Chapter 4

Fuzzy soft compact topological spaces

4.1 Introduction

In fuzzy topological spaces, fuzzy compactness was first introduced by Chang[24] but it is well known by now that Chang's fuzzy compactness does not satisfy the Tychonoff property. Later on Lowen[69] gave an alternative definition of fuzzy compactness and proved that it satisfies the Tychonoff property(cf.[70]).

In fuzzy soft topological spaces, compactness has been introduced earlier by Gain et al.[43], Osmanoğlu and Tokat [83], Sreedevi and Ravi Shankar [105]. They have extended Chang's definition to the case of fuzzy soft topological spaces. Varol et al.[116] introduced $L-$fuzzy soft topological spaces using Šhostak approach[104] and defined $L-$fuzzy soft compactness in these spaces in terms of Aygün's fuzzy compactness[7].

In this chapter, we have introduced and studied fuzzy soft compactness in fuzzy soft topological spaces which is an extension of the fuzzy compactness in a fuzzy topological space given by Lowen[69].

Several basic desirable results have been obtained. In particular, we have proved the counterparts of the Alexanders's subbase lemma and Tychonoff theorem for fuzzy soft topological spaces.

The contents of this chapter, in the form of a research paper, has been accepted for publication in 'Journal of Mathematics'.

Throughout the chapter we mean a fuzzy soft topological spaces in the sense of Definition 1.22.

Now we recall some definitions and results which will be used in this chapter.

Proposition 4.1. *[59] Let $(\varphi, \psi) : (X, \tau) \to (Y, \delta)$ be a fuzzy soft mapping and f_{A_1} and f_{A_2} be fuzzy soft sets over X such that $f_{A_1} \sqsubseteq f_{A_2}$. Then $(\varphi, \psi)f_{A_1} \sqsubseteq (\varphi, \psi)f_{A_2}$.*

Proposition 4.2. *[59] Let $f_A \in \mathcal{F}(X, E)$ and $g_B \in \mathcal{F}(X, K)$. Then*

1. *$(\varphi, \psi)((\varphi, \psi)^{-1}g_B) \sqsubseteq g_B$, the equality holds if (φ, ψ) is surjective.*
2. *$f_A \sqsubseteq (\varphi, \psi)^{-1}((\varphi, \psi)f_A)$, the equality holds if (φ, ψ) is injective.*

Proposition 4.3. *[59] Let $(\varphi, \psi) : (X, \tau) \to (Y, \delta)$ be a fuzzy soft mapping and $\{f_{A_i} : i \in \Omega\}$ be a family of fuzzy soft sets over X. Then,*

1. *$(\varphi, \psi)(\bigsqcup_{i\in\Omega} f_{A_i}) = \bigsqcup_{i\in\Omega}(\varphi, \psi)f_{A_i}$.*
2. *$(\varphi, \psi)(\sqcap_{i\in\Omega} f_{A_i}) \sqsubseteq \sqcap_{i\in\Omega}(\varphi, \psi)f_{A_i}$.*

Proposition 4.4. *[59] Let $(\varphi, \psi) : (X, \tau) \to (Y, \delta)$ be a fuzzy soft mapping and $\{g_{B_i} : i \in \Omega\}$ be a family of fuzzy soft sets over Y. Then,*

1. *$(\varphi, \psi)^{-1}(\bigsqcup_{i\in\Omega} g_{B_i}) = \bigsqcup_{i\in\Omega}(\varphi, \psi)^{-1}g_{B_i}$.*
2. *$(\varphi, \psi)^{-1}(\sqcap_{i\in\Omega} g_{B_i}) = \sqcap_{i\in\Omega}(\varphi, \psi)^{-1}g_{B_i}$.*

Definition 4.5. Let f_A and g_B be fuzzy soft sets over X such that $f_A \sqsupseteq g_B$. Then $f_A - g_B$ is the fuzzy soft set over X given by

$$(f_A - g_B)(e) = f_A(e) - g_B(e), \forall e \in E.$$

Definition 4.6. [119] A family of sets is said to be of *finite character* iff each finite subset of a member of the family is also a member, and each set belongs to this family if each of its finite subsets belong to it.

Lemma 4.7 (TUKEY). *[119] Each nonempty family of sets of finite character has a maximal element.*

4.2 Fuzzy soft compact topological spaces

In this section, we extend the definition of fuzzy compactness in fuzzy topological spaces given in [69], for fuzzy soft topological spaces.

Definition 4.8. Let (X, τ) be a fuzzy soft topological space relative to the parameters set E. Then a fuzzy soft set f_A over X is said to be *fuzzy soft compact* if for any family $\beta \subseteq \tau$ such that $\bigsqcup_{g_B \in \beta} g_B \sqsupseteq f_A$ and $\forall \epsilon$ such that $f_A \sqsupseteq \epsilon_E$, there exists a finite subfamily β_o of β such that $\bigsqcup_{g_B \in \beta_o} g_B \sqsupseteq f_A - \epsilon_E$.

Definition 4.9. A fuzzy soft topological space (X, τ) relative to the parameters set E is said to be *fuzzy soft compact* if each constant fuzzy soft set over X is fuzzy soft compact i.e., for $\alpha \in [0, 1]$, if there exists a family β of fuzzy soft open sets over X such that $\bigsqcup_{f_A \in \beta} f_A \sqsupseteq \alpha_E$, then $\forall \epsilon \in (0, \alpha)$, there exists a finite subfamily β_o of β such that $\bigsqcup_{f_A \in \beta_o} f_A \sqsupseteq (\alpha - \epsilon)_E$.

Proposition 4.10. *Let (X, τ) and (Y, δ) be fuzzy soft topological spaces relative to parameters sets E and K respectively, $(\varphi, \psi) : (X, \tau) \to (Y, \delta)$ be a fuzzy soft continuous mapping and $f_A \in \mathcal{F}(X, E)$ be fuzzy soft compact. Then $(\varphi, \psi) f_A$ is also fuzzy soft compact.*

Proof. Let $\beta \subseteq \delta$ be such that

$$\begin{aligned} \bigsqcup_{g_B \in \beta} g_B &\sqsupseteq (\varphi, \psi) f_A \\ \Rightarrow \bigsqcup_{g_B \in \beta} (\varphi, \psi)^{-1} g_B &\sqsupseteq f_A. \end{aligned}$$

Since $\{(\varphi, \psi)^{-1} g_B\}_{g_B \in \beta}$ is a family of fuzzy soft open sets over X and f_A is fuzzy soft compact, so $\forall \epsilon$ such that $f_A \sqsupseteq \epsilon_E$, there exists a finite subfamily $\beta_o \subseteq \beta$ such that

$$\bigsqcup_{g_B \in \beta_o} (\varphi, \psi)^{-1} g_B \sqsupseteq f_A - \epsilon_E$$

Then applying (φ, ψ) on both sides, we get

$$\bigsqcup_{g_B \in \beta_o} g_B \sqsupseteq (\varphi, \psi)(f_A - \epsilon_E) = (\varphi, \psi) f_A - \epsilon_K$$

which implies that $(\varphi, \psi)f_A$ is fuzzy soft compact. □

From the fact that (φ, ψ) is surjective if φ and ψ both are surjective(cf.[115]), each constant fuzzy soft set α_K over Y is the image of constant fuzzy soft set α_E over X. Hence we have the following result:

Corollary 4.11. *Let (X, τ) and (Y, δ) be fuzzy soft topological spaces, where (X, τ) is fuzzy soft compact and (φ, ψ) be a surjective fuzzy soft continuous mapping from (X, τ) to (Y, δ). Then (Y, δ) is fuzzy soft compact.*

As in the case of soft topological spaces ([9]), here we have the following:

Definition 4.12. Let (X, τ) be a fuzzy soft topological space relative to the parameters set E. Then for $e \in E$, the *$e-$parameter fuzzy topological space* is given by (X, τ_e), where $\tau_e = \{f_A(e) : f_A \in \tau\}$.

The following proposition is a counterpart of the Theorem 4.1 in [9].

Proposition 4.13. *Let (X, τ) be a fuzzy soft topological space relative to the parameters set E, which is finite. Then (X, τ) is fuzzy soft compact if each $e-$parameter fuzzy topological space is fuzzy compact.*

Proof. Suppose that each $e-$parameter fuzzy topological space is fuzzy compact. Then to show that α_E, $\alpha \in [0, 1]$ is fuzzy soft compact, consider a family β of fuzzy soft open sets over X such that

$$\begin{aligned} & \alpha_E \sqsubseteq \bigsqcup_{f_A \in \beta} f_A \\ \Rightarrow \quad & \alpha_E(e) \subseteq \bigcup_{f_A \in \beta} f_A(e), \quad \forall e \in E \\ \Rightarrow \quad & \alpha_X \subseteq \bigcup_{f_A \in \beta} f_A(e), \quad \forall e \in E. \end{aligned}$$

Then for $e \in E$, by fuzzy compactness of (X, τ_e), for $\epsilon \in (0, \alpha)$, there exists a finite subfamily β_o^e of β such that $(\alpha - \epsilon)_X \subseteq \bigcup_{f_A \in \beta_o^e} f_A(e)$.
Now set $\beta_o = \bigcup_{e \in E} \beta_o^e$. Then β_o is a finite subfamily of β such that $(\alpha - \epsilon)_X \subseteq \bigcup_{f_A \in \beta_o} f_A(e)$, $\forall e \in E$. Hence $(\alpha - \epsilon)_E \sqsubseteq \bigsqcup_{f_A \in \beta_o} f_A$, which shows that (X, τ) is fuzzy soft compact. □

Now we consider the mappings (cf.[101]) $h : \mathcal{F}(X,E) \to I^{X\times E}$, where $I^{X\times E}$ is the set of all fuzzy sets in $X \times E$, defined as follows:

$$h(f_A)(x,e) = f_A(e)(x), \quad \forall f_A \in \mathcal{F}(X,E)$$

and $g : I^{X\times E} \to \mathcal{F}(X,E)$ as follows:

$$g(U) = f_E^U, \forall U \in I^{X\times E}, \text{ where } f_E^U(e)(x) = U(x,e), \forall e \in E, \forall x \in X.$$

We observe that $hog = id_{I^{X\times E}}$ and $goh = id_{\mathcal{F}(X,E)}$.

Now we state the following theorem proved in [101]:

Theorem 4.14. *[101] $(\mathcal{F}(X,E), \sqcup, \sqcap, ^c)$ is isomorphic to $(I^{X\times E}, \cup, \cap, ^c)$, where $I^{X\times E}$ denotes the set of all fuzzy sets in $X \times E$.*

As mentioned in [101], if (X,τ) is a fuzzy soft topological space relative to the parameters set E, then $(X \times E, h(\tau))$ is a fuzzy topological space and also if $(X \times E, \mathcal{T})$ is a fuzzy topological space, then $(X, g(\mathcal{T}))$ is a fuzzy soft topological space relative to the parameters set E, where $h(\tau) = \{h(f_A) : f_A \in \tau\}$ and $g(\mathcal{T}) = \{g(U) : U \in \mathcal{T}\}$.

Proposition 4.15. *A fuzzy soft topological space (X,τ) relative to the parameters set E is fuzzy soft compact iff $(X \times E, h(\tau))$ is fuzzy compact.*

Proof. First, suppose that (X,τ) is fuzzy soft compact. Then to show that $(X \times E, h(\tau))$ is fuzzy compact, consider a family $\beta \subseteq h(\tau)$ of fuzzy open sets in $X \times E$ such that

$$\begin{aligned} & \alpha_{X\times E} \subseteq \bigcup_{\nu\in\beta} \nu \\ \Rightarrow \quad & g(\alpha_{X\times E}) \sqsubseteq g(\bigcup_{\nu\in\beta} \nu) \\ \Rightarrow \quad & \alpha_E \sqsubseteq \bigsqcup_{\nu\in\beta} g(\nu). \end{aligned}$$

Note that if $\nu \in \beta \subseteq h(\tau)$, then $g(\nu) \in \tau$. So by fuzzy soft compactness of (X, τ), for $\epsilon \in (0, \alpha)$, there exists a finite subfamily β_o of β such that

$$\begin{aligned} & (\alpha-\epsilon)_E \sqsubseteq \bigsqcup_{\nu \in \beta_o} g(\nu) \\ \Rightarrow\ & h((\alpha-\epsilon)_E) \subseteq h(\bigsqcup_{\nu \in \beta_o} g(\nu)) \\ \Rightarrow\ & (\alpha-\epsilon)_{X\times E} \subseteq \bigcup_{\nu \in \beta_o} (hog)(\nu) \\ \Rightarrow\ & (\alpha-\epsilon)_{X\times E} \subseteq \bigcup_{\nu \in \beta_o} \nu \end{aligned}$$

which proves the fuzzy compactness of $(X \times E, h(\tau))$.

Conversely, assume that $(X \times E, h(\tau))$ is fuzzy compact. To show that (X, τ) is fuzzy soft compact, we have to show that $\alpha_E, \alpha \in [0, 1]$ is fuzzy soft compact. Let $\beta \subseteq \tau$ such that

$$\begin{aligned} & \alpha_E \sqsubseteq \bigsqcup_{f_A \in \beta} f_A \\ \Rightarrow\ & h(\alpha_E) \subseteq h(\bigsqcup_{f_A \in \beta} f_A) \\ \Rightarrow\ & \alpha_{X\times E} \subseteq \bigcup_{f_A \in \beta} h(f_A). \end{aligned}$$

Then by the fuzzy compactness of $(X \times E, h(\tau))$, for $\epsilon \in (0, \alpha)$, there exists a finite subfamily β_o of β such that

$$\begin{aligned} & (\alpha-\epsilon)_{X\times E} \subseteq \bigcup_{f_A \in \beta_o} h(f_A) \\ \Rightarrow\ & g((\alpha-\epsilon)_{X\times E}) \sqsubseteq g(\bigcup_{f_A \in \beta_o} h(f_A)) \\ \Rightarrow\ & (\alpha-\epsilon)_E \sqsubseteq \bigsqcup_{f_A \in \beta_o} (goh)(f_A) \\ \Rightarrow\ & (\alpha-\epsilon)_E \sqsubseteq \bigsqcup_{f_A \in \beta_o} f_A \end{aligned}$$

which proves that the fuzzy soft topological space (X, τ) is fuzzy soft compact. □

Now we prove the counterparts of the well known Alexander's subbase lemma and the Tychonoff theorem for fuzzy soft topological spaces. The proofs for which are based on the proofs of the corresponding results given in [69] and [70] respectively.

Theorem 4.16. *Let (X,τ) be a fuzzy soft topological space relative to the parameters set E. Then (X,τ) is fuzzy soft compact iff for any subbase σ for τ, if there is a family $\beta \subseteq \sigma$ such that $\bigsqcup_{f_A\in\beta} f_A \sqsupseteq \alpha_E$, then for $\epsilon \in (0,\alpha)$, there exists a finite subfamily β_o of β such that $\bigsqcup_{f_A\in\beta_o} f_A \sqsupseteq (\alpha-\epsilon)_E$.*

Proof. First assume that (X,τ) is fuzzy soft compact. Choose $\beta \subseteq \sigma$ such that $\bigsqcup_{f_A\in\beta} f_A \sqsupseteq \alpha_E$, $\alpha \in [0,1]$. Now since $\sigma \subseteq \tau$ and (X,τ) is fuzzy soft compact, so for $\epsilon \in (0,\alpha)$, there exists a finite subfamily β_o of β such that $\bigsqcup_{f_A\in\beta_o} f_A \sqsupseteq (\alpha-\epsilon)_E$.

Conversely, to show that (X,τ) is fuzzy soft compact, we have to show that if for $\beta \subseteq \tau$, there exist α and $\epsilon \in (0,\alpha)$ such that there does not exist any finite subfamily β_o of β such that $\bigsqcup_{f_A\in\beta_o} f_A \sqsupseteq (\alpha-\epsilon)_E$, then it must follow that $\bigsqcup_{f_A\in\beta} f_A$ does not contain α_E.

Consider the family

$$\mathcal{C} = \{\beta \subseteq \tau : \text{there does not exist any finite subfamily } \beta_o \text{ of } \beta \text{ such that } \bigsqcup_{f_A\in\beta_o} f_A \sqsupseteq (\alpha-\epsilon)_E\}.$$

Then $\mathcal{C}$ is of finite character. Now it follows from the Tukey's lemma that $\forall\beta \in \mathcal{C}$, there exists a maximal element $\beta' \in \mathcal{C}$ containing β.

Next, we show that if $f_A \in \beta'$ and $f_{A_1}, f_{A_2}, ..., f_{A_n} \in \tau$ such that $f_A \sqsupseteq f_{A_1} \sqcap f_{A_2} \sqcap ... \sqcap f_{A_n}$, then there exists some k, $k = 1,2,...,n$ such that $f_{A_k} \in \beta'$. For this we proceed as follows:

If we take $f_{A_1} \in \tau$ such that $f_{A_1} \notin \beta'$, then since $\beta' \in \mathcal{C}$ is maximal, for the family $\{f_{A_1}\} \cup \beta'$, there exists a finite subfamily $\{f_{A_1}, g_{B_1}, g_{B_2}, ..., g_{B_p}\}$, where $g_{B_i} \in \beta'$, $\forall i = 1,2,...,p$, such that $f_{A_1} \sqcup g_{B_1} \sqcup g_{B_2} \sqcup ... \sqcup g_{B_p} \sqsupseteq (\alpha-\epsilon)_E$, which implies that if for $e \in E$ and $x \in X$,

$$\sup\{g_{B_1}(e)(x), g_{B_2}(e)(x), ..., g_{B_p}(e)(x)\} < \alpha - \epsilon,$$

then

$$f_{A_1}(e)(x) \geq \alpha - \epsilon. \tag{4.1}$$

Similarly, if we take another $f_{A_2} \in \tau$ such that $f_{A_2} \notin \beta'$, then again since $\beta' \in \mathcal{C}$ is maximal, for the family $\{f_{A_2}\} \cup \beta'$, there exists a finite subfamily $\{f_{A_2}, g'_{B_1}, g'_{B_2}, ..., g'_{B_q}\}$, where $g'_{B_i} \in \beta'$, $\forall i = 1, 2, ..., q$, such that $f_{A_2} \sqcup g'_{B_1} \sqcup g'_{B_2} \sqcup ... \sqcup g'_{B_q} \sqsupseteq (\alpha - \epsilon)_E$, which implies that if for $e' \in E$ and $x' \in X$,

$$\sup\{g_{B_1}(e')(x'), g_{B_2}(e')(x'), ..., g_{B_q}(e')(x')\} < \alpha - \epsilon,$$

then

$$f_{A_2}(e')(x') \geq \alpha - \epsilon. \tag{4.2}$$

Now we show that

$$(f_{A_1} \sqcap f_{A_2}) \sqcup g_{B_1} \sqcup g_{B_2} \sqcup ... \sqcup g_{B_p} \sqcup g'_{B_1} \sqcup g'_{B_2} \sqcup ... \sqcup g'_{B_q} \sqsupseteq (\alpha - \epsilon)_E$$

as follows:

If for $k \in E$ and $y \in X$, $(g_{B_1} \sqcup g_{B_2} \sqcup ... \sqcup g_{B_p} \sqcup g'_{B_1} \sqcup g'_{B_2} \sqcup ... \sqcup g'_{B_q})(k)(y) < \alpha - \epsilon$ which implies that $\sup_{j=1,2,...,p} g_{B_j}(k)(y) < \alpha - \epsilon$ and $\sup_{i=1,2,...,q} g'_{B_i}(k)(y) < \alpha - \epsilon$, then from (4.1) and (4.2), we get $f_{A_1}(k)(y) \geq \alpha - \epsilon$ and $f_{A_2}(k)(y) \geq \alpha - \epsilon$. This implies that $(f_{A_1} \sqcap f_{A_2})(k)(y) \geq \alpha - \epsilon$ and hence $f_{A_1} \sqcap f_{A_2} \notin \beta'$. Thus, in general, if $f_{A_1}, f_{A_2}, ..., f_{A_n}$ do not belong to β', then $f_{A_1} \sqcap f_{A_2} \sqcap ... \sqcap f_{A_n}$ does not belong to β' implying that there is no fuzzy soft open set containing $f_{A_1} \sqcap f_{A_2} \sqcap ... f_{A_n}$ which belong to β'. Thus we have shown that if f_{A_i}, $i = 1, 2, ..., n$ do not belong to β', then no fuzzy soft open set f_A such that $f_A \sqsupseteq f_{A_1} \sqcap f_{A_2} \sqcap ... \sqcap f_{A_n}$ belongs to β'. Equivalently, if $f_A \in \beta'$ such that $f_A \sqsupseteq f_{A_1} \sqcap f_{A_2} \sqcap ... f_{A_n}$, then there exists some k, $k = 1, 2, ..., n$ such that $f_{A_k} \in \beta'$.

Next, consider $\beta' \cap \sigma$. Then, from the assumption of the theorem, we have $\bigsqcup_{f_A \in \beta' \cap \sigma} f_A$ does not contain α_E. Now we show $\bigsqcup_{f_A \in \beta'} f_A \sqsubseteq \bigsqcup_{g_B \in \beta' \cap \sigma} g_B$. Since, $\forall f_A \in \beta'$, $\forall e \in E$ and $\forall x \in X$ such that $f_A(e)(x) > 0$ and $\forall a < f_A(e)(x)$, then $e_{f_A(e)(x) - a} \in f_A$ and hence using Proposition 2.7, there exist $f^a_{A_1}$, $f^a_{A_2}$, ..., $f^a_{A_n} \in \sigma$ such that $f^a_{A_1} \sqcap f^a_{A_2} \sqcap ... \sqcap f^a_{A_n} \sqsubseteq f_A$ and $f^a_{A_1}(e)(x) \wedge f^a_{A_2}(e)(x) \wedge ... \wedge f^a_{A_n}(e)(x) > f_A(e)(x) - a$. Since $f_A \in \beta'$ and β' is maximal, it follows that there exists some k, $k = 1, 2, ..., n$ such that $f^a_{A_k} \in \beta'$. Thus, $\forall a > 0$, there exists $f^a_{A_k}$ such that $f^a_{A_k}(e)(x) > f_A(e)(x) - a$, $\forall e \in E$, $\forall x \in X$ and $f^a_{A_k} \in \beta' \cap \sigma$. Now fix e and x both. Then $\forall f_A \in \beta'$ such that $f_A(e)(x) > 0$ and $a < f_A(e)(x)$, where $a > 0$, there exists $f_A^a \in \beta' \cap \sigma$ such that

$$\begin{aligned} & f_A^a(e)(x) > f_A(e)(x) - a \\ \Rightarrow\ & (\bigsqcup_{g_B \in \beta' \cap \sigma} g_B)(e)(x) \geq f_A(e)(x) \\ \Rightarrow\ & \bigsqcup_{f_A \in \beta'} f_A \sqsubseteq \bigsqcup_{g_B \in \beta' \cap \sigma} g_B. \end{aligned}$$

This implies that $\bigsqcup_{f_A \in \beta'} f_A$ does not contain α_E and hence $\bigsqcup_{f_A \in \beta} f_A$ does not contain α_E. Thus, (X, τ) is fuzzy soft compact. □

Theorem 4.17. *If $\{(X_i, \tau_i) : i \in \Omega\}$ is a family of fuzzy soft topological spaces relative to the parameters sets E_i, respectively. Then the product fuzzy soft topological space $(X, \tau) = \prod_{i \in \Omega}(X_i, \tau_i)$ is fuzzy soft compact if and only if each coordinate fuzzy soft topological space (X_i, τ_i) is fuzzy soft compact.*

Proof. Let us first assume that each coordinate space (X_i, τ_i), $i \in \Omega$, is fuzzy soft compact. From the previous theorem, to show that (X, τ) is fuzzy soft compact, it is sufficient to show that for any family $\beta \subseteq \sigma = \{(p_{X_i}, q_{E_i})^{-1}(f_{A_i}) : i \in \Omega, f_{A_i} \in \tau_i\}$, if there exist α, ϵ, where $\alpha > \epsilon > 0$ such that $\forall$ finite subfamily β_o of β, $\bigsqcup_{f_A \in \beta_o} f_A$ does not contain $(\alpha - \epsilon)_E$, then it must follow that $\bigsqcup_{f_A \in \beta} f_A$ does not contain α_E.

Let β be such a family. Then $\forall j \in \Omega$, put $\beta_j = \{f_{A_j} \in \tau_j : (p_{X_j}, q_{E_j})^{-1}(f_{A_j}) \in \beta\}$. Then $\forall$ finite subfamily $(\beta_j)_o$ of β_j, $\{(p_{X_j}, q_{E_j})^{-1}(f_{A_j}) : f_{A_j} \in (\beta_j)_o\}$ is a finite subfamily of β. Hence, from our assumption there exist some $e \in E$ and $x \in X$ such that

$$\begin{aligned} (\bigsqcup_{f_{A_j} \in (\beta_j)_o} (p_{X_j}, q_{E_j})^{-1} f_{A_j})(e)(x) &< \alpha - \epsilon \\ \Rightarrow \sup_{f_{A_j} \in (\beta_j)_o} f_{A_j}(e_j)(x_j) &< (\alpha - \epsilon/2) - \epsilon/2 \end{aligned}$$

Since the above inequality holds $\forall$ finite subfamily $(\beta_j)_o$ of β_j, so from the fuzzy soft compactness of (X_j, τ_j), there exist $e'_j \in E_j$ and $x'_j \in X_j$ such that

$$\sup_{f_{A_j} \in \beta_j} f_{A_j}(e'_j)(x'_j) < (\alpha - \epsilon/2).$$

The same inequality holds for all $j \in \Omega$. Finally, if we set $e = (e'_j)_{j\in\Omega}$ and $x = (x'_j)_{j\in\Omega}$, then

$$\begin{aligned}
(\bigsqcup_{f_A\in\beta} f_A)(e)(x) &= \sup_{f_A\in\beta} f_A(e)(x) \\
&= \sup_{j\in\Omega} \sup_{f_A\in\beta\cap(p_{X_j},q_{E_j})^{-1}(\tau_j)} f_A(e)(x) \\
&= \sup_{j\in\Omega} \sup_{f_{A_j}\in\beta_j} ((p_{X_j},q_{E_j})^{-1} f_{A_j})(e)(x) \\
&= \sup_{j\in\Omega} \sup_{f_{A_j}\in\beta_j} f_{A_j}(e'_j)(x'_j) \\
&\leq \alpha - \epsilon/2 < \alpha
\end{aligned}$$

$$\Rightarrow \quad \bigsqcup_{f_A\in\beta} f_A \text{ does not contain } \alpha_E.$$

The converse part follows using Corollary 4.11 and the fact that (p_{X_j}, q_{E_j}) are fuzzy soft continuous and surjective maps, $\forall j \in \Omega$. □

4.3 Conclusion

In this chapter, we have introduced the notion of compactness in fuzzy soft topological spaces, as a generalization of the corresponding concept given by Lowen[69] for fuzzy topological spaces. Further, we have established the counterpart of Alexander's subbase lemma for fuzzy soft topological spaces and using it, we have proved the Tychonoff theorem for fuzzy soft compact topological spaces.