$ *such that* $x \neq y$*.*

Proof. A fuzzy topology is orderable if it is generated by asymmetric part of a total fuzzy partial order relation. Since it has already been shown in the Proposition 5.15 that a fuzzy relation is asymmetric part of a total fuzzy partial order relation if and only if it is asymmetric, negatively transitive and connecting so the required conditions follow from Theorem 5.11. □

Corresponding to the definition 5.3 in [53], we give here the following:

Definition 5.18. Let X be a non empty set. A fuzzy relation $\mathcal{R}$ on a set X is said to be a *selection* if it is antireflexive and for every $x, y \in X$, $x \neq y$ either $\mathcal{R}(x, y) = 1$ and $\mathcal{R}(y, x) = 0$ or $\mathcal{R}(x, y) = 0$ and $\mathcal{R}(y, x) = 1$.

Theorem 5.19. *If (X, τ) is a fuzzy topological space such that τ is generated by a selection, then it has a subbase $\{U_x, V_x : x \in X\}$ such that the following conditions are satisfied:*

1. $U_y(x) = V_x(y)$*, for each* $x, y \in X$*.*
2. $\{U_x, V_x, \{x\}\}$ *is a fuzzy partition of* X*, for each* $x \in X$*.*

Proof. Assume that τ is generated by a selection. Since in Theorem 5.9, we have already shown that if τ is generated by a fuzzy relation $\mathcal{R}$ then it has a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$, so we only need to show that if a fuzzy relation $\mathcal{R}$ is a selection then $\{U_x, V_x, \{x\}\}$ is a fuzzy partition of X, for each $x \in X$. Since $\mathcal{R}$ is a selection it is antireflexive (i.e., $\mathcal{R}(x,x) = 0$, for each $x \in X$) implying that $U_x(x) = 0$ and $V_x(x) = 0$, for each $x \in X$ and for $x, y \in X, x \neq y$ either $\mathcal{R}(x,y) = 1$ and $\mathcal{R}(y,x) = 0$ or $\mathcal{R}(x,y) = 0$ and $\mathcal{R}(y,x) = 1$. This implies that for $x, y \in X$, $x \neq y$ either $V_x(y) = 1$ and $U_x(y) = 0$ or $U_x(y) = 1$ and $V_x(y) = 0$. So the only thing which remains to prove is that if $U_x(z_1) = V_x(z_2) = 1$, for some $z_1, z_2 \in X$, then $U_x(z_2) = V_x(z_1)$. Since $\mathcal{R}(z_1, x) = U_x(z_1) = 1$, so $x \neq z_1$ and therefore $\mathcal{R}(x, z_1) = 0$. Similarly, $\mathcal{R}(z_2, x) = 0$. Therefore $U_x(z_2) = \mathcal{R}(z_2, x) = \mathcal{R}(x, z_1) = V_x(z_1)$. □

Definition 5.20. [40] A fuzzy relation $\mathcal{R}$ on a set X is said to be *fuzzy interval order* if

1. $\max\{\mathcal{R}(x,y), \mathcal{R}(y,x)\} = 1$, for each $x, y \in X$.
2. $\min\{\mathcal{R}(x,y), \mathcal{R}(z,w)\} \leq \max\{\mathcal{R}(x,w), \mathcal{R}(z,y)\}$, for each $x, y, z, w \in X$.

Definition 5.21. [40] A fuzzy relation $\mathcal{R}$ on a set X is said to be *fuzzy semiorder* if

1. It is a fuzzy interval order.
2. $\min\{\mathcal{R}(x,y), \mathcal{R}(y,w)\} \leq \max\{\mathcal{R}(x,z), \mathcal{R}(z,w)\}$, for each $x, y, z, w \in X$.

Theorem 5.22. *Let (X, τ) be a fuzzy topological space. Then τ is generated by a fuzzy interval order if and only if it has a subbase $\{U_x, V_x : x \in X\}$ satisfying the following properties:*

1. *$U_y(x) = V_x(y)$, for each $x, y \in X$.*
2. *$V_x \cup U_x = 1_X$, for each $x \in X$.*
3. *$U_x \times U_y \subseteq (U_y \times 1_X) \cup (1_X \times U_x)$, for each $x, y \in X$.*

If in addition, $V_x \cap U_y \neq \phi \Rightarrow V_x \cup U_y = 1_X$, for each $x, y \in X$, then τ is generated by a fuzzy semiorder.

Proof. Let (X, τ) be a fuzzy topological space which is generated by a fuzzy interval order $\mathcal{R}$. Since it has already been proved in Theorem 5.9 that a fuzzy topology τ is generated by some fuzzy relation if and only if it has a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$ for each $x, y \in X$, so we only need to show that if the fuzzy relation $\mathcal{R}$ is a fuzzy interval order then the conditions (2) and (3) of the theorem are satisfied. Let $\mathcal{R}$ be a fuzzy interval order. Then

$$\max\{\mathcal{R}(x,y), \mathcal{R}(y,x)\} = 1 \text{ and } \min\{\mathcal{R}(x,y), \mathcal{R}(z,w)\} \leq \max\{\mathcal{R}(x,w), \mathcal{R}(z,y)\}, \quad \text{for each, } x, y, z, w \in X$$

$$\begin{aligned}
\Rightarrow\ & \max\{V_x(y), U_x(y)\} = 1 \text{ and } \min\{U_y(x), U_w(z)\} \leq \max\{U_w(x), U_y(z)\}, \\
& \quad \text{for each } x, y, z, w \in X \\
\Rightarrow\ & V_x \cup U_x = 1_X \text{ and } \min\{U_y(x), U_w(z)\} \leq \max\{\min\{U_w(x), 1\}, \min\{1, U_y(z)\}\}, \\
& \quad \text{for each } x, y, z, w \in X \\
\Rightarrow\ & V_x \cup U_x = 1_X \text{ and } U_x \times U_y \subseteq (U_y \times 1_X) \cup (1_X \times U_x), \\
& \text{for each } x, y \in X.
\end{aligned}$$

Conversely, assume that there exists a subbase $\{U_x, V_x : x \in X\}$ for τ satisfying the hypothesis (1)-(3) of the theorem. Since it has been already shown in the Theorem 5.9 that if τ has a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$, then τ is generated by the fuzzy relation $\mathcal{R}$, where $\mathcal{R} : X \times X \to I$ is given by $\mathcal{R}(x, y) = U_y(x) = V_x(y)$, for each $(x, y) \in X \times X$. We only need to show that $\mathcal{R}$ is a fuzzy interval order. Since

$$\begin{aligned}
& U_x \cup V_x = 1_X, \quad \text{for each } x \in X \\
\Rightarrow\ & \max\{U_x(y), V_x(y)\} = 1, \quad \text{for each } x,\ y \in X \\
\Rightarrow\ & \max\{\mathcal{R}(y,x), \mathcal{R}(x,y)\} = 1, \quad \text{for each } x,\ y \in X.
\end{aligned}$$

Also,

$$\begin{aligned}
& U_x \times U_y \subseteq (U_y \times 1_X) \cup (1_X \times U_x), \quad \text{for each } x, y \in X \\
\Rightarrow\ & (U_x \times U_y)(z, w) \leq \{(U_y \times 1_X) \cup (1_X \times U_x)\}(z, w), \quad \text{for each } x, y, z, w \in X \\
\Rightarrow\ & \min\{\mathcal{R}(z,x), \mathcal{R}(w,y)\} \leq \max\{\mathcal{R}(z,y), \mathcal{R}(w,x)\}, \quad \text{for each } x, y, z, w \in X.
\end{aligned}$$

This implies that the fuzzy relation $\mathcal{R}$ is a fuzzy interval order.

In addition, assume that $V_x \cap U_y \neq \phi \Rightarrow V_x \cup U_y = 1_X$, for each $x, y \in X$, then we have to show that the fuzzy relation $\mathcal{R}$ is a fuzzy semiorder. Since the condition $V_x \cap U_y \neq \phi \Rightarrow V_x \cup U_y = 1_X$ implies that if $(V_x \cap U_y)(z) > 0$ for some $z \in X$, then $(V_x \cup U_y)(w) = 1$, for each $w \in X$, so we have $(V_x \cap U_y)(z) \leq (V_x \cup U_y)(w)$, for each $x, y, z, w \in X$. This implies that $\min\{\mathcal{R}(x, z), \mathcal{R}(z, y)\} \leq \max\{\mathcal{R}(x, w), \mathcal{R}(w, y)\}$ for each $x, y, z, w \in X$ and hence $\mathcal{R}$ is a fuzzy semiorder. □

Now we define the adjoint of a fuzzy relation as a generalization of the corresponding concept given in [53].

Definition 5.23. Let $\mathcal{R}$ be a fuzzy relation on a non empty set X. Then the *adjoint* $\mathcal{R}^a$ of $\mathcal{R}$ is defined as the complement of the transpose of $\mathcal{R}$ i.e., $\mathcal{R}^a = (\mathcal{R}^t)^c$.

We note that the adjoint operator $\mathcal{R}^a$ is idempotent, i.e., $(\mathcal{R}^a)^a = \mathcal{R}$.

Theorem 5.24. *Let $\mathcal{R}$ be a fuzzy relation on a set X. Then the following properties are satisfied:*

1. *If $\{L_x, R_x\}, \{L_x^t, R_x^t\}, \{L_x^c, R_x^c\}, \{L_x^a, R_x^a\}$ represent the lower and upper contours of the element $x \in X$ with respect to $\mathcal{R}$, $\mathcal{R}^t, \mathcal{R}^c, \mathcal{R}^a$ respectively. Then*

   $L_x = R_x^t$, $R_x = L_x^t$, *for each* $x \in X$.

   $L_x^c = 1_X \setminus L_x = R_x^a$, $R_x^c = 1_X \setminus R_x = L_x^a$, *for each* $x \in X$.

2. $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^t}$.

3. $\tau_{\mathcal{R}^a} = \tau_{\mathcal{R}^c}$.

Proof. 1. Since $L_x(y) = \mathcal{R}(y, x) = \mathcal{R}^t(x, y) = R_x^t(y)$, for each $y \in Y$ and $R_x(y) = \mathcal{R}(x, y) = \mathcal{R}^t(y, x) = L_x^t(y)$, for each $y \in Y$. Therefore, $L_x = R_x^t$ and $R_x = L_x^t$.

Since $L_x^c(y) = \mathcal{R}^c(y, x) = 1 - \mathcal{R}(y, x) = (1_X \setminus L_x)(y) = (1_X \setminus R_x^t)(y) = 1 - R_x^t(y) = R_x^a(y)$, for each $y \in X$ and $R_x^c(y) = \mathcal{R}^c(x, y) = 1 - \mathcal{R}(x, y) = (1_X \setminus R_x)(y) = (1_X \setminus L_x^t)(y) = 1 - L_x^t(y) = L_x^a(y)$, for each $y \in X$, so we have $L_x^c = 1_X \setminus L_x = R_x^a$ and $R_x^c = 1_X \setminus R_x = L_x^a$.

2. Since $\tau_{\mathcal{R}}$ and $\tau_{\mathcal{R}^t}$ are respectively generated by the families $\{L_x, R_x : x \in X\}$ and $\{L^t_x, R^t_x : x \in X\}$ and $L_x = R^t_x$, $R_x = L^t_x$, for each $x \in X$ (by part (1)), so $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^t}$.

3. Since $\tau_{\mathcal{R}^a}$ and $\tau_{\mathcal{R}^c}$ are respectively generated by the families $\{L^a_x, R^a_x : x \in X\}$ and $\{L^c_x, R^c_x : x \in X\}$ and also $L^c_x = R^a_x$ and $R^c_x = L^a_x$, for each $x \in X$ (by part (1)), so $\tau_{\mathcal{R}^a} = \tau_{\mathcal{R}^c}$.

□

5.3 Fuzzy bitopological spaces generated by a fuzzy relation

In [53], authors had introduced and studied bitopological spaces generated by binary relations. In this section, we introduce and study fuzzy bitopological spaces generated by fuzzy relations.

Definition 5.25. [94] A *fuzzy bitopological space* is a triple (X, τ_1, τ_2), where X is a non empty set and τ_1, τ_2 are any fuzzy topologies on X.

The following is a generalization of the corresponding concept given in [53].

Definition 5.26. A fuzzy bitopological space (X, τ_1, τ_2) is said to be generated by a fuzzy relation $\mathcal{R}$ on X if $\tau_{\mathcal{R}} = \tau_1$ and $\tau_{\mathcal{R}^a} = \tau_2$

Example 5.7. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{x, y\}$, given by*

$\mathcal{R}$	x	y
x	*0.5*	*0.7*
y	*0.3*	*0.4*

It is easy to verify that the adjoint $\mathcal{R}^a$ of $\mathcal{R}$ is the fuzzy relation on $X = \{x, y\}$, given by

$\mathcal{R}^a$	x	y
x	*0.5*	*0.7*
y	*0.3*	*0.6*

Then in view of Example 5.1, the fuzzy topology τ_1 generated by a fuzzy relation $\mathcal{R}$ is given by:

$$\tau_{\mathcal{R}} = \{0_X, 1_X, L_x, L_y, R_x, R_y, \frac{0.5}{x} + \frac{0.4}{y}, \frac{0.3}{x} + \frac{0.3}{y}, \frac{0.7}{x} + \frac{0.7}{y}\}.$$

Now, the fuzzy topology τ_2 generated by $\mathcal{R}^a$ is generated by the subbasis $\{L_x^a, L_y^a, R_x^a, R_y^a\}$, where L_x^a, L_y^a, R_x^a and R_y^a are given as follows:

$$L_x^a = \frac{0.5}{x} + \frac{0.3}{y}, \quad L_y^a = \frac{0.7}{x} + \frac{0.6}{y}, \quad R_x^a = \frac{0.5}{x} + \frac{0.7}{y}, \quad R_y^a = \frac{0.3}{x} + \frac{0.6}{y}$$

and hence

$$\tau_2 = \{0_X, 1_X, L_x^a, L_y^a, R_x^a, R_y^a, \frac{0.3}{x} + \frac{0.3}{y}, \frac{0.5}{x} + \frac{0.6}{y}, \frac{0.7}{x} + \frac{0.7}{y}\}.$$

Therefore, (X, τ_1, τ_2) is a fuzzy bitopological space generated by $\mathcal{R}$.

Now we are going to obtain a characterization of a fuzzy bitopological space which is generated by some fuzzy relation.

Theorem 5.27. *A fuzzy bitopological space (X, τ_1, τ_2) is generated by a fuzzy relation if and only if there exist collections $\{U_x : x \in X\}$ and $\{V_x : x \in X\}$ of τ_1-fuzzy open sets such that*

1. *$\{U_x : x \in X\} \cup \{V_x : x \in X\}$ is a subbase of τ_1.*
2. *$\{1_X \setminus U_x : x \in X\} \cup \{1_X \setminus V_x : x \in X\}$ is a subbase for τ_2.*
3. *$U_y(x) = V_x(y)$, for each $x, y \in X$.*

Proof. First suppose that (X, τ_1, τ_2) is generated by a fuzzy relation. Let $\mathcal{R}$ be a fuzzy relation such that $\tau_1 = \tau_{\mathcal{R}}$ and $\tau_2 = \tau_{\mathcal{R}^a}$. By definition of $\tau_{\mathcal{R}}$, we know that the family of contours $\{L_x\}_{x\in X} \cup \{R_x\}_{x\in X}$ is a subbase for τ_1. Moreover, by part (1) and part (3) of Theorem 5.24, we have that $\{1_X \setminus L_x : x \in X\} \cup \{1_X \setminus R_x : x \in X\}$ is a subbase for $\tau_2 = \tau_{\mathcal{R}^a} = \tau_{\mathcal{R}^c}$. Also we have that $L_y(x) = R_x(y)$, for each $x, y \in X$. Put $U_x = L_x$ and $V_x = R_x$, then the required conditions (1)-(3) are satisfied.

Conversely, assume that there exist collections $\{U_x : x \in X\}$ and $\{V_x : x \in X\}$ of τ_1-fuzzy open sets satisfying conditions (1)-(3). Then to show that (X, τ_1, τ_2) is generated by some fuzzy relation, define a fuzzy relation $\mathcal{R} : X \times X \to I$ by

$\mathcal{R}(x,y) = U_y(x) = V_x(y)$, for each $(x,y) \in X \times X$. Then by using Theorem 5.9, $\mathcal{R}$ generates τ_1. Also note that the lower contour L_x and the upper contour R_x of the element $x \in X$ are U_x and V_x, respectively. Then by the hypothesis (2) given in the theorem, Theorem 5.24(part (1) and part(3)), $\mathcal{R}^a$ generates τ_2. This completes the proof. □

Now we prove the following theorem where we mean fuzzy T_1-ness and fuzzy T_0-ness in the sense of Srivastava et al.[107] and Lowen et al.[71], respectively.

Theorem 5.28. *Let (X, τ_1, τ_2) be a fuzzy bitopological space that is generated by some fuzzy relation. Then the following properties hold:*

1. *The fuzzy topology τ_1 is fuzzy T_1 iff the fuzzy topology τ_2 is fuzzy T_1.*
2. *The fuzzy topology τ_1 is fuzzy T_0 iff the fuzzy topology τ_2 is fuzzy T_0.*

Proof. 1. Let $\mathcal{R}$ be a fuzzy relation such that $\tau_1 = \tau_{\mathcal{R}}$ and $\tau_2 = \tau_{\mathcal{R}^a}$. Suppose that τ_1 is fuzzy T_1. To show that τ_2 is fuzzy T_1, choose two distinct fuzzy points x'_r and y'_s in X. Let $s_1 \in (0,1)$ be such that $s_1 > s$. Now x'_{1-r} and y'_{1-s_1} are two distinct fuzzy points in X. Since τ_1 is fuzzy T_1, there exist two fuzzy open sets $U, V \in \tau_1$ such that $x'_{1-r} \in U$, $y'_{1-s_1} \notin U$, $x'_{1-r} \notin V$, $y'_{1-s_1} \in V$. Since $\tau_1 = \tau_{\mathcal{R}}$, so U and V can be written respectively in the following form: $U = \bigcup_{i \in \Omega} \bigcap_{j \in I_1} U_{ij}$ and $V = \bigcup_{i \in \Omega_1} \bigcap_{j \in I_2} V_{ij}$, where U_{ij} and V_{ij} are of the form L_x or R_y, I_1 and I_2 are finite. Now

$$\begin{aligned}
& x'_{1-r} \in U \\
\Rightarrow\ & x'_{1-r} \in \bigcup_{i \in \Omega} \bigcap_{j \in I_1} U_{ij} \\
\Rightarrow\ & x'_{1-r} \in \bigcap_{j \in I_1} U_{i_1 j}, \quad \text{for some } i_1 \in \Omega \\
\Rightarrow\ & 1 - r < \min_{j \in I_1} U_{i_1 j}(x') \\
\Rightarrow\ & 1 - r < U_{i_1 j}(x'), \quad \text{for each } j \in I_1 \\
\Rightarrow\ & r > 1 - U_{i_1 j}(x'), \quad \text{for each } j \in I_1 \\
\Rightarrow\ & x'_r \notin U^c_{i_1 j}, \quad \text{for each } j \in I_1
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
& y'_{1-s_1} \notin U \\
\Rightarrow\ & 1-s > 1-s_1 \geq \sup_{i\in\Omega} \min_{j\in I_1} U_{ij}(y') \\
\Rightarrow\ & 1-s > \min_{j\in I_1} U_{ij}(y'), \quad \text{for each } i \in \Omega \\
\Rightarrow\ & 1-s > U_{i_1 j_1}(y'), \quad \text{for some } j_1 \in I_1 (\text{Since } I_1 \text{ is finite}) \\
\Rightarrow\ & s < 1 - U_{i_1 j_1}(y') \\
\Rightarrow\ & y'_s \in U^c_{i_1 j_1}. \qquad (5.5)
\end{aligned}$$

Therefore in view of (5.4) and (5.5), $x'_r \notin U^c_{i_1 j_1}$ and $y'_s \in U^c_{i_1 j_1}$. Now using Theorem 5.24(1), $U^c_{i_1 j_1}$ is of the form L^a_x or R^a_y. Let $U^c_{i_1 j_1} = A^a_{i_1 j_1}$, then $A^a_{i_1 j_1} \in \tau_2$ is such that $x'_r \notin A^a_{i_1 j_1}$ and $y'_s \in A^a_{i_1 j_1}$.

Next consider two distinct fuzzy points x'_{1-r_1} and y'_{1-s} in X where $r_1 > r$. Since τ_1 is fuzzy T_1, there exist two fuzzy open sets $U' = \bigcup_{i\in\Omega'} \bigcap_{j\in J_1} U'_{ij}$, $V' = \bigcup_{i\in\Omega'_1} \bigcap_{j\in J_2} V'_{ij}$ in τ_1 such that $x'_{1-r_1} \in U'$, $y'_{1-s} \notin U'$, $x'_{1-r_1} \notin V'$, $y'_{1-s} \in V'$. Now $x'_{1-r_1} \notin V'$, $y'_{1-s} \in V'$ implies, as in the previous case that $x'_r \in (V'_{i_2 j_2})^c$ and $y'_s \notin (V'_{i_2 j_2})^c$ where $(V'_{i_2 j_2})^c$ is of the form L^a_x or R^a_y. Now put $(V'_{i_2 j_2})^c = B^a_{i_2 j_2}$. Then $B^a_{i_2 j_2} \in \tau_2$ is such that $x'_r \in B^a_{i_2 j_2}$ and $y'_s \notin B^a_{i_2 j_2}$, showing that τ_2 is fuzzy T_1.

Conversely, assume that τ_2 is fuzzy T_1. To show that τ_1 is fuzzy T_1, choose two distinct fuzzy points x'_r and y'_s in X. Since τ_2 is fuzzy T_1, so for two distinct fuzzy points x'_{1-r} and y'_{1-s_1} in X where $0 < s < s_1 < 1$, there exist two fuzzy open sets U^a, $V^a \in \tau_2$ such that $x'_{1-r} \in U^a$, $y'_{1-s_1} \notin U^a$, $x'_{1-r} \notin V^a$, $y'_{1-s_1} \in V^a$. Since $\tau_2 = \tau_{\mathcal{R}^a}$, so U^a and V^a can be written respectively in the following form: $U^a = \bigcup_{i\in\mathcal{A}} \bigcap_{j\in I'_1} U^a_{ij}$ and $V^a = \bigcup_{i\in\mathcal{A}_1} \bigcap_{j\in I'_2} V^a_{ij}$, where U^a_{ij} and V^a_{ij} are of the form L^a_x or R^a_y, I'_1 and I'_2 are finite. Then

$$\begin{aligned}
& x'_{1-r} \in U^a \\
\Rightarrow\ & x'_{1-r} \in \bigcup_{i\in\mathcal{A}} \bigcap_{j\in I'_1} U^a_{ij} \\
\Rightarrow\ & x'_{1-r} \in \bigcap_{j\in I'_1} U^a_{i'_1 j}, \quad \text{for some } i'_1 \in \mathcal{A} \\
\Rightarrow\ & 1-r < \min_{j\in I'_1} U^a_{i'_1 j}(x')
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\ 1-r < U^a_{i'_1 j}(x'), \quad \text{for each } j \in I'_1\\
&\Rightarrow\ r > 1-U^a_{i'_1 j}(x'), \quad \text{for each } j \in I'_1\\
&\Rightarrow\ x'_r \notin (U^a_{i'_1 j})^c, \quad \text{for each } j \in I'_1
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
& y'_{1-s_1} \notin U^a\\
\Rightarrow\ & 1-s > 1-s_1 \geq \sup_{i\in\mathcal{A}} \min_{j\in I'_1} U^a_{ij}(y')\\
\Rightarrow\ & 1-s > \min_{j\in I'_1} U^a_{ij}(y'), \quad \text{for each } i \in \mathcal{A}\\
\Rightarrow\ & 1-s > U^a_{i'_1 j'_1}(y'), \quad \text{for some } j'_1 \in I'_1 (\text{Since } I'_1 \text{ is finite})\\
\Rightarrow\ & s < 1-U^a_{i'_1 j'_1}(y')\\
\Rightarrow\ & y'_s \in (U^a_{i'_1 j'_1})^c
\end{aligned} \tag{5.7}$$

So in view of (5.6),(5.7) and Theorem 5.24(1), there exists a member $(U^a_{i'_1 j'_1})^c$ of τ_1 such that $x'_r \notin (U^a_{i'_1 j'_1})^c$ and $y'_s \in (U^a_{i'_1 j'_1})^c$. Similarly it can be shown that there exists a member $B_{i'_2 j'_2} \in \tau_1$, where $B_{i'_2 j'_2}$ is of the form L_x or R_y, such that $x'_r \in B_{i'_2 j'_2}$ and $y'_s \notin B_{i'_2 j'_2}$, showing that τ_1 is fuzzy T_1.

2. Suppose that τ_1 is fuzzy T_0. To show that τ_2 is fuzzy T_0, choose $x', y' \in X, x' \neq y'$. Since τ_1 is fuzzy T_0, there exists a fuzzy open set $U \in \tau_1$ such that $U(x') \neq U(y')$. Now, $U = \bigcup_{i\in\Omega} \bigcap_{j\in I_1} U_{ij}$, where U_{ij} is of the form L_x or R_y and I_1 is finite. Then,

$$\begin{aligned}
& U(x') \neq U(y')\\
\Rightarrow\ & \Big(\bigcup_{i\in\Omega}\bigcap_{j\in I_1} U_{ij}\Big)(x') \neq \Big(\bigcup_{i\in\Omega}\bigcap_{j\in I_1} U_{ij}\Big)(y')\\
\Rightarrow\ & \Big(\bigcap_{j\in I_1} U_{i_1 j}\Big)(x') \neq \Big(\bigcap_{j\in I_1} U_{i_1 j}\Big)(y'), \quad \text{for some } i_1 \in \Omega\\
\Rightarrow\ & \min_{j\in I_1} U_{i_1 j}(x') \neq \min_{j\in I_1} U_{i_1 j}(y')\\
\Rightarrow\ & U_{i_1 j_1}(x') \neq U_{i_1 j_1}(y'), \quad \text{for some } j_1 \in I_1\\
\Rightarrow\ & 1-U_{i_1 j_1}(x') \neq 1-U_{i_1 j_1}(y'), \quad \text{for some } j_1 \in I_1\\
\Rightarrow\ & U^c_{i_1 j_1}(x') \neq U^c_{i_1 j_1}(y'),
\end{aligned}$$

where $U^c_{i_1 j_1}$ is of the form L^a_x or R^a_y(by using Theorem 5.24(1)). This implies that τ_2 is fuzzy T_0.

Conversely, assume that τ_2 is fuzzy T_0. To show that τ_1 is fuzzy T_0, choose $x', y' \in X, x' \neq y'$. Since τ_2 is fuzzy T_0, there exists a fuzzy open set $U^a \in \tau_2$ such that $U^a(x') \neq U^a(y')$. Now, $U^a = \bigcup\limits_{i\in\Omega'}\bigcap\limits_{j\in J} U^a_{ij}$, where U^a_{ij} is of the form L^a_x or R^a_y and J is finite. Then,

$$\begin{aligned}
& U^a(x') \neq U^a(y') \\
\Rightarrow\ & (\bigcup_{i\in\Omega'}\bigcap_{j\in J} U^a_{ij})(x') \neq (\bigcup_{i\in\Omega'}\bigcap_{j\in J} U^a_{ij})(y') \\
\Rightarrow\ & (\bigcap_{j\in J} U^a_{i'_1 j})(x') \neq (\bigcap_{j\in J} U^a_{i'_1 j})(y'), \quad \text{for some}\, i'_1 \in \Omega' \\
\Rightarrow\ & \min_{j\in J} U^a_{i'_1 j}(x') \neq \min_{j\in J} U^a_{i'_1 j}(y'), \\
\Rightarrow\ & U^a_{i'_1 j'_1}(x') \neq U^a_{i'_1 j'_1}(y'), \quad \text{for some}\, j'_1 \in J \\
\Rightarrow\ & 1 - U^a_{i'_1 j'_1}(x') \neq 1 - U^a_{i'_1 j'_1}(y'), \quad \text{for some}\, j'_1 \in J \\
\Rightarrow\ & (U^a_{i'_1 j'_1})^c(x') \neq (U^a_{i'_1 j'_1})^c(y')
\end{aligned}$$

Now put $(U^a_{i'_1 j'_1})^c = A_{i'_1 j'_1}$, then $A_{i'_1 j'_1}$ is of the form L_x or R_y(by using Theorem 5.24(1)). Hence $A_{i'_1 j'_1} \in \tau_1$ is such that $A_{i'_1 j'_1}(x') \neq A_{i'_1 j'_1}(y')$ which implies that τ_1 is fuzzy T_0.

□

5.4 Conclusion

In this chapter, we have introduced the concepts of a fuzzy topological space and a fuzzy bitopological space generated by a fuzzy relation as an extension of the corresponding concepts in [62] and [53] respectively for the crisp case. Several results have been proved. In particular, we have obtained necessary and sufficient condition when a fuzzy topology τ on X is generated by a fuzzy relation, characterizations for a fuzzy topology generated by a fuzzy interval order, a preorderable fuzzy topology, an orderable fuzzy topology and a fuzzy bitopological space generated by a fuzzy relation. Further, it has been proved that if (X, τ_1, τ_2) is the fuzzy bitopological space generated by a fuzzy relation $\mathcal{R}$, then the fuzzy topology τ_1 is fuzzy T_i iff τ_2 is fuzzy T_i, $i = 0, 1$.