Chapter 2

Hausdorff fuzzy soft topological spaces

2.1 Introduction

Topological structure of fuzzy soft sets has been introduced and studied by Tanay and Kandemir[113]. It was further studied by Mahanta and Das[72], Varol and Aygün[115] and Çetkin and Aygün[20], etc. Mahanta and Das[72] had introduced fuzzy soft points and studied the concept of neighbourhoods of a fuzzy soft point in a fuzzy soft topological space. They have also introduced and studied fuzzy soft closure, fuzzy soft interior, separation axioms and connectedness in fuzzy soft topological spaces.

In this chapter, we have given an alternative definition of a 'fuzzy soft point'and 'belonging of a fuzzy soft point to a fuzzy soft set'. Using these concepts, we have introduced and studied the notion of Hausdorff separation axiom in fuzzy soft topological spaces. Several basic desirable results have been proved. In particular, we have obtained a characterization of a Hausdorff fuzzy soft topological space, in terms of the diagonal set. It has been further shown that Hausdorffness in a fuzzy soft topological space satisfies productive, projective and hereditary properties.

Throughout the chapter we mean a fuzzy soft topological space in the sense of Definition 1.21.

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First, we introduce the notion of fuzzy soft subspaces of a fuzzy soft topological space.

Theorem 2.1. *Let (X, τ) be a fuzzy soft topological space relative to the parameters set E and $G \subseteq E$. Then (X, τ_G) is a fuzzy soft topological space relative to the parameters set G, where*

$$\tau_G = \{f_A \mid_G : f_A \in \tau\}.$$

Proof. (X, τ_G) is a fuzzy soft topological space relative to the parameters set G follows from the following:

1. Since $0_E, 1_E \in \tau$ and $0_G = 0_E \mid_G, 1_G = 1_E \mid_G$, therefore 0_G and $1_G \in \tau_G$.
2. Let $f_{G_1}, f_{G_2} \in \tau_G$. Since $f_{G_1} = f_{A_1} \mid_G$ and $f_{G_2} = f_{A_2} \mid_G$, where $f_{A_1}, f_{A_2} \in \tau$, so $(f_{G_1} \sqcap f_{G_2}) = (f_{A_1} \mid_G) \sqcap (f_{A_2} \mid_G) = (f_{A_1} \sqcap f_{A_2}) \mid_G$. This implies that $f_{G_1} \sqcap f_{G_2} \in \tau_G$, since $f_{A_1} \sqcap f_{A_2} \in \tau$.
3. Let $\{f_{G_i} : i \in \Omega\}$ be an arbitrary family of members of τ_G. So $f_{G_i} = f_{A_i} \mid_G$, where $f_{A_i} \in \tau$. Then $\bigsqcup_{i \in \Omega} f_{G_i} = (\bigsqcup_{i \in \Omega} f_{A_i}) \mid_G$, which implies that $\bigsqcup_{i \in \Omega} f_{G_i} \in \tau_G$, since $\bigsqcup_{i \in \Omega} f_{A_i} \in \tau$.

□

Definition 2.2. If (X, τ) is a fuzzy soft topological space relative to the parameters set E and $G \subseteq E$, then (X, τ_G) defined in the above theorem is called a *fuzzy soft subspace* of (X, τ).

Mahanta and Das ([72]) had given the following definitions of 'fuzzy soft points'and 'belonging of a fuzzy soft point to a fuzzy soft set':

Definition 2.3. ([72]) A fuzzy soft set g_A is said to be a *fuzzy soft point*, denoted by e_{g_A}, if for the element $e \in A$, $g_A(e) \neq 0_X$ and $g_A(e') = 0_X$, $\forall e' \in A - \{e\}$.

Definition 2.4. ([72]) A fuzzy soft point e_{g_A} is said to be in a fuzzy soft set h_A, denoted by $e_{g_A} \widetilde{\in} h_A$, if for the element $e \in A$, $g_A(e) \leq h_A(e)$.

We observe from the above definitions that the result given by the authors [72], in Theorem 2.5(v) i.e.,

$$e_{g_A} \tilde{\in} \bigsqcup \{h_{\lambda B} : \lambda \in \Lambda\} \Leftrightarrow \exists\, \lambda \in \Lambda \text{ such that } e_{g_A} \tilde{\in} h_{\lambda B}$$

does not hold good. A counter example is given as follows:

Example 2.1. *Consider the fuzzy soft point* e_{g_A} *such that*

$$e_{g_A}(e) = \alpha_X, \quad \alpha \in (0,1)$$

and the family $\{h_{\lambda B} : 0 < \lambda < \alpha\}$ *of fuzzy soft sets over* X *such that*

$$h_{\lambda B}(e') = \begin{cases} (\alpha - \lambda)_X, & \text{if } e' = e \\ 0_X, & \text{otherwise.} \end{cases}$$

Then $e_{g_A} \tilde{\in} \bigsqcup_{0<\lambda<\alpha} h_{\lambda B}$ *but* $e_{g_A} \tilde{\notin} h_{\lambda B}$ *for any* λ *such that* $0 < \lambda < \alpha$.

To retain the above result, in the Definition 2.4, '$g_A(e) \leq h_A(e)$' must be replaced by '$g_A(e) < h_A(e)$ (i.e., $g_A(e)(x) < h_A(e)(x)$, $\forall x \in X$)'. In view of this modification, any fuzzy soft point $e_{g_A} \tilde{\in} h_A$ only if $h_A(e)(x) > 0, \forall x \in X$. In this situation, no pair of distinct fuzzy soft points e_{g_A} and e_{k_A} can be separated by disjoint fuzzy soft open sets, which is a requirement in the definition of Hausdorffness(cf.[72]) in a fuzzy soft topological space.

Therefore, we give an alternative definition of a 'fuzzy soft point' and 'belonging of a fuzzy soft point to a fuzzy soft set', as follows:

Definition 2.5. A *fuzzy soft point* e_{x_λ} over X is a fuzzy soft set over X defined as follows:

$$e_{x_\lambda}(e') = \begin{cases} x_\lambda, & \text{if } e' = e \\ 0_X, & \text{if } e' \in E - \{e\}, \end{cases}$$

where x_λ is the fuzzy point in X with support x and value λ, $\lambda \in (0,1)$([106]).

A fuzzy soft point e_{x_λ} is said to *belong* to a fuzzy soft set f_A, denoted by $e_{x_\lambda} \in f_A$ if $\lambda < f_A(e)(x)$. Two fuzzy soft points e_{x_λ} and e'_{y_s} are said to be *distinct* if $x \neq y$ or $e \neq e'$.

Example 2.2. *Let* $X = \{x^1, x^2\}$ *and* $E = \{e^1, e^2\}$ *be a universe set and a parameters set for the universe* X*, respectively. Then the fuzzy soft point* $(e^1)_{(x^1)_{0.5}}$ *is the fuzzy soft set over* X *given by*

$$(e^1)_{(x^1)_{0.5}}(e) = \begin{cases} (x^1)_{0.5}, & \text{if } e = e^1 \\ 0_X, & \text{if } e = e^2. \end{cases}$$

Proposition 2.6. *Let $\{f_{A_i} : i \in \Omega\}$ be a family of fuzzy soft sets over X , then $e_{x_\lambda} \in \bigsqcup_{i\in\Omega} f_{A_i}$ iff $e_{x_\lambda} \in f_{A_i}$, for some $i \in \Omega$.*

Proof. First, suppose that $e_{x_\lambda} \in f_{A_i}$, for some $i \in \Omega$. Then,

$$\begin{aligned} & \lambda < f_{A_i}(e)(x) \\ \Rightarrow \quad & \lambda < f_{A_i}(e)(x) \leqslant \sup_{j\in\Omega} f_{A_j}(e)(x) \\ \Rightarrow \quad & e_{x_\lambda} \in \bigsqcup_{j\in\Omega} f_{A_j} \end{aligned}$$

Conversely, let $e_{x_\lambda} \in \bigsqcup_{j\in\Omega} f_{A_j}$, then

$$\begin{aligned} & \lambda < (\bigsqcup_{j\in\Omega} f_{A_j})(e)(x) \\ \Rightarrow \quad & \lambda < \sup_{j\in\Omega} f_{A_j}(e)(x) \\ \Rightarrow \quad & \lambda < f_{A_i}(e)(x), \text{ for some } i \in \Omega \\ \Rightarrow \quad & e_{x_\lambda} \in f_{A_i}. \end{aligned}$$

□

Proposition 2.7. *A fuzzy soft set f_A over X is the union of all the fuzzy soft points belonging to it i.e.,*

$$f_A = \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}.$$

Proof. It is easy to see that $\bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\} \sqsubseteq f_A$.

Conversely, to show that $f_A \sqsubseteq \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}$. First we note that $f_A(e')(x') = 0$, if $e' \notin A$ or $x' \notin \text{supp} f_A(e')$. Next consider the case when $e' \in A, x' \in \text{supp} f_A(e')$. Then,

$$\begin{aligned} \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}(e')(x') &= \sup\{e_{x_\lambda}(e')(x') : e_{x_\lambda} \in f_A\} \\ &= \sup\{e'_{x'_\lambda}(e')(x') : e'_{x'_\lambda} \in f_A\} \\ &= \sup\{\lambda : e'_{x'_\lambda} \in f_A\}, \\ &= f_A(e')(x') \end{aligned}$$

Thus, $f_A \sqsubseteq \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}$. Hence $f_A = \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}$. □

Proposition 2.8. *Let (X, τ) be a fuzzy soft topological space relative to the parameters set E. Then a fuzzy soft set f_A is fuzzy soft open iff for each $e_{x_r} \in f_A$, there exists a basic fuzzy soft open set g_B such that $e_{x_r} \in g_B \sqsubseteq f_A$.*

Proof. First, suppose that the fuzzy soft set f_A over X is open and $\mathcal{B}$ denotes a base for τ. Then $f_A = \bigsqcup_{i\in\Omega}(g_B)_i$, where Ω is an index set and $(g_B)_i \in \mathcal{B}$, for each $i \in \Omega$. Let $e_{x_r} \in f_A$. Then, $e_{x_r} \in \bigsqcup_{i\in\Omega}(g_B)_i \Rightarrow e_{x_r} \in (g_B)_i \sqsubseteq f_A$, for some $i \in \Omega$.

Conversely, assume that for each $e_{x_r} \in f_A$, there exists a basic fuzzy soft open set $(g_B)_{e_{x_r}}$ such that

$$e_{x_r} \in (g_B)_{e_{x_r}} \sqsubseteq f_A.$$

Now, taking union, we get

$$\bigsqcup\{e_{x_r} : e_{x_r} \in f_A\} \sqsubseteq \bigsqcup(g_B)_{e_{x_r}} \sqsubseteq f_A$$

implying that

$$f_A = \bigsqcup_{e_{x_r}\in f_A}(g_B)_{e_{x_r}}.$$

Hence f_A is fuzzy soft open. □

2.2 Hausdorff fuzzy soft topological spaces

Mahanta and Das ([72]) had introduced Hausdorffness in a fuzzy soft topological space using the definitions of a 'fuzzy soft point' and 'belonging of a fuzzy soft point to a fuzzy soft set', in his sense. Here we define Hausdorffness in a fuzzy soft topological space in terms of the modified definitions of a 'fuzzy soft point' and 'belonging', given in Definition 2.5.

Definition 2.9. Let (X, τ) be a fuzzy soft topological space relative to the parameters set E. Then (X, τ) is said to be *Hausdorff* if for each pair of distinct fuzzy soft points e_{x_λ}, e'_{y_s} over X, there exist fuzzy soft open sets f_A and g_B such that $e_{x_\lambda} \in f_A$, $e'_{y_s} \in g_B$ and $f_A \sqcap g_B = 0_E$.

Now we give an example of a Hausdorff fuzzy soft topological space as follows:

Example 2.3. *Let $X = \{x^1, x^2\}$ and $E = \{e^1, e^2\}$ be a universe set and a parameters set for the universe X, respectively. Consider the collection τ of fuzzy soft sets over X,*

$$\tau = \{0_E, 1_E, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}, F_{14}\},$$

where F_i's are as follows:

$$F_1(e^1) = \{\frac{x^1}{1}, \frac{x^2}{0}\}, F_1(e^2) = \{\frac{x^1}{0}, \frac{x^2}{0}\}; \quad F_2(e^1) = \{\frac{x^1}{0}, \frac{x^2}{1}\}, F_2(e^2) = \{\frac{x^1}{0}, \frac{x^2}{0}\};$$

$$F_3(e^1) = \{\frac{x^1}{0}, \frac{x^2}{0}\}, F_3(e^2) = \{\frac{x^1}{1}, \frac{x^2}{0}\}; \quad F_4(e^1) = \{\frac{x^1}{0}, \frac{x^2}{0}\}, F_4(e^2) = \{\frac{x^1}{0}, \frac{x^2}{1}\};$$

$$F_5(e^1) = \{\frac{x^1}{1}, \frac{x^2}{1}\}, F_5(e^2) = \{\frac{x^1}{0}, \frac{x^2}{0}\}; \quad F_6(e^1) = \{\frac{x^1}{0}, \frac{x^2}{1}\}, F_6(e^2) = \{\frac{x^1}{1}, \frac{x^2}{0}\};$$

$$F_7(e^1) = \{\frac{x^1}{0}, \frac{x^2}{0}\}, F_7(e^2) = \{\frac{x^1}{1}, \frac{x^2}{1}\}; \quad F_8(e^1) = \{\frac{x^1}{1}, \frac{x^2}{0}\}, F_8(e^2) = \{\frac{x^1}{0}, \frac{x^2}{1}\};$$

$$F_9(e^1) = \{\frac{x^1}{1}, \frac{x^2}{0}\}, F_9(e^2) = \{\frac{x^1}{1}, \frac{x^2}{0}\}; \quad F_{10}(e^1) = \{\frac{x^1}{0}, \frac{x^2}{1}\}, F_{10}(e^2) = \{\frac{x^1}{0}, \frac{x^2}{1}\};$$

$$F_{11}(e^1) = \{\frac{x^1}{1}, \frac{x^2}{1}\}, F_{11}(e^2) = \{\frac{x^1}{1}, \frac{x^2}{0}\}; \quad F_{12}(e^1) = \{\frac{x^1}{0}, \frac{x^2}{1}\}, F_{12}(e^2) = \{\frac{x^1}{1}, \frac{x^2}{1}\};$$

$$F_{13}(e^1) = \{\frac{x^1}{1}, \frac{x^2}{0}\}, F_{13}(e^2) = \{\frac{x^1}{1}, \frac{x^2}{1}\}; \quad F_{14}(e^1) = \{\frac{x^1}{1}, \frac{x^2}{1}\}, F_{14}(e^2) = \{\frac{x^1}{0}, \frac{x^2}{1}\}.$$

Then, clearly τ is fuzzy soft topology over X. Also, for every pair of distinct fuzzy soft points, there exist disjoint fuzzy soft open sets over X containing them. Hence (X, τ) is a Hausdorff fuzzy soft topological space.

Theorem 2.10. *A fuzzy soft topological space (X, τ) relative to the parameters set E is Hausdorff iff the fuzzy soft set f_{Δ_E} over $X \times X$ is fuzzy soft closed, where f_{Δ_E} is given by :*

$$f_{\Delta_E}(e_1, e_2) = \begin{cases} \chi_{\Delta_X}, & \text{if } e_1 = e_2 \\ 0_{X \times X}, & \text{if } e_1 \neq e_2. \end{cases}$$

Proof. First, let us assume that (X, τ) is Hausdorff. To show that f_{Δ_E} is fuzzy soft closed, equivalently, $(f_{\Delta_E})^c$ is fuzzy soft open, choose a fuzzy soft point $(e, e')_{(x,y)_\lambda} \in (f_{\Delta_E})^c$. Now e_{x_λ} and e'_{y_λ} are distinct fuzzy soft points over X. Since (X, τ) is Hausdorff, there exist fuzzy soft open sets f_A and g_B such that

$$e_{x_\lambda} \in f_A, \; e'_{y_\lambda} \in g_B \text{ and } f_A \sqcap g_B = 0_E.$$

Now, consider $f_A \times g_B$. Then

$$(e, e')_{(x,y)_\lambda} \in f_A \times g_B \sqsubseteq (f_{\Delta_E})^c$$

as shown below:

Since $e_{x_\lambda} \in f_A$ and $e'_{y_\lambda} \in g_B$, so we have

$$\begin{aligned} & \lambda < f_A(e)(x) \text{ and } \lambda < g_B(e')(y) \\ \Rightarrow \quad & \lambda < \min\{f_A(e)(x), g_B(e')(y)\} = (f_A \times g_B)(e, e')(x, y) \\ \Rightarrow \quad & (e, e')_{(x,y)_\lambda} \in f_A \times g_B \end{aligned}$$

Next, for $f_A \times g_B \sqsubseteq (f_{\Delta_E})^c$, we proceed as follows:
Case 1: $e_1 \neq e_2$, this inclusion is trivially satisfied.
Case 2: $e_1 = e_2$, we need only to show that

$$(f_A \times g_B)(e_1, e_1)(x, x) = 0, \text{ for each } e_1 \in E,\ x \in X,$$

which is true, since we have

$$\begin{aligned} & f_A \sqcap g_B = 0_E \\ \Rightarrow \quad & (f_A(e_1) \cap g_B(e_1))(x) = 0, \text{ for each } e_1 \in E,\ x \in X \\ \Rightarrow \quad & \min\{f_A(e_1)(x), g_B(e_1)(x)\} = 0, \text{ for each } e_1 \in E,\ x \in X \\ \Rightarrow \quad & (f_A \times g_B)(e_1, e_1)(x, x) = 0, \text{ for each } e_1 \in E,\ x \in X. \end{aligned}$$

Conversely, let f_{Δ_E} be fuzzy soft closed. To show that (X, τ) is fuzzy soft Hausdorff, let e_{x_r} and e'_{y_s} be two distinct fuzzy soft points over X. Then $(e, e')_{(x,y)_\lambda} \in (f_{\Delta_E})^c$, where $\lambda = \max(r, s)$. Now since $(f_{\Delta_E})^c$ is fuzzy soft open, there exists a basic fuzzy soft open set, say $f_A \times g_B$, such that

$$\begin{aligned} & (e, e')_{(x,y)_\lambda} \in f_A \times g_B \sqsubseteq (f_{\Delta_E})^c \\ \Rightarrow \quad & \lambda < (f_A \times g_B)(e, e')(x, y) \\ \Rightarrow \quad & \lambda < \min\{f_A(e)(x), g_B(e')(y)\} \\ \Rightarrow \quad & \lambda < f_A(e)(x) \text{ and } \lambda < g_B(e')(y) \\ \Rightarrow \quad & r \leq \lambda < f_A(e)(x) \text{ and } s \leq \lambda < g_B(e')(y) \\ \Rightarrow \quad & e_{x_r} \in f_A \text{ and } e'_{y_s} \in g_B \end{aligned}$$

Further, since $f_A \times g_B \sqsubseteq (f_{\Delta_E})^c$, we have

$$\begin{aligned} &(f_A \times g_B)(e_1, e_1)(x, x) = 0, \text{ for each } e_1 \in E,\ x \in X \\ \Rightarrow\ &\min\{f_A(e_1)(x), g_B(e_1)(x)\} = 0, \text{ for each } e_1 \in E,\ x \in X \\ \Rightarrow\ &f_A(e_1) \cap g_B(e_1) = 0_X, \text{ for each } e_1 \in E \\ \Rightarrow\ &f_A \sqcap g_B = 0_E. \end{aligned}$$

□

Theorem 2.11. *If $\{(X_i, \tau_i) : i \in \Omega\}$ is a family of fuzzy soft topological spaces relative to the parameters sets E_i, then the product fuzzy soft topological space $(X, \tau) = \prod_{i\in\Omega}(X_i, \tau_i)$ is Hausdorff iff each coordinate fuzzy soft topological space (X_i, τ_i) is Hausdorff.*

Proof. First, let us assume that each (X_i, τ_i), $i \in \Omega$ is Hausdorff. Let $(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r}$ and $(\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s}$ be any pair of distinct fuzzy soft points over X. Then $\prod_{j\in\Omega} x_j \neq \prod_{j\in\Omega} y_j$ or $\prod_{j\in\Omega} e_j \neq \prod_{j\in\Omega} e'_j$. Let $\prod_{j\in\Omega} x_j \neq \prod_{j\in\Omega} y_j$, then $x_i \neq y_i$ for some $i \in \Omega$. Consider two fuzzy soft points $(e_i)_{(x_i)_r}$ and $(e'_i)_{(y_i)_s}$ over X_i which are distinct as $x_i \neq y_i$. Since (X_i, τ_i) is Hausdorff, so there exist two fuzzy soft open sets f_{A_i} and g_{B_i} such that

$$(e_i)_{(x_i)_r} \in f_{A_i}, \quad (e'_i)_{(y_i)_s} \in g_{B_i} \text{ and } f_{A_i} \sqcap g_{B_i} = 0_{E_i}.$$

Now, consider two fuzzy soft open sets over X as follows:

$$f_A = \prod_{j\in\Omega} f^1_{A_j} \text{ and } g_B = \prod_{j\in\Omega} g^1_{B_j},$$

where $f^1_{A_j} = 1_{E_j} = g^1_{B_j}$, $j \neq i$ and $f^1_{A_i} = f_{A_i}$, $g^1_{B_i} = g_{B_i}$. It is easy to see that f_A and g_B are disjoint fuzzy soft open sets such that $(\prod_{i\in\Omega} e_i)_{(\prod_{i\in\Omega} x_i)_r} \in f_A$ and $(\prod_{i\in\Omega} e'_i)_{(\prod_{i\in\Omega} y_i)_s} \in g_B$. The other case can be handled similarly.

Conversely, let us assume that the fuzzy soft product space (X, τ) is Hausdorff. Now, let $(e_i)_{(x_i)_r}$ and $(e'_i)_{(y_i)_s}$ be two distinct fuzzy soft points over X_i. Then $x_i \neq y_i$ or $e_i \neq e'_i$. Let $x_i \neq y_i$. Consider two fuzzy soft points $(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r}$ and $(\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s}$ over X, where $\prod_{j\in\Omega} x_j$ and $\prod_{j\in\Omega} y_j$ have identical j^{th} coordinates for $j \neq i$ and have i^{th} coordinates as x_i and y_i respectively and $\prod_{j\in\Omega} e_j$ and $\prod_{j\in\Omega} e'_j$ have identical j^{th} coordinates for $j \neq i$ and have i^{th} coordinates as e_i and e'_i respectively.

Then $(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r}$ and $(\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s}$ are distinct fuzzy soft points over X. Since (X,τ) is Hausdorff, there exist two fuzzy soft open sets g_A and h_B such that

$$(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r} \in g_A,\ (\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s} \in h_B \text{ and } g_A \sqcap h_B = 0_E.$$

Now, since g_A and h_B are fuzzy soft open, so we can find basic fuzzy soft open sets $\prod_{j\in\Omega} g_{A_j}$, $\prod_{j\in\Omega} h_{B_j}$ such that

$$(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r} \in \prod_{j\in\Omega} g_{A_j} \sqsubseteq g_A$$

and

$$(\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s} \in \prod_{j\in\Omega} h_{B_j} \sqsubseteq h_B$$

Now,

$$\begin{aligned}
&(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r} \in \prod_{j\in\Omega} g_{A_j}\\
\Rightarrow\quad & r < \inf_j g_{A_j}(e_j)(x_j)\\
\Rightarrow\quad & r < g_{A_j}(e_j)(x_j),\ \forall j \in \Omega \qquad (2.1)\\
\Rightarrow\quad & r < g_{A_i}(e_i)(x_i)\\
\Rightarrow\quad & (e_i)_{(x_i)_r} \in g_{A_i}
\end{aligned}$$

Similarly, $(e'_i)_{(y_i)_s} \in h_{B_i}$.
Since, $r \in (0,1)$, so from (2.1), we get

$$g_{A_j}(e_j)(x_j) > 0, \quad \forall j \in \Omega. \qquad (2.2)$$

Similarly,

$$h_{B_j}(e'_j)(y_j) > 0, \quad \forall j \in \Omega. \qquad (2.3)$$

Next, we have to show that

$$g_{A_i} \sqcap h_{B_i} = 0_{E_i}$$

Suppose on the contrary that,

$$g_{A_i} \sqcap h_{B_i} \neq 0_{E_i}$$

Then there exist $p_i \in E_i$, $z_i \in X_i$ such that

$$g_{A_i}(p_i)(z_i) > 0 \quad \text{and} \quad h_{B_i}(p_i)(z_i) > 0 \tag{2.4}$$

Construct a fuzzy soft point $(\prod_{j\in\Omega} e_j^{''})_{(\prod_{j\in\Omega} z_j^1)_\lambda}$ over X such that

$$e_j^{''} = \begin{cases} e_j, & \text{if } j \neq i \\ p_i, & \text{if } j = i, \end{cases}$$

and

$$z_j^1 = \begin{cases} x_j, & \text{if } j \neq i \\ z_i, & \text{if } j = i, \end{cases}$$

Now, for $z = \prod_{j\in\Omega} z_j^1$, from (2.2) and (2.4), we get

$$\prod_{j\in\Omega} g_{A_j}(\prod_{j\in\Omega} e_j^{''})(z) \quad = \quad \inf_j g_{A_j}(e_j^{''})(z_j^1) > 0$$

Similarly, from (2.3) and (2.4), we get

$$\prod_{j\in\Omega} h_{B_j}(\prod_{j\in\Omega} e_j^{''})(z) > 0$$

This gives us, $g_A(\prod_{j\in\Omega} e_j^{''})(z) > 0$, since $\prod_{j\in\Omega} g_{A_j} \sqsubseteq g_A$. Similarly, $h_B(\prod_{j\in\Omega} e_j^{''})(z) > 0$, since $\prod_{j\in\Omega} h_{B_j} \sqsubseteq h_B$
implying that

$$g_A \sqcap h_B \neq 0_E,$$

which is a contradiction.

The other case can be handled similarly. □

Proposition 2.12. *Subspace of a Hausdorff fuzzy soft topological space is also Hausdorff.*

The proof is easy, hence is omitted.

Theorem 2.13. *[115] Let $(\varphi, \psi) : (X_1, \tau_1) \to (X_2, \tau_2)$ be a fuzzy soft mapping and $\mathcal{B}$ be a base for τ_2. Then (φ, ψ) is fuzzy soft continuous iff $(\varphi, \psi)^{-1} f_B \in \tau_1, \forall f_B \in \mathcal{B}$.*

Now we prove the following theorem:

Theorem 2.14. *Let (φ, ψ) and (φ', ψ') be two fuzzy soft continuous maps between fuzzy soft topological spaces (X_1, τ_1) and (X_2, τ_2) relative to the parameters sets E, E' respectively, where (X_2, τ_2) is Hausdorff. Then the fuzzy soft set h_A over X_1 defined as follows:*

$$h_A(e)(x) = \begin{cases} 1, & \textit{if } e \in A_1, x \in B_1 \\ 0, & \textit{otherwise}, \end{cases}$$

where

$$A_1 = \{e \in E : \psi(e) = \psi'(e)\} \textit{ and } B_1 = \{x \in X_1 : \varphi(x) = \varphi'(x)\},$$

is fuzzy soft closed.

Proof. Here $(\varphi, \psi) : (X_1, \tau_1) \to (X_2, \tau_2)$ and $(\varphi', \psi') : (X_1, \tau_1) \to (X_2, \tau_2)$ are two fuzzy soft continuous maps. Now we define

$$((\varphi, \psi), (\varphi', \psi')) : (X_1, \tau_1) \to (X_2 \times X_2, \tau_2 \times \tau_2)$$

as the fuzzy soft map given by

$$((\varphi, \psi), (\varphi', \psi'))h_A = ((\varphi, \varphi'), (\psi, \psi'))h_A, \text{ for each } h_A \in \mathcal{F}(X_1, E),$$

where $(\psi, \psi') \colon E \to E' \times E'$, $(\varphi, \varphi') \colon X_1 \to X_2 \times X_2$ are given by

$$(\varphi, \varphi')(x) = (\varphi(x), \varphi'(x)), \text{ for each } x \in X_1, \ (\psi, \psi')(e) = (\psi(e), \psi'(e)), \text{ for each } e \in E.$$

Now we show that $((\varphi, \psi),(\varphi', \psi'))$ is fuzzy soft continuous. For this, consider a basic fuzzy soft open set $f_{A'} \times g_{B'}$ over $X_2 \times X_2$. Then,

$$\begin{aligned} &((\varphi, \psi), (\phi', \psi'))^{-1}(f_{A'} \times g_{B'})(e)(x) \\ &= ((\varphi, \varphi'), (\psi, \psi'))^{-1}(f_{A'} \times g_{B'})(e)(x), \text{ for each } e \in E, x \in X_1 \\ &= (f_{A'} \times g_{B'})(\psi(e), \psi'(e))(\varphi(x), \varphi'(x)), \text{ for each } e \in E, x \in X_1 \\ &= \min\{f_{A'}(\psi(e))(\varphi(x)), g_{B'}(\psi'(e))(\varphi'(x))\}, \text{ for each } e \in E, x \in X_1 \\ &= ((\varphi, \psi)^{-1} f_{A'} \sqcap (\varphi', \psi')^{-1} g_{B'})(e)(x), \text{ for each } e \in E, x \in X_1 \end{aligned}$$

Since, (φ, ψ) and (φ', ψ') both are fuzzy soft continuous maps from (X_1, τ_1) to (X_2, τ_2), so we have $(\varphi, \psi)^{-1} f_{A'} \sqcap (\varphi', \psi')^{-1} g_{B'} \in \tau_1$. Hence $((\varphi, \psi), (\varphi', \psi'))$ is fuzzy soft continuous. Therefore $((\varphi, \psi), (\varphi', \psi'))^{-1} f_{\Delta_{E'}}$ is a fuzzy soft closed over X_1, since $f_{\Delta_{E'}}$ is fuzzy soft closed over $X_2 \times X_2$. Now we show that $((\varphi, \psi), (\varphi', \psi'))^{-1} f_{\Delta_{E'}} = h_A$, as follows:

$$\begin{aligned}
&((\varphi, \psi), (\varphi', \psi'))^{-1}(f_{\Delta_{E'}})(e)(x) \\
&= ((\varphi, \varphi'), (\psi, \psi'))^{-1}(f_{\Delta_{E'}})(e)(x), \text{ for each } e \in E, x \in X_1 \\
&= (f_{\Delta_{E'}})(\psi(e), \psi'(e))(\varphi(x), \varphi'(x)), \text{ for each } e \in E, x \in X_1 \\
&= \begin{cases} 1, & \text{if } \psi(e) = \psi'(e) \text{ and } \varphi(x) = \varphi'(x) \\ 0, & \text{otherwise} \end{cases} \\
&= h_A(e)(x), \text{ for each } e \in E, x \in X_1.
\end{aligned}$$

□

2.3 Conclusion

In this chapter, we have introduced the notion of subspace of a fuzzy soft topological space and modified Mahanta's definitions of a 'fuzzy soft point' and 'belonging of a fuzzy soft point to a fuzzy soft set'. Then, by using these modified definitions, we have introduced Hausdorff(T_2) separation axiom in fuzzy soft topological spaces. Further, we have obtained a characterization for Hausdorff fuzzy soft topological spaces in terms of the fuzzy soft set f_{Δ_E}, where

$$f_{\Delta_E}(e_1, e_2) = \begin{cases} \chi_{\Delta_X}, & \text{if } e_1 = e_2 \\ 0_{X \times X}, & \text{if } e_1 \neq e_2 \end{cases}$$

and it has been shown that Hausdorffness in fuzzy soft topological spaces satisfies productive, projective and hereditary properties.