Chapter 7

Abelian Theorems of the Bessel Wavelet Transform

7.1 Introduction

Abelian theorems of integral transforms are widely used in solving the boundary values problems of mathematical physics. Mathematicians studied Abelian theorems for different types of integral transforms. Griffith[10] proved an Abelian theorem by exploiting the theory of Hankel transform in ordinary sense. Zemanian introduced Abelian theorems for the Hankel transform in ordinary and distributional sense both. Using the technique of Fourier transform, Pathak[20] discussed the Abelian theorems of the wavelet transform in both ordinary and distributional sense.

Motivated from the above results, in the present chapter we prove the Abelian theorems of the Bessel wavelet transform in ordinary and distributional sense both.

7.2 Abelian Theorems for the Bessel Wavelet Transform of Functions

In the present section, initial and final value theorems for the Bessel wavelet transform are given.

We assume that

$$(h_\mu\psi)(\omega) = 0(\omega^\mu), \qquad \omega \to 0^+ \tag{7.2.1}$$

and set

$$\int_0^\infty (h_\mu\psi)(\omega)\omega^{1/2-\eta} = H(\eta), \qquad \frac{3}{2} < \eta < \mu + 2. \tag{7.2.2}$$

Theorem 7.2.1. *Let* $\frac{3}{2} < \eta < \mu + 2$. *Assume that* $\omega^{1/2-\eta}(h_\mu\psi)(\omega) \in L^1(0,\infty)$, $|(h_\mu\psi)(\omega)| < M, M > 0$ *and* $\omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega) \in L^1(\delta,\infty), \forall\delta > 0$. *If*

$$\lim_{\omega\to 0} \omega^{\eta-\frac{1}{2}}(h_\mu f)(\omega) = \alpha \tag{7.2.3}$$

then

$$\lim_{a\to\infty} a^{3/2-\eta}(B_\psi f)(b,a) = \alpha H(\eta). \tag{7.2.4}$$

Proof. Using the arguments of (7.2.1) and (7.2.2), we get

$$\begin{aligned}
&\left|a^{3/2-\eta}(B_\psi f)(b,a) - \alpha H(\eta)\right| \\
&= \left|a^{3/2-\eta}\int_0^\infty (b\omega)^{1/2}J_\mu(b\omega)\overline{(h_\mu\psi)}(a\omega)\omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega)\right. \\
&\quad \left. - \alpha\int_0^\infty \overline{(h_\mu\psi)}(a\omega)(a\omega)^{1/2-\eta}a d\omega\right| \\
&= \left|\int_0^\infty \left[(b\omega)^{1/2}J_\mu(b\omega)\,\omega^{-\mu+\eta-1}(h_\mu f)(\omega) - \alpha\right](a\omega)^{1/2-\eta}\overline{(h_\mu\psi)}(a\omega)a\,d\omega\right| \\
&\le \int_0^\infty \left|(b\omega)^{1/2}J_\mu(b\omega)\,\omega^{-\mu+\eta-1}(h_\mu f)(\omega) - \alpha\right|\left|(a\omega)^{1/2-\eta}\overline{(h_\mu\psi)}(a\omega)\right|a\,d\omega \\
&\le \sup_{0<\omega<\delta}\left|(b\omega)^{1/2}J_\mu(b\omega)\,\omega^{-\mu+\eta-1}(h_\mu f)(\omega) - \alpha\right|\int_0^\delta \left|(a\omega)^{1/2-\eta}\overline{(h_\mu\psi)}(a\omega)\right|a\,d\omega \\
&\quad + Ma^{3/2-\eta}\int_\delta^\infty \left|(b\omega)^{1/2}J_\mu(b\omega)\,\omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega) - \alpha\omega^{\frac{1}{2}-\eta}\right|d\omega.
\end{aligned}$$

$$\begin{aligned}
&\left|a^{3/2-\eta}(B_\psi f)(b,a) - \alpha H(\eta)\right| \\
&\leq \sup_{0<\omega<\delta} \left|(b\omega)^{1/2} J_\mu(b\omega)\, \omega^{-\mu+\eta-1}(h_\mu f)(\omega) - \alpha\right| \int_0^\infty \left|(a\omega)^{1/2-\eta}\overline{(h_\mu\psi)}(a\omega)\right| a\, d\omega \\
&\quad + M a^{3/2-\eta} \int_\delta^\infty \left|(b\omega)^{1/2} J_\mu(b\omega)\, \omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega) - \alpha\omega^{\frac{1}{2}-\eta}\right| d\omega. \qquad (7.2.5)
\end{aligned}$$

In (7.2.5), the first of the above two integrals is convergent. Therefore, for given ϵ the first term can be made less than $\frac{\epsilon}{2}$ by choosing δ small enough. Since $\eta > \frac{3}{2}$, keeping δ fixed the second term in (7.2.5) can be made less than $\frac{\epsilon}{2}$ for all sufficiently large a. □

Theorem 7.2.2. *Let* $\frac{3}{2} < \eta < \mu + 2$, $\mu > 0$. *Assume that* $\omega^{\frac{1}{2}-\eta}(h_\mu\psi)(\omega) \in L^1(0,\infty)$ *and* $\omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega) \in L^1(0,X), X > 0$. *If*

$$\lim_{\omega\to\infty} (b\omega)^{1/2} J_\mu(b\omega)\omega^{\eta-\frac{1}{2}}(h_\mu f)(\omega) = \alpha, \qquad (7.2.6)$$

then

$$\lim_{a\to 0} a^{3/2-\eta}(B_\psi f)(b,a) = \alpha H(\eta). \qquad (7.2.7)$$

Proof. As in the previous theorem, for $X > 0$, we have

$$\begin{aligned}
&\left|a^{3/2-\eta}(B_\psi f)(b,a) - \alpha H(\eta)\right| \\
&\leq a \int_{\omega<X} \left|\left[(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\right] (a\omega)^{\frac{1}{2}-\eta}\overline{(h_\mu\psi)}(a\omega)\right| d\omega \\
&\quad + a \int_{\omega>X} \left|\left[(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega)\omega^{\eta-\frac{1}{2}} - \alpha\right] (a\omega)^{\frac{1}{2}-\eta}\overline{(h_\mu\psi)}(a\omega)\right| d\omega \\
&\leq a^{3/2-\eta} \int_0^X \left|(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu+\eta-1}(h_\mu f)(\omega) - \alpha\right| \left|\omega^{\frac{1}{2}-\eta}\overline{(h_\mu\psi)}(a\omega)\right| d\omega \\
&\quad + \sup_{\omega>X} \left|(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu+\eta-1}(h_\mu f)(\omega) - \alpha\right| \int_0^\infty \left|\omega^{\frac{1}{2}-\eta}(h_\mu\psi)(\omega)\right| d\omega. \qquad (7.2.8)
\end{aligned}$$

In view of the asymptotic behaviour (7.2.1), there exists a constant $A > 0$ such that $|(h_\mu\psi)(a\omega)| \leq A(a\omega)^\mu$.

Hence

$$\begin{aligned}
&\left|a^{3/2-\eta}(B_\psi f)(b,a) - \alpha H(\eta)\right| \\
&\le \; A \; a^{3/2-\eta+\mu} \int_0^X \left|(b\omega)^{1/2} \, J_\mu(b\omega) \; \omega^{-\mu+\eta-1} \; (h_\mu f)(\omega) - \alpha\right| \;\; \omega^{-\eta+\mu+\frac{1}{2}} \; d\omega \\
&\quad + \sup_{\omega > X} \left|(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu+\eta-1}(h_\mu f)(\omega) - \alpha\right| \int_0^\infty \left|\omega^{\frac{1}{2}-\eta}(h_\mu\psi)(\omega)\right| d\omega \\
&\le \; A \, a^{3/2-\eta+\mu} \int_0^X \left|(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega) - \alpha\omega^{\frac{1}{2}-\eta}\right| \omega^\mu d\omega \\
&\quad + \sup_{\omega > X} \left|(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu+\eta-1}(h_\mu f)(\omega) - \alpha\right| \int_0^\infty \left|\omega^{\frac{1}{2}-\eta}(h_\mu\psi)(\omega)\right| d\omega.
\end{aligned} \tag{7.2.9}$$

Since both the integrals on the right hand side of (7.2.9) are convergent and second term is independent of a, for given $\epsilon > 0$, the second term can be made less than $\epsilon/2$ by choosing X sufficiently large. Then there will exist $B > 0$ such that when $\eta < \frac{3}{2} + \mu$ the first term is less than $\epsilon/2$ for $0 < a < B$. □

7.3 Abelian Theorems for Bessel wavelet Transform of Distributions

In this section, we study the Bessel wavelet transform of distributions and its various properties.

Now, we take $\psi(x) \in \mathcal{H}_\mu(I)$. Then $h_\mu\psi \in \mathcal{H}_\mu(I)$ and $(b\omega)^{1/2}J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega) \in \mathcal{H}_\mu(I)$. Let $f \in \mathcal{H}_\mu^{'}(I)$. Then $h_\mu f \in \mathcal{H}_\mu^{'}(I)$.

We can define the Bessel wavelet transform of $f \in \mathcal{H}_\mu^{'}(I)$ by the following way:

$$(B_\psi f)(b,a) = \left\langle (h_\mu f)(\omega), (b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega) \right\rangle. \tag{7.3.1}$$

Theorem 7.3.1. *The function* $(B_\psi f)(b,a)$ *is differentiable and for each* $i, j \in \mathbb{N}_0$, *we have*

$$\left(b^{-1}\frac{\partial}{\partial b}\right)^j \left(a^{-1}\frac{\partial}{\partial a}\right)^i b^{-\mu-\frac{1}{2}}(B_\psi f)(b,a)$$
$$= \left\langle (h_\mu f)(\omega), \left(a^{-1}\frac{\partial}{\partial a}\right)^i (-1)^i \omega^{2j}(b\omega)^{-(\mu+j)} J_{\mu+j}(b\omega)\overline{(h_\mu\psi)}(a\omega)\right\rangle. \quad (7.3.2)$$

Proof. First we prove the differentiability of $(B_\psi f)(b,a)$ with respect to the variable $a > 0$. For $h > 0$, we have

$$\frac{1}{h}\left[(B_\psi f)(b,a+h) - (B_\psi f)(b,a)\right] - \left\langle (h_\mu f)(\omega), \frac{\partial}{\partial a}\omega^{-\mu-\frac{1}{2}}(b\omega)^{1/2}J_\mu(b\omega)\overline{(h_\mu\psi)}(a\omega)\right\rangle$$
$$= \Bigg\langle (h_\mu f)(\omega),$$
$$(b\omega)^{1/2}J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}\left\{\frac{1}{h}\left(\overline{(h_\mu\psi)}((a+h)\omega) - \overline{(h_\mu\psi)}(a\omega)\right) - \frac{\partial}{\partial a}\overline{(h_\mu\psi)}(a\omega)\right\}\Bigg\rangle.$$

Now, we show that
$(b\omega)^{1/2}J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}\left\{\frac{1}{h}\left(\overline{(h_\mu\psi)}((a+h)\omega) - \overline{(h_\mu\psi)}(a\omega)\right) - \frac{\partial}{\partial a}\overline{(h_\mu\psi)}(a\omega)\right\} \to 0$
as $h \to 0$ in $\mathcal{H}_\mu(I)$.
Denoting $\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)$ by $\overline{\hat{\Psi}}(a\omega)$ and $\left(\omega^{-1}\frac{\partial}{\partial\omega}\right)^r\left(\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)\right)$ by $\overline{\hat{\Psi}_r}(a\omega)$, we have

$$\left|\omega^k\left(\omega^{-1}\frac{\partial}{\partial\omega}\right)^m \omega^{-\mu-\frac{1}{2}}\left[(b\omega)^{1/2}J_\mu(b\omega)\left\{\frac{1}{h}\left(\overline{\hat{\Psi}}((a+h)\omega) - \overline{\hat{\Psi}}(\omega)\right) - \frac{\partial}{\partial a}\overline{\hat{\Psi}}(\omega)\right\}\right]\right|$$
$$= \Bigg|\omega^k b^{1/2}\sum_{r=0}^{m}\binom{m}{r}\left(\omega^{-1}\frac{\partial}{\partial\omega}\right)^{m-r}\left(\omega^{-\mu}J_\mu(b\omega)\right)$$
$$\left(\omega^{-1}\frac{\partial}{\partial\omega}\right)^r\left\{\frac{1}{h}\left(\overline{\hat{\Psi}}((a+h)\omega) - \overline{\hat{\Psi}}(\omega)\right) - \frac{\partial}{\partial a}\overline{\hat{\Psi}}(\omega)\right\}\Bigg|$$
$$= \Bigg|\omega^k b^{1/2}\sum_{r=0}^{m}\binom{m}{r}(-1)^{m-r}\omega^{-(\mu+m-r)}b^{m-r}J_{\mu+m-r}(b\omega)$$
$$\left\{\frac{1}{h}\left(\overline{\hat{\Psi}_r}((a+h)\omega) - \overline{\hat{\Psi}_r}(\omega)\right) - \frac{\partial}{\partial a}\overline{\hat{\Psi}_r}(\omega)\right\}\Bigg|$$

$$
\begin{aligned}
&= \sum_{r=0}^{m}\binom{m}{r} b^{\mu+2m-2r+\frac{1}{2}}\left|(b\omega)^{-(\mu+m-r)}J_{\mu+m-r}(b\omega)\right|\omega^k \\
&\qquad\qquad\left|\left\{\frac{1}{h}\left(\overline{\hat{\Psi}_r}((a+h)\omega)-\overline{\hat{\Psi}_r}(\omega)\right)-\frac{\partial}{\partial a}\overline{\hat{\Psi}_r}(\omega)\right\}\right| \\
&\leq \sum_{r=0}^{m}\binom{m}{r} b^{\mu+2m-2r+\frac{1}{2}}\,M\,\omega^k\frac{1}{h}\left|\int_a^{a+h}\frac{\partial}{\partial t}\overline{\hat{\Psi}_r}(t\omega)dt-\int_a^{a+h}\frac{\partial}{\partial a}\overline{\hat{\Psi}_r}(a\omega)dt\right| \\
&= \sum_{r=0}^{m}\binom{m}{r} b^{\mu+2m-2r+\frac{1}{2}}\,M\,\omega^k\left|\frac{1}{h}\int_a^{a+h}\int_a^{h}\left(\frac{\partial}{\partial u}\right)^2\overline{\hat{\Psi}_r}(u\omega)du\,dt\right| \\
&\leq \sum_{r=0}^{m}\binom{m}{r} b^{\mu+2m-2r+\frac{1}{2}}\,M\,h\sup_{a\leq u\leq a+h}\left|\omega^k\left(\frac{\partial}{\partial u}\right)^2\left(\omega^{-1}\frac{\partial}{\partial\omega}\right)^r\left(\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)\right)\right| \\
&\leq \sum_{r=0}^{m}\binom{m}{r} b^{\mu+2m-2r+\frac{1}{2}}\,M\,h\sup_{z\in I}\left|z^{k+4}\left(z^{-1}\frac{\partial}{\partial z}\right)^{r+2}z^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(z)\right|\sup_{a\leq u\leq a+h}u^{2r+\mu-k-\frac{3}{2}} \\
&\leq h\sum_{r=0}^{m}\binom{m}{r} b^{\mu+2m-2r+\frac{1}{2}}\,M\,\gamma^{\mu}_{k+4,r+2}(\overline{h_\mu\psi})\sup_{a\leq u\leq a+h}u^{2r+\mu-k-\frac{3}{2}}.
\end{aligned}
$$

Thus, we get

$$
\lim_{h\to 0}\frac{(B_\psi f)(b,a+h)-(B_\psi f)(b,a)}{h}=\left\langle (h_\mu f)(\omega),\frac{\partial}{\partial a}\omega^{-\mu-\frac{1}{2}}(b\omega)^{1/2}J_\mu(b\omega)\overline{(h_\mu\psi)}(a\omega)\right\rangle.
$$

Similarly, we can prove the differentiability with respect to the variable b and in general we have (7.3.2). □

Next, we obtain asymptotic order of $(B_\psi f)(b,a)$.

Theorem 7.3.2. *Let $(B_\psi f)(b,a)$ be the wavelet transform of $f\in\mathcal{H}'_\mu(I)$ defined by (7.3.1). Then, for large k, we have*

$$
\begin{aligned}
(B_\psi f)(b,a) &= O(a^{-2k}b^{2N}), && a\to 0\\
&= O(a^{2N}), && a\to\infty\\
&= O\left((1+a^2)^k a^{2(N-k)}\right), && b\to 0\\
&= O\left(a^{-2k}(1+a^2)^k b^{2N}\right), && b\to\infty,
\end{aligned}
$$

where $N\geq k+\mu+\frac{1}{2}$.

Proof. From the boundedness property of generalized functions [33, p.111] there exists a constant $C > 0$ and a non-negative integer k depending on f such that

$$\begin{aligned}
|(B_\psi f)(b,a)| &\leq C \sup_\omega \left| (1+\omega^2)^k \left(\omega^{-1}\frac{d}{d\omega}\right)^k \omega^{-\mu-\frac{1}{2}}(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)\right| \\
&= C \sup_\omega \left| (1+\omega^2)^k b^{1/2} \sum_{s=0}^{k} \binom{k}{s} \left(\omega^{-1}\frac{d}{d\omega}\right)^{k-s} \left(\omega^{-\mu} J_\mu(b\omega)\right)\right. \\
&\qquad \left. \left(\omega^{-1}\frac{d}{d\omega}\right)^{s} \left(\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)\right) \right| \\
&= C \sup_\omega \left| (1+\omega^2)^k \, b^{2k-2s+\mu+1/2} \sum_{s=0}^{k} \binom{k}{s} (b\omega)^{-(\mu+k-s)} J_{\mu+k-s}(b\omega)\right. \\
&\qquad \left. \left(\omega^{-1}\frac{d}{d\omega}\right)^{s} \left(\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)\right) \right|.
\end{aligned}$$

Let $\left|(b\omega)^{-(\mu+k-s)} J_{\mu+k-s}(b\omega)\right| \leq M, \;\; M > 0$. Then we have

$$\begin{aligned}
|(B_\psi f)(b,a)| &\leq CM \sup_\omega \left| \sum_{s=0}^{k}\sum_{r=0}^{k} \binom{k}{s}\binom{k}{r} \omega^{2r} \left(\omega^{-1}\frac{d}{d\omega}\right)^{s} \left(\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)\right) \right| \\
&\quad b^{2k-2s+\mu+1/2} \\
&= C' \sum_{s=0}^{k}\sum_{r=0}^{k} \binom{k}{s}\binom{k}{r} \left| \sup_\omega z^{2r} \left(z^{-1}\frac{d}{dz}\right)^{s} \left(z^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(z)\right) \right| \\
&\quad a^{2s-2r+\mu+1/2}\, b^{2k-2s+\mu+1/2} \\
&= C' \sum_{s=0}^{k}\sum_{r=0}^{k} \binom{k}{s}\binom{k}{r} \gamma^{\mu}_{2r,s}\left(\overline{(h_\mu\psi)}(\omega)\right) a^{2s-2r+\mu+1/2}\, b^{2k-2s+\mu+1/2} \\
&= C' \sum_{s=0}^{k}\sum_{r=0}^{k} \binom{k}{s}\binom{k}{r} a^{2s-2r+\mu+1/2}\, b^{2k-2s+\mu+1/2} \\
&= C' \sum_{r=0}^{k} \binom{k}{r} a^{-2r}\,(a^2+b^2)^k a^{\mu+\frac{1}{2}}\, b^{\mu+\frac{1}{2}} \\
&= C'(1+a^{-2})^k (a^2+b^2)^k a^{\mu+\frac{1}{2}}\, b^{\mu+\frac{1}{2}}.
\end{aligned}$$

Since $a^{\mu+\frac{1}{2}}\, b^{\mu+\frac{1}{2}} \leq (a^2+b^2)^{\mu+\frac{1}{2}}$,

$$|(B_\psi f)(b,a)| \leq C'(1+a^{-2})^k (a^2+b^2)^{k+\mu+\frac{1}{2}}.$$

Choosing $N \geq k + \mu + \frac{1}{2}$, we have

$$|(B_\psi f)(b,a)| \leq C^{'}(1+a^{-2})^k (a^2+b^2)^N. \tag{7.3.3}$$

From (7.3.3), we get the result. □

Theorem 7.3.3. *Let $\psi \in \mathcal{H}_\mu(I)$ and $f \in \mathcal{H}_\mu^{'}(I)$ be a distribution of compact support in I. Then*

$$(B_\psi f)(b,a) = \left\langle (h_\mu f)(\omega), (b\omega)^{1/2} J_\mu(b\omega)\, \omega^{-\mu-\frac{1}{2}} \overline{(h_\mu \psi)}(a\omega) \right\rangle,$$

is a smooth function on $I \times I$ and satisfies

$$(B_\psi f)(b,a) = O(a^\mu (1+b)^N), \qquad a \to 0. \tag{7.3.4}$$

Proof. Let $\psi \in \mathcal{H}_\mu(I)$. Then from [38, Theorem 5.4-1], $h_\mu \psi \in \mathcal{H}_\mu(I)$ and as a function of ω, $(b\omega)^{1/2} J_\mu(b\omega)\overline{(h_\mu\psi)}(\omega) \in \mathcal{E}(I)$. Let $f \in \mathcal{H}_\mu^{'}(I)$, then $h_\mu f \in \mathcal{H}_\mu^{'}(I)$ is of compact support $K \subseteq I$. Now, we take $\lambda \in D(I)$ such that $\lambda(\omega) = 1$ in a neighbourhood of K.

Therefore,

$$\begin{aligned}(B_\psi f)(b,a) &= \left\langle (h_\mu f)(\omega), (b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}} \overline{(h_\mu\psi)}(a\omega) \right\rangle \\ &= \left\langle (h_\mu f)(\omega), \lambda(\omega)(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}} \overline{(h_\mu\psi)}(a\omega) \right\rangle.\end{aligned}$$

By Theorem 7.3.1, $(B_\psi f)(b,a)$ is infinitely differentiable with respect to the variables b and a.

Using the boundedness property [33, p.111], we have

$$\begin{aligned}|(B_\psi f)(b,a)| &= \left|\left\langle (h_\mu f)(\omega), \lambda(\omega)(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}} \overline{(h_\mu\psi)}(a\omega) \right\rangle\right| \\ &\leq C \max_r \sup_{\omega\in K} \left| D_\omega^r \left\{ \lambda(\omega)(b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}} \overline{(h_\mu\psi)}(a\omega) \right\}\right| \\ &= C\, b^{1/2} \max_r \sup_{\omega\in K} \sum_{n=0}^{r} \binom{r}{n} \left|D_\omega^{r-n}\lambda(\omega)\right| \left| D_\omega^n \left\{ \omega^{-\mu} J_\mu(b\omega)\overline{(h_\mu\psi)}(a\omega) \right\}\right|\end{aligned}$$

$$
\begin{aligned}
&= C\, b^{1/2} \max_r \sup_{\omega \in K} \sum_{n=0}^{r} \binom{r}{n} \left| D_\omega^{r-n} \lambda(\omega) \right| \\
&\qquad \sum_{s=0}^{n} \binom{n}{s} \left| D_\omega^{n-s} \left(\omega^{-\mu} J_\mu(b\omega) \right) D_\omega^{s} \left(\overline{(h_\mu \psi)}(a\omega) \right) \right| \\
&= C\, b^{1/2} \max_r \sup_{\omega \in K} \sum_{n=0}^{r} \binom{r}{n} \left| D_\omega^{r-n} \lambda(\omega) \right| \\
&\qquad \sum_{s=0}^{n} \binom{n}{s} \left| (-1)^{n-s} b^{n-s} \omega^{-\mu} J_{\mu+n-s}(b\omega) \right| \left| D_\omega^{s} \left(\overline{(h_\mu \psi)}(a\omega) \right) \right| .
\end{aligned}
$$

Assume that

$$
D_\omega^s [\overline{(h_\mu \psi)}(a\omega)] = O(\omega^\mu), \quad \omega \to 0,\ \forall s \in \mathbb{N}_0. \tag{7.3.5}
$$

Then

$$
\begin{aligned}
|(B_\psi f)(b,a)| &\le C'\ b^{1/2} \max_r \sup_{\omega \in K} \sum_{n=0}^{r} \sum_{s=0}^{n} \binom{r}{n} \binom{n}{s} \left| D_\omega^{r-n} \lambda(\omega) \right| \\
&\qquad \left| b^{n-s+\mu} (b\omega)^{-\mu} J_{\mu+n-s}(b\omega) \right| a^{\mu+s} \omega^\mu \\
&\le C''\ \max_r \sum_{n=0}^{r} \sum_{s=0}^{n} \binom{r}{n} \binom{n}{s} b^{\eta-s+\mu+1/2} a^{\mu+s} \\
&= C''\ \max_r \sum_{n=0}^{r} \binom{r}{n} \left(\sum_{s=0}^{n} \binom{n}{s} b^{n-s} a^s \right) b^{\mu+1/2} a^\mu \\
&= C''\ \max_r \sum_{n=0}^{r} \binom{r}{n} (a+b)^n\, b^{1/2+\mu} a^\mu \\
&= C''\ \max_r (1+a+b)^r\, b^{1/2+\mu} a^\mu \\
&\le C''\ \max_r (1+a+b)^{r+\mu+1/2} a^\mu \\
&\le C''\ \max_r (1+a+b)^N a^\mu, \text{ where } N \ge r+\mu+1/2.
\end{aligned} \tag{7.3.6}
$$

From (7.3.6), we get the required result. □

The initial value theorem for the distributional Bessel wavelet transform is given below:

Theorem 7.3.4. *Let $h_\mu f \in \mathcal{H}'_\mu(I)$. Then it can be decomposed into $h_\mu f = h_\mu f_1 + h_\mu f_2$, where $h_\mu f_1$ is an ordinary function and $h_\mu f_2 \in \mathcal{E}'(I)$ is of order k. Now, we* *take the real numbers* μ *and* η *such that* $\frac{3}{2}+2N<\eta<2+\mu$, *where* $N\geq k+\mu+\frac{1}{2}$. *Now, we again assume that* $\omega^{1-\eta}(h_\mu\psi)(\omega)\in L^1(I)$ *and* $\omega^{-\mu-\frac{1}{2}}(h_\mu f_1)(\omega)\in L^1(\delta,\infty)$ $\forall\delta>0$. *Then* $(B_\psi f)(b,a)$ *is the distributional Bessel wavelet transform of* f *which is defined by (7.3.1) can be written in the following form*

$$\lim_{a\to\infty} a^{\frac{3}{2}-\eta}(B_\psi f)(b,a)=H(\eta)\lim_{\omega\to 0}\omega^{\eta-\frac{1}{2}}(h_\mu f)(\omega). \tag{7.3.7}$$

Proof. From Theorems 7.3.1 and 7.3.2 ,

$$(B_\psi f_2)(b,a)=\left\langle (h_\mu f_2)(\omega),(b\omega)^{1/2}J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)\right\rangle$$

is an infinitely differentiable function on $I\times I$ and $(B_\psi f_2)(b,a)=O(a^{2N}), a\to\infty$. Hence, there exists a constant $A>0$ such that

$$\left|a^{\frac{3}{2}-\eta}(B_\psi f_2)(b,a)\right|\leq Aa^{2N+\frac{3}{2}-\eta}\to 0, \quad \text{as } a\to\infty.$$

Also, since the support of $h_\mu f_2\in\mathcal{E}'(I)$ is a compact support of I,

$$\lim_{\omega\to 0}\omega^{\eta-\frac{1}{2}}\,(h_\mu\psi)(\omega)=0.$$

Thus theorem follows by an application of Theorem 7.2.1 with $(h_\mu f)(\omega)$ replaced by $(h_\mu f_1)(\omega)$. □

Final value theorem for the distributional Bessel wavelet transform is the following:

Theorem 7.3.5. *Let* $\frac{3}{2}<\eta<\mu+2$. *Assume that* $h_\mu f\in\mathcal{H}'_\mu(I)$ *can be decomposed into* $h_\mu f=h_\mu f_1+h_\mu f_2$, *where* $h_\mu f_1$ *is an ordinary function satisfying* $\omega^{-\mu-\frac{1}{2}}(h_\mu f_1)(\omega)\in L^1(0,X)$ $\forall X>0$ *and* $h_\mu f_2\in\mathcal{E}'(I)$. *If* $(B_\psi f)(b,a)$ *is the distributional Bessel wavelet transform of* f, *then*

$$\lim_{a\to 0} a^{\frac{3}{2}-\eta}(B_\psi f)(b,a)=H(\eta)\lim_{\omega\to\infty}(b\omega)^{1/2}J_\mu(b\omega)\,\omega^{\eta-\frac{1}{2}}(h_\mu f)(\omega). \tag{7.3.8}$$

Proof. From Theorems 7.3.1 and 7.3.3,

$$(B_\psi f_2)(b,a) = \left\langle (h_\mu f_2)(\omega), (b\omega)^{1/2} J_\mu(b\omega)\omega^{-\mu-\frac{1}{2}}\overline{(h_\mu\psi)}(a\omega)\right\rangle$$

is a smooth function on $I \times I$ and $(B_\psi f_2)(b,a) \leq Aa^\mu(1+b)^N$, for $a \to 0$, A being a large constant.
Since $\frac{3}{2} - \eta + \mu > 0$,

$$a^{\frac{3}{2}-\eta}\left|(B_\psi f)(b,a)\right| \leq Aa^{\mu-\eta+\mu}(1+b)^N \to 0 \quad \text{as } a \to 0.$$

The final result follows with the help of Theorem 7.2.2 with $h_\mu f$ replaced by $h_\mu f_1$. □

7.4 An Application

We apply the preceding theorems to the Bessel wavelet transform defined by function $\psi(x) = x^{\mu+\frac{1}{2}}e^{-\alpha x^2}$ with $\alpha > 0$, Re $\mu > -1$. The Hankel transform of ψ is $(h_\mu\psi)(\omega) = \frac{\omega^{\mu+\frac{1}{2}}}{(2\alpha)^{\mu+1}}e^{-(\frac{\omega^2}{4\alpha})}$ and $(h_\mu\psi)(\omega) = O(\omega^{\mu+1})$, $\omega \to 0$.
Assuming $\alpha = 1$, we have

$$\begin{aligned} H(\eta) &= \int_0^\infty (h_\mu\psi)(\omega)\omega^{1/2-\eta}d\omega \\ &= \int_0^\infty \frac{\omega^{\mu+\frac{1}{2}}}{2^{\mu+1}}e^{-\frac{\omega^2}{4}}\omega^{1/2-\eta}d\omega \\ &= 2^{-\eta-1}\Gamma\left(\frac{\mu-\eta+2}{2}\right), \quad \eta < \mu+2 \end{aligned} \tag{7.4.1}$$

and

$$\begin{aligned} (B_\psi f)(b,a) &= \int_0^\infty (b\omega)^{1/2}J_\mu(b\omega)\frac{(a\omega)^{\mu+\frac{1}{2}}}{2^{\mu+1}}e^{-\frac{(a\omega)^2}{4}}\omega^{-\mu-\frac{1}{2}}(h_\mu f)(\omega) \\ &= \frac{a^{\mu+\frac{1}{2}}}{2^{\mu+1}}\int_0^\infty (b\omega)^{1/2}J_\mu(b\omega)e^{-\frac{(a\omega)^2}{4}}(h_\mu f)(\omega). \end{aligned} \tag{7.4.2}$$

Therefore, by a modification of Theorem 7.2.1 for $\eta < \mu + 2$ and $e^{-\frac{(a\omega)^2}{4}}(h_\mu f)(\omega) \in L^1(\delta, \infty)$, we have

$$\lim_{a\to\infty} a^{3/2-\eta}(B_\psi f)(b,a) = 2^{-\eta-1}\Gamma\left(\frac{\mu-\eta+2}{2}\right)\lim_{\omega\to 0}\omega^{\eta-\frac{1}{2}}(h_\mu f)(\omega). \tag{7.4.3}$$

By Theorem 7.2.2, $\eta < \mu + 2$ and $e^{-\frac{(a\omega)^2}{4}}(h_\mu f)(\omega) \in L^1(0, X)$, we have

$$\lim_{a\to 0} a^{3/2-\eta}(B_\psi f)(b,a) = 2^{-\eta-1}\Gamma\left(\frac{\mu-\eta+2}{2}\right)\lim_{\omega\to\infty}(b\omega)^{1/2}J_\mu(b\omega)\omega^{\eta-\frac{1}{2}}(h_\mu f)(\omega). \tag{7.4.4}$$

Since in the present case kernel $(h_\mu\psi)(\omega)$ is exponentially decreasing, hence conditions of validity of initial and final value theorems are relaxed. Results corresponding to Theorem 7.3.4 and Theorem 7.3.5 can be obtained using results (7.4.3) and (7.4.4).