Chapter 6

Bessel Wavelet Transform on the Spaces with Exponential Growth

6.1 Introduction

Hankel transform is an important tool which is used by many mathematician to solve various problems on different types of functional and distributional spaces. Zemanian[35, 38], Koh[14, 15], Pandey[8], Pathak[17] and others investigated the aforesaid spaces in many research papers and studied their properties.

The space of type $H_\mu(I)$ which is introduced by Zemanian[38]. He showed that the Hankel transform h_μ is an automorphism on $H_\mu(I)$. He also proved that the generalized Hankel transformation $h_\mu^{'}$ is an automorphism on $H_\mu^{'}(I)$. Several other properties of the Hankel transformation are studied by Zemanian [34, 37] and others. The spaces of exponential growth of U_μ types were given by Pathak[17] and he obtained their algebraic & topological properties using the theory of Hankel transform.

To study the Hankel transform on the space of exponential growth, Betancor and Mesa[2] defined the $\chi_\mu(I)$ and $\mathcal{Q}_\mu(I)$ spaces. Involving Hankel transform and

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Hankel convolution theory, it is shown that Hankel transformation is an isomorphism from $\chi_\mu(I)$ onto $\mathcal{Q}_\mu(I)$ space for $\mu \geq -\frac{1}{2}$. From the Hankel convolution tool, he also discussed many other properties.

In the present chapter, motivated from the work of Betancor and Mesa[2], the Bessel wavelet transforms on $\chi_\mu(I)$ and $\mathcal{Q}_\mu(I)$ spaces are investigated and it is shown that Bessel wavelet transforms $B_\psi : \chi_\mu(I) \longrightarrow \chi_\mu(I \times I)$, $B_\psi : \mathcal{Q}_\mu(I) \longrightarrow \mathcal{Q}_\mu(I \times I)$ are linear and continuous. Applying the above continuity properties of the Bessel wavelet transform, some properties of Hankel convolution are studied. The theory of Bessel wavelet transform on $\chi_\mu(I)$ space is suitably used on Fredholm integral equation associated with Hankel convolution and obtained the solution of integral equations.

Now, we are stating the various definitions and properties which are related to our present work.

The space $\chi_\mu(I)$ consists of all smooth complex-valued function $\phi(x)$, $x \in I$ satisfies the following norm

$$\gamma^\mu_{k,m}(\phi) = \sup_{x\in(0,\infty)} \left| e^{kx} \left(x^{-1}\frac{d}{dx}\right)^m (x^{-\mu-\frac{1}{2}}\phi(x)) \right| < \infty, \qquad (6.1.1)$$

for every $k, m \in \mathbb{N}_0$.

The semi norm for $\phi \in \chi_\mu(I)$ is given by

$$\eta^\mu_{k,m}(\phi) = \sup_{x\in(0,\infty)} \left| e^{kx} x^{-\mu-\frac{1}{2}} S^m_\mu \phi(x) \right|, \qquad k, m \in \mathbb{N}_0, \qquad (6.1.2)$$

where $S_\mu = x^{-\mu-\frac{1}{2}}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx}x^{-\mu-\frac{1}{2}}$, induces on $\chi_\mu(I)$ the same topology as defined by $\left\{\gamma^\mu_{k,m}\right\}_{k,m\in\mathbb{N}_0}$.

The space of multipliers of $\chi_\mu(I)$ is denoted by θ_{χ_μ} and is defined as a function $f \in \theta_{\chi_\mu}$ whenever $f\phi \in \chi_\mu(I)$ for every $\phi \in \chi_\mu(I)$. Thus a function $f \in \theta_{\chi_\mu}$ if and only if

(i) f is smooth on I, and

(ii) for every $m \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $C > 0$ such that

$$\left| \left(\frac{1}{x} \frac{d}{dx} \right)^m f(x) \right| \leq Ce^{kx}, \ x \in I.$$

The space $\mathcal{Q}_\mu(I)$ consists of all complex-valued functions Φ which satisfy the following two conditions

(i) $z^{-\mu-\frac{1}{2}}\Phi(z)$ is an even entire function.

(ii) For every $k, m \in \mathbb{N}_0$, the following norm is given by

$$\omega^\mu_{k,m}(\Phi) = \sup_{|Im\, z| \leq k} (1 + |z|^2)^m |z^{-\mu-\frac{1}{2}}\phi(z)| < \infty. \tag{6.1.3}$$

Bessel functions J_μ satisfies the following boundedness properties which are very useful in our investigation.

$$\text{(i).} \ \left| z^{-\mu} J_\mu(z) \right| \ \leq \ Ce^{|Im\, z|}, \quad z \in \mathbb{C} \tag{6.1.4}$$

$$\text{(ii).} \ |z^{1/2} H^{(1)}_\mu(z)| \ \leq \ Ce^{-Im\, z}, \quad z \in \mathbb{C}, |z| \geq 1, \tag{6.1.5}$$

where $H^{(1)}_\mu$ denotes the Hankel function of the first kind of order μ and C is a positive constant depending on μ in (6.1.4) and (6.1.5).

6.2 The Bessel Wavelet Transform on the Spaces χ_μ(I) and $\mathcal{Q}_\mu$(I)

In this section, the properties of Bessel wavelet transform on $\chi_\mu(I)$ and $\mathcal{Q}_\mu(I)$ type spaces are studied.

Lemma 6.2.1. *If $\psi \in \chi_\mu(I), I = (0, \infty)$, then we have the following estimate*

$$\left|\left(a^{-1}\frac{d}{da}\right)^q (h_\mu\psi)(a(\xi + i(k+1))\right| \leq \sum_{r=0}^{q} \binom{q}{r} C_\mu a^{\mu+\frac{1}{2}-2r} |\xi + i(k+1)|^{\mu-2r+2q+\frac{1}{2}}$$
$$\gamma^\mu_{k+1,0}(\psi)\Gamma(2\mu - 2r + 2q + 1),$$

where $a > 0, \mu > r - q - \frac{1}{2}$ and $C_\mu = C \times A_{\mu,r}$ with arbitrary constant C and $A_{\mu,r} = (\mu + \frac{1}{2})(\mu + \frac{1}{2} - 2)...(\mu + \frac{1}{2} - 2(r-1))$.

Proof. We have

$$\begin{aligned}
\left(v^{-1}\frac{d}{dv}\right)^q (h_\mu\psi)(v) &= \left(v^{-1}\frac{d}{dv}\right)^q \int_0^\infty (vy)^{1/2} J_\mu(vy)\psi(y)dy \\
&= \int_0^\infty \left(v^{-1}\frac{d}{dv}\right)^q (vy)^{-\mu} J_\mu(vy)(vy)^{\mu+1/2}\psi(y)dy \\
&= \int_0^\infty \left(v^{-1}\frac{d}{dv}\right)^q \left(v^{-\mu} J_\mu(vy) v^{\mu+1/2}\right) y^{-\mu} y^{\mu+1/2}\psi(y)dy \\
&= \int_0^\infty \sum_{r=0}^{q} \binom{q}{r} \left(v^{-1}\frac{d}{dv}\right)^{q-r} \left(v^{-\mu} J_\mu(vy)\right) \left(v^{-1}\frac{d}{dv}\right)^r \\
&\quad\; v^{\mu+1/2} y^{-\mu} y^{\mu+1/2}\psi(y)dy. \qquad (6.2.1)
\end{aligned}$$

Since

$$\left(v^{-1}\frac{d}{dv}\right)^{q-r} \left(v^{-\mu} J_\mu(vy)\right) = (-1)^{q-r} v^{-(\mu+q-r)}\, y^{q-r}\, J_{\mu+q-r}(vy) \qquad (6.2.2)$$

and

$$\left(v^{-1}\frac{d}{dv}\right)^r v^{\mu+1/2} = (\mu + \frac{1}{2})(\mu + \frac{1}{2} - 2)....(\mu + \frac{1}{2} - 2(r-1))v^{\mu+\frac{1}{2}-2r}, \qquad (6.2.3)$$

putting values of (6.2.2) and (6.2.3) in (6.2.1), we get

$$\begin{aligned}
&\left(v^{-1}\frac{d}{dv}\right)^q (h_\mu\psi)(v) \\
&= \int_0^\infty \sum_{r=0}^{q} \binom{q}{r} \Bigg((-1)^{q-r} v^{-(\mu+q-r)} y^{q-r}\, J_{\mu+q-r}(vy) \\
&\qquad (\mu+\frac{1}{2})(\mu+\frac{1}{2}-2)...(\mu+\frac{1}{2}-2(r-1)) v^{\mu+\frac{1}{2}-2r} \Bigg)\, y^{-\mu} y^{\mu+1/2} \psi(y) dy \\
&= \sum_{r=0}^{q} \int_0^\infty \binom{q}{r} (-1)^{q-r} (vy)^{-(\mu+q-r)} J_{\mu+q-r}(vy)\, y^{\mu+2q-2r+\frac{1}{2}}\, A_{\mu,r}\ v^{\mu+\frac{1}{2}-2r}\, \psi(y) dy \\
&= \sum_{r=0}^{q} \binom{q}{r} (-1)^{q-r} A_{\mu,r}\ v^{\mu+\frac{1}{2}-2r} \int_0^\infty (vy)^{-(\mu+q-r)} J_{\mu+q-r}(vy)\, y^{\mu+2q-2r+\frac{1}{2}}\, \psi(y) dy.
\end{aligned}$$

Taking the absolute value of the above equation, we have

$$\begin{aligned}
\left|\left(v^{-1}\frac{d}{dv}\right)^q (h_\mu\psi)(v)\right| \le \sum_{r=0}^{q} \binom{q}{r} A_{\mu,r} |v|^{\mu+\frac{1}{2}-2r} \int_0^\infty \Big| (vy)^{-(\mu+q-r)} J_{\mu+q-r}(vy) \\
y^{\mu+2q-2r+\frac{1}{2}} \psi(y)\Big|\, dy.
\end{aligned}$$

From (6.1.4), we get the following estimate

$$\begin{aligned}
&\left|\left(v^{-1}\frac{d}{dv}\right)^q (h_\mu\psi)(v)\right| \\
&\le \sum_{r=0}^{q} \binom{q}{r} A_{\mu,r}\ |v|^{\mu+\frac{1}{2}-2r} C \int_0^\infty e^{ky} y^{\mu+2q-2r+\frac{1}{2}}\ |\psi(y)|\, dy, \qquad if\ |Im\, v| \le k \\
&\le \sum_{r=0}^{q} \binom{q}{r} A_{\mu,r}\, C\ |v|^{\mu+\frac{1}{2}-2r} \int_0^\infty e^{ky} y^{2\mu+2q-2r+1}\, y^{-\mu-\frac{1}{2}}\, |\psi(y)|\, dy \\
&\le \sum_{r=0}^{q} \binom{q}{r} C_\mu\ |v|^{\mu+\frac{1}{2}-2r} \sup_{y\in(0,\infty)} \left| e^{(k+1)y} y^{-\mu-\frac{1}{2}} \psi(y)\right| \int_0^\infty e^{-y} y^{2\mu+2q-2r+1} dy \\
&\le \sum_{r=0}^{q} \binom{q}{r} C_\mu\ |v|^{\mu+\frac{1}{2}-2r}\ \gamma^{\mu}_{k+1,0}(\psi)\, \Gamma(2\mu-2r+2q+1).
\end{aligned}$$

Putting $v = a\,(\xi + i(k+1))$, we have

$$\left|\left([a\,(\xi + i(k+1))]^{-1}\frac{d}{d[a\,(\xi + i(k+1))]}\right)^q (h_\mu\psi)\,(a\,(\xi + i(k+1)))\right|$$
$$\leq \sum_{r=0}^{q}\binom{q}{r}C_\mu a^{\mu-2r+\frac{1}{2}}\,|\xi + i(k+1)|^{\mu+\frac{1}{2}-2r}\,\gamma^\mu_{k+1,0}(\psi)\,\Gamma(2\mu - 2r + 2q + 1).$$

Hence

$$\left|\left(a^{-1}\frac{d}{da}\right)^q (h_\mu\psi)\,(a(\xi + i(k+1)))\right|$$
$$\leq \sum_{r=0}^{q}\binom{q}{r}C_\mu\, a^{\mu-2r+\frac{1}{2}}\,|\xi + i(k+1)|^{\mu-2r+2q+\frac{1}{2}}\,\gamma^\mu_{k+1,0}(\psi)\Gamma(2\mu - 2r + 2q + 1).$$

□

Theorem 6.2.2. *Bessel wavelet transform B_ψ is a continuous linear map from $\chi_\mu(I)$ to $\chi_\mu(I \times I)$ for $\mu \geq -\frac{1}{2}$.*

Proof. Let $\phi \in \chi_\mu(I)$. Then by using (1.4.4), we have

$$(-1)^q\left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a)$$
$$= (-1)^q\left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}\int_0^\infty (bx)^{1/2}J_\mu(bx)x^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(x)\,(h_\mu\psi)\,(ax)dx$$
$$= (-1)^q\int_0^\infty\left(b^{-1}\frac{d}{db}\right)^q\left(b^{-\mu}J_\mu(bx)\right)x^{1/2}\left(x^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(x)\right)(h_\mu\psi)\,(ax)dx$$
$$= (-1)^{2q}\int_0^\infty\left\{b^{-\mu-q}x^q J_{\mu+q}(bx)\right\}x^{1/2}\left(x^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(x)\right)(h_\mu\psi)\,(ax)dx$$
$$= \int_0^\infty b^{-\mu-q}x^{q+\frac{1}{2}}J_{\mu+q}(bx)\left(x^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(x)\right)(h_\mu\psi)\,(ax)dx.$$

Using [2, Theorem 2.1], we get

$$(-1)^q\left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a) = \frac{1}{2}\int_{-\infty}^\infty b^{-\mu-q}(\xi+i\eta)^{q+\frac{1}{2}}H^{(1)}_{\mu+q}(b(\xi+i\eta))$$
$$\left((\xi+i\eta)^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(\xi+i\eta)\right)(h_\mu\psi)\,(a(\xi+i\eta))d\xi.$$

Therefore,

$$\begin{aligned}
&\left|(-1)^q \left(a^{-1}\frac{d}{da}\right)^l \left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a)\right| \\
&\leq \frac{1}{2}\int_{-\infty}^{\infty} \left|b^{-\mu-q}\,(\xi+i\eta)^{q+\frac{1}{2}}\, H^{(1)}_{\mu+q}(b(\xi+i\eta))\left((\xi+i\eta)^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(\xi+i\eta)\right)\right| \\
&\qquad\qquad \left|\left(a^{-1}\frac{d}{da}\right)^l (h_\mu\psi)\,(a(\xi+i\eta))\right| d\xi \\
&= \int_{-\infty}^{\infty} b^{-\mu-q-\frac{1}{2}}\, \left|(b(\xi+i\eta))^{\frac{1}{2}} H^{(1)}_{\mu+q}(b(\xi+i\eta))\right| \\
&\qquad \left|(\xi+i\eta)^q \left((\xi+i\eta)^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(\xi+i\eta)\right)\right| \left|\left(a^{-1}\frac{d}{da}\right)^l (h_\mu\psi)\,(a(\xi+i\eta))\right| d\xi \\
&\leq b^{-\mu-q-\frac{1}{2}}\, C\, e^{-b\eta}\int_{-\infty}^{\infty} \left|(\xi+i\eta)^q \left((\xi+i\eta)^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(\xi+i\eta)\right)\right| \\
&\qquad\qquad\qquad \left|\left(a^{-1}\frac{d}{da}\right)^l (h_\mu\psi)\,(a(\xi+i\eta))\right| d\xi.
\end{aligned}$$

Using Lemma 6.2.1, we get

$$\begin{aligned}
&\left|(-1)^q \left(a^{-1}\frac{d}{da}\right)^l \left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a)\right| \\
&\leq b^{-\mu-q-\frac{1}{2}} C\, e^{-b\eta}\int_{-\infty}^{\infty} \left|(\xi+i\eta)^q \left[(\xi+i\eta)^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(\xi+i\eta)\right]\right| \\
&\quad \left(\sum_{r=0}^{l}\binom{l}{r} C_\mu\, a^{\mu-2r+\frac{1}{2}}\, |\xi+i\eta|^{\mu+\frac{1}{2}-2r-2l}\, \gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1)\right) d\xi \\
&\leq C\, b^{-\mu-q-\frac{1}{2}} a^{\mu-2r+\frac{1}{2}} e^{-b\eta} \sum_{r=0}^{l}\binom{l}{r} C_\mu\, \gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1) \\
&\qquad \int_{-\infty}^{\infty} \left|(\xi+i\eta)^{\mu-2r+2l+q+\frac{1}{2}} \left[(\xi+i\eta)^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(\xi+i\eta)\right]\right| d\xi.
\end{aligned}$$

Putting $\eta = k+1$, we have

$$\left| e^{kb}\left(a^{-1}\frac{d}{da}\right)^{l}\left(b^{-1}\frac{d}{db}\right)^{q} b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a)\right|$$
$$\leq C.C_\mu\, b^{-\mu-q-\frac{1}{2}} a^{\mu-2r+\frac{1}{2}} e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1)$$
$$\int_{-\infty}^{\infty}|\xi+i(k+1)|^{\mu-2r+2l+q+\frac{1}{2}}\left|(\xi+i(k+1))^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(\xi+i(k+1))\right| d\xi.$$

Now let $z = \xi + i(k+1)$, we have

$$\left| e^{kb}\left(a^{-1}\frac{d}{da}\right)^{l}\left(b^{-1}\frac{d}{db}\right)^{q} b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a)\right|$$
$$\leq C'_\mu b^{-\mu-q-\frac{1}{2}} a^{\mu-2r+\frac{1}{2}} e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1)$$
$$\int_{-\infty}^{\infty}|z|^{\mu-2r+2l+q+\frac{1}{2}}\left|z^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(z)\right| dz$$
$$\leq C'_\mu b^{-\mu-q-\frac{1}{2}} a^{\mu-2r+\frac{1}{2}} e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1)$$
$$\int_{-\infty}^{\infty}\left(1+|z|^2\right)^{-m}|z|^{\mu-2r+2l+q+\frac{1}{2}}\left(1+|z|^2\right)^{m}\left|z^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(z)\right| dz$$
$$\leq C'_\mu b^{-\mu-q-\frac{1}{2}} a^{\mu-2r+\frac{1}{2}} e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1)$$
$$\sup_{|Im\, z|\leq k+1}\left(1+|z|^2\right)^{m}\left|z^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(z)\right|\int_{-\infty}^{\infty}\left(1+|z|^2\right)^{-m}|z|^{\mu-2r+2l+q+\frac{1}{2}} dz.$$

The last integral is convergent for large value of m. Then

$$\left| e^{kb}\left(a^{-1}\frac{d}{da}\right)^{l}\left(b^{-1}\frac{d}{db}\right)^{q} b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a)\right| \leq C'_\mu b^{-\mu-q-\frac{1}{2}} a^{\mu-2r+\frac{1}{2}} e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)$$
$$\Gamma(2\mu-2r+2l+1)\sup_{|Im\, z|\leq k+1}\left(1+|z|^2\right)^{m}\left|z^{-\mu-\frac{1}{2}}\,(h_\mu\phi)\,(z)\right|.$$

Using [2, Theorem 2.1], we get

$$\left| e^{kb}\left(a^{-1}\frac{d}{da}\right)^{l}\left(b^{-1}\frac{d}{db}\right)^{q} b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a)\right|$$
$$\leq C'_\mu b^{-\mu-q-\frac{1}{2}}a^{\mu-2r+\frac{1}{2}}e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)\Gamma(2\mu-2r+2l+1)$$
$$\left\{\eta^{\mu}_{k+2,0}(\phi)+\eta^{\mu}_{k+2,m}(\phi)\right\}.$$

□

Theorem 6.2.3. *The Bessel wavelet transform B_ψ is a continuous linear mapping from $\mathcal{Q}_\mu(I)$ to $\mathcal{Q}_\mu(I\times I)$.*

Proof. Let $\phi\in\mathcal{Q}_\mu(I)$. Suppose $(B_\psi\phi)(z,a)=\Phi(z,a)$, where $z=b+ib'$, $b,b'\in I$ and $a\in I$. From (1.4.4), we have

$$\begin{aligned}
&a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\\
&=a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\int_0^\infty (zx)^{\frac{1}{2}}J_\mu(zx)\left(h_\mu\phi\right)(x)x^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)dx\\
&=a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\int_0^\infty (zx)^{-\mu}J_\mu(zx)(zx)^{\mu+\frac{1}{2}}\left(h_\mu\phi\right)(x)x^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)dx\\
&=\int_0^\infty (zx)^{-\mu}J_\mu(zx)x^{\mu+\frac{1}{2}}\left(h_\mu\phi\right)(x)(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)dx.
\end{aligned}$$

Therefore,

$$\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right|\leq\int_0^\infty\left|(zx)^{-\mu}J_\mu(zx)\right|\left|x^{\mu+\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\left|(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|dx.$$

Using (6.1.4), we have

$$\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right|\leq C\int_0^\infty e^{x|Imz|}\left|x^{\mu+\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\left|(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|dx.$$

For $|Im\,z| \leq k$, we can write the above expression

$$\begin{aligned}
&\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right| \\
&\leq C\int_0^\infty e^{xk}\left|x^{\mu+\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\left|(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|dx \\
&\leq C\int_0^\infty e^{-axk}x^{2\mu+1}\left|e^{xk}x^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\left|e^{axk}(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|dx \\
&\leq C\sup_{x\in I}\left|e^{xk}x^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\sup_{x\in I}\left|e^{axk}(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|\int_0^\infty e^{-axk}x^{2\mu+1}dx.
\end{aligned}$$

Then

$$\begin{aligned}
&\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right| \\
&\leq C\sup_{x\in I}\left|e^{xk}x^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\sup_{x\in I}\left|e^{axk}(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|\frac{\Gamma(2\mu+1)}{(ak)^{2\mu+1}}.
\end{aligned}$$

In the view of [2, Theorem 2.1, pp.38-39], we have

$$\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right| \leq C\,\omega^\mu_{k+1,l}(\phi)\,\omega^\mu_{k+1,l}(\psi)\frac{\Gamma(2\mu+1)}{(ak)^{2\mu+1}}, \tag{6.2.4}$$

for $x > 1$ and $l > \mu + \frac{3}{2}$.

For $x \in (0,1)$, using the arguments of [2, Theorem 2.1, pp.38-39], we have

$$\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right| \leq C\omega^\mu_{1,n}(\phi)\,\omega^\mu_{1,n}(\psi)\frac{\Gamma(2\mu+1)}{(ak)^{2\mu+1}}, \tag{6.2.5}$$

where $n \in \mathbb{N}$ and $n > \mu + 1$.

Taking (6.2.4) and (6.2.5), B_ψ is a continuous mapping from $\mathcal{Q}_\mu(I)$ to $\mathcal{Q}_\mu(I\times I)$. □

Lemma 6.2.4. *Let $\psi \in \chi_\mu(I)$ be a Bessel wavelet, then it can be written in the terms of Hankel transform as*

$$\tau_b\psi_a(x) = b^{\mu+\frac{1}{2}}h_\mu\left[(bu)^{-\mu}J_\mu(bu)(h_\mu\psi)(au)\right](x), \tag{6.2.6}$$

where a, b are dilation and translation parameters, respectively.

Proof. From (1.4.3), we have

$$\begin{aligned}
\tau_b\psi_a(x) &= a^{\mu-\frac{1}{2}}\psi\left(\frac{x}{a},\frac{b}{a}\right)\\
&= a^{\mu-\frac{1}{2}}\int_0^\infty \psi(z)\,D_\mu\left(\frac{x}{a},\frac{b}{a},z\right)dz\\
&= a^{\mu-\frac{1}{2}}\int_0^\infty \psi(z)\left(\int_0^\infty t^{-\mu-\frac{1}{2}}\left(\frac{xt}{a}\right)^{\frac{1}{2}}J_\mu\left(\frac{xt}{a}\right)\left(\frac{bt}{a}\right)^{\frac{1}{2}}J_\mu\left(\frac{bt}{a}\right)\right.\\
&\qquad\qquad\left.(zt)^{\frac{1}{2}}J_\mu(zt)dt\right)dz.
\end{aligned}$$

Putting $\frac{t}{a}=u$, we have

$$\begin{aligned}
\tau_b\psi_a(x) &= a^{\mu-\frac{1}{2}}\int_0^\infty \psi(z)\left(\int_0^\infty (au)^{-\mu-\frac{1}{2}}(xu)^{\frac{1}{2}}J_\mu(xu)(bu)^{\frac{1}{2}}J_\mu(bu)\right.\\
&\qquad\qquad\left.(zau)^{\frac{1}{2}}J_\mu(zau)adu\right)dz\\
&= \int_0^\infty\left(\int_0^\infty (zau)^{\frac{1}{2}}J_\mu(zau)\psi(z)dz\right)u^{-\mu-\frac{1}{2}}(xu)^{\frac{1}{2}}J_\mu(xu)(bu)^{\frac{1}{2}}J_\mu(bu)du\\
&= \int_0^\infty u^{-\mu-\frac{1}{2}}(xu)^{\frac{1}{2}}J_\mu(xu)(bu)^{\frac{1}{2}}J_\mu(bu)(h_\mu\psi)(au)du\\
&= b^{\mu+\frac{1}{2}}h_\mu\left[(bu)^{-\mu}J_\mu(bu)(h_\mu\psi)(au)\right](x).
\end{aligned}$$

□

Theorem 6.2.5. *If* $f\in\chi_\mu^{'}(I)$ *and* $\psi\in\chi_\mu(I)$ *then* $(1+a^{2r})^{-1}b^{-\mu-\frac{1}{2}}\left(B_\psi f\right)(b,a)\in\theta_{\chi_\mu}$*, where* $\chi_\mu^{'}(I)$ *and* θ_{χ_μ} *denote the dual and multiplier of* $\chi_\mu(I)$*, respectively.*

Proof. Suppose $f\in\chi_\mu^{'}(I)$ and $\psi\in\chi_\mu(I)$. Since $f\in\chi_\mu^{'}$, from [2] there exist $r\in\mathbb{N}$ and essentially bounded functions f_k on I, $0\le k\le r$, such that

$$f=\sum_{k=0}^{r}S_\mu^k(e^{rx}x^{-\mu-\frac{1}{2}}f_k).$$

Therefore,

$$\begin{aligned}
\left(B_\psi f\right)(b,a) &= \left(f\#\overline{\psi_a}\right)(b) \\
&= \langle f, \tau_b\psi_a\rangle \\
&= \left\langle \sum_{k=0}^{r} S_\mu^k(e^{rx}x^{-\mu-\frac{1}{2}}f_k), \tau_b\psi_a \right\rangle \\
&= \left\langle \sum_{k=0}^{r} e^{rx}x^{-\mu-\frac{1}{2}}f_k, \tau_b(S_\mu^k\psi_a) \right\rangle \\
&= \sum_{k=0}^{r}\int_0^\infty e^{rx}x^{-\mu-\frac{1}{2}}f_k(x)\tau_b(S_\mu^k\psi_a)(x)dx.
\end{aligned}$$

From Lemma 6.2.4 and [16], we have

$$\begin{aligned}
\left(B_\psi f\right)(b,a) &= \sum_{k=0}^{r}\int_0^\infty e^{rx}x^{-\mu-\frac{1}{2}}f_k(x)b^{\mu+\frac{1}{2}} \\
&\qquad h_\mu\left[(bt)^{-\mu}J_\mu(bt)\,h_\mu(S_\mu^k\psi)(at)\right](x)dx \\
&= \sum_{k=0}^{r}\int_0^\infty e^{rx}x^{-\mu-\frac{1}{2}}f_k(x)b^{\mu+\frac{1}{2}} \\
&\qquad h_\mu\left[(bt)^{-\mu}J_\mu(bt)(at)^{2k}\,(h_\mu\psi)(at)\right](x)dx, \\
b^{-\mu-\frac{1}{2}}\left(B_\psi f\right)(b,a) &= \sum_{k=0}^{r}a^{2k}\int_0^\infty e^{rx}x^{-\mu-\frac{1}{2}}f_k(x) \\
&\qquad h_\mu\left[(bt)^{-\mu}J_\mu(bt)t^{2k}\,(h_\mu\psi)(at)\right](x)dx, \\
\left(b^{-1}\frac{d}{db}\right)^n b^{-\mu-\frac{1}{2}}\left(B_\psi f\right)(b,a) &= \sum_{k=0}^{r}a^{2k}\int_0^\infty e^{rx}x^{-\mu-\frac{1}{2}}f_k(x) \\
&\qquad h_\mu\left[(bt)^{-\mu-n}J_{\mu+n}(bt)t^{2(k+n)}\,(h_\mu\psi)(at)\right](x)dx.
\end{aligned}$$

Now, we can estimate the following expression

$$\begin{aligned}
&\left|\left(b^{-1}\frac{d}{db}\right)^n b^{-\mu-\frac{1}{2}}\left(B_\psi f\right)(b,a)\right| \\
&\leq \sum_{k=0}^{r}a^{2k}\int_0^\infty \left|e^{rx}x^{-\mu-\frac{1}{2}}f_k(x)h_\mu\left[(bt)^{-\mu-n}J_{\mu+n}(bt)(h_\mu\psi)(at)t^{2(k+n)}\right](x)\right|dx
\end{aligned}$$

$$\leq \sup_{x\in I}\left|e^{(r+1)x}x^{-\mu-\frac{1}{2}}h_\mu\left[(bt)^{-\mu-n}J_{\mu+n}(bt)t^{2(k+n)}(h_\mu\psi)(at)\right](x)\right|$$
$$\sum_{k=0}^{r}a^{2k}\int_0^\infty\left|e^{-x}f_k(x)\right|dx$$
$$\leq \sup_{x\in I}\left|e^{(r+1)x}x^{-\mu-\frac{1}{2}}h_\mu\left[(bt)^{-\mu-n}J_{\mu+n}(bt)t^{2(k+n)}(h_\mu\psi)(at)\right](x)\right|$$
$$ra^{2r}\int_0^\infty\left|e^{-x}f_k(x)\right|dx.$$

This implies that

$$\left|\left(b^{-1}\frac{d}{db}\right)^n b^{-\mu-\frac{1}{2}}\left(B_\psi f\right)(b,a)\right|$$
$$\leq C^{'}a^{2r}\sup_{x\in I}\left|e^{(r+1)x}x^{-\mu-\frac{1}{2}}h_\mu\left[(bt)^{-\mu-n}J_{\mu+n}(bt)t^{2(k+n)}(h_\mu\psi)(at)\right](x)\right|.$$

Since $\psi\in\chi_\mu(I)$, then from [2, Theorem 2.1] $(h_\mu\psi)(at)\in\mathcal{Q}_\mu(I)$. From [2, p.40] $(bt)^{-\mu-n}J_{\mu+n}(bt)\in\theta_{\mathcal{Q}_\mu}$. Therefore, from the definition of multipliers $t^{2(n+k)}(bt)^{-\mu-n}J_{\mu+n}(bt)(h_\mu\psi)(at)\in\mathcal{Q}_\mu(I)$ and there exist $l,m\in\mathbb{N}$ such that

$$\left|(b^{-1}\frac{d}{db})^n(b^{-\mu-1/2}\left(f\#\overline{\psi_a}\right)(b))\right|\leq C^{'}a^{2r}\omega^\mu_{l,m}\left(t^{2(n+k)}(bt)^{-\mu-n}J_{\mu+n}(bt)(h_\mu\psi)(at)\right).$$

Using (6.1.4), we have

$$\left|(b^{-1}\frac{d}{db})^n(b^{-\mu-1/2}\left(f\#\overline{\psi_a}\right)(b))\right|\leq C^{'}a^{2r}\omega^\mu_{l,m+k+n}\left(h_\mu\psi(at)\right)e^{lb}$$
$$\leq C^{'}C_{\mu,m,k,n}\,a^{2r}e^{lb},$$
$$\left|(b^{-1}\frac{d}{db})^n(1+a^{2r})^{-1}b^{-\mu-1/2}(B_\psi f)(b,a)\right|\leq D^{'}e^{lb}.$$

This shows that $(1+a^{2r})^{-1}b^{-\mu-1/2}(B_\psi f)(b,a)\in\theta_{\chi_\mu}$. □

6.3 Applications

In this section, motivated from [7, p.214], we introduce the Fredholm integral equation associated with Hankel convolution on $\chi_\mu(I)$ space.
The Fredholm integral equation is defined by

$$\int_0^\infty f(t)g(x,t)dt + \lambda f(x) = u(x), \tag{6.3.1}$$

where $g(x)$ and $u(x)$ are given functions and λ is a known parameter.
From (1.3.4), we can write (6.3.1) as,

$$(f\#g)(x) + \lambda f(x) = u(x). \tag{6.3.2}$$

Theorem 6.3.1. *Let $f \in L^1(0,\infty)$ and $g \in L^1(0,\infty)$. Then solution of* (6.3.2) *is*

$$f(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) \frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu g)(\xi) + \lambda)} d\xi. \tag{6.3.3}$$

Proof. Hankel transform of (6.3.2) is

$$h_\mu[(f\#g)](\xi) + \lambda(h_\mu f)(\xi) = (h_\mu u)(\xi).$$

Using (1.3.10), we get

$$\begin{aligned}
\xi^{-\mu-\frac{1}{2}}(h_\mu f)(\xi)(h_\mu g)(\xi) + \lambda(h_\mu f)(\xi) &= (h_\mu u)(\xi) \\
(h_\mu f)(\xi)\left[\xi^{-\mu-\frac{1}{2}}(h_\mu g)(\xi) + \lambda\right] &= (h_\mu u)(\xi) \\
(h_\mu f)(\xi) &= \frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu g)(\xi) + \lambda)}.
\end{aligned} \tag{6.3.4}$$

From the inversion formula (1.0.3), we can find the solution of (6.3.2)

$$f(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) \frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu g)(\xi) + \lambda)} d\xi.$$

□

Theorem 6.3.2. *Let $f, g \in \chi_\mu(I)$. Then*

$$(f\#g)(x) + \lambda f(x) \in \chi_\mu(I). \tag{6.3.5}$$

Proof. From [2, Proposition 3.2], the Hankel convolution is a continuous linear mapping from $\chi_\mu \times \chi_\mu$ into χ_μ. This implies that $(f\#g) \in \chi_\mu(I)$.
Since $f \in \chi_\mu(I)$, therefore

$$(f\#g)(x) + \lambda f(x) \in \chi_\mu(I).$$

□

The above theorem can be justified with the following example.

Example 6.3.1. *We take $u(x) = x^{\mu+\frac{1}{2}}e^{-ax^2}$ with $Re\, a > 0, Re\, \mu > -1$ and $f = g$ in (6.3.1). Then from (6.3.4),we have*

$$\begin{aligned}[(h_\mu f)(\xi)]^2 &= \xi^{\mu+\frac{1}{2}}\ h_\mu \left(x^{\mu+\frac{1}{2}}e^{-ax^2}\right)(\xi) \\ &= \frac{\xi^{2\mu+1}}{(2a)^{\mu+1}}e^{-\xi^2/4a}.\end{aligned}$$

From [9, p.29], we have

$$f(x) = 2^{\frac{\mu+1}{2}}x^{\mu+1}e^{-2ax^2}.$$

Thus, $x^{\mu+1}e^{-2ax^2} \in \chi_\mu(I)$ and the solution $f(x) = 2^{\frac{\mu+1}{2}}x^{\mu+1}e^{-2ax^2} \in \chi_\mu(I)$.

Theorem 6.3.3. *Let $\psi \in \chi_\mu \subset L^1(0,\infty)$ be a Bessel wavelet. Then*

$$f(b) = \int_0^\infty J_\mu(b\xi)(b\xi)^{1/2}\frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu\psi_a)(\xi) + \lambda)}d\xi. \tag{6.3.6}$$

Proof. Putting $g = \psi_a(b)$ in (6.3.2), we have

$$\begin{aligned}(f\#\psi_a)(b) + \lambda f(b) &= u(b) \\ (B_\psi f)(b,a) + \lambda f(b) &= u(b). \end{aligned} \tag{6.3.7}$$

Taking the Hankel transform to the both sides of (6.3.7), we get

$$\begin{aligned}\xi^{-\mu-\frac{1}{2}}(h_\mu f)(\xi)(h_\mu \psi_a)(\xi)+\lambda(h_\mu f)(\xi) &= (h_\mu u)(\xi)\\ (h_\mu f)(\xi)\left[\xi^{-\mu-\frac{1}{2}}(h_\mu\psi_a)(\xi)+\lambda\right] &= (h_\mu u)(\xi)\\ (h_\mu f)(\xi) &= \frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu\psi_a)(\xi)+\lambda)}. \qquad (6.3.8)\end{aligned}$$

With the help of inversion formula of Hankel transform, we have

$$f(b)=\int_0^\infty J_\mu(b\xi)(b\xi)^{1/2}\frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu\psi_a)(\xi)+\lambda)}d\xi.$$

□

Example 6.3.2. *Solve the integral equation*

$$\int_0^\infty f(t)\psi\left(\frac{t}{a},\frac{b}{a}\right)dt = u(b),$$

where $\psi(x)=2^\nu\Gamma(\nu+1)x^{-\nu-\frac{1}{2}}J_{\mu+\nu+1}(x)$.

Solution. From (6.3.8) and [9, p.26], we get

$$\begin{aligned}(h_\mu f)(\xi) &= (h_\mu u)(\xi)\frac{1}{\xi^{-\mu-\frac{1}{2}}h_\mu(2^\nu\Gamma(\nu+1)x^{-\nu-\frac{1}{2}}J_{\mu+\nu+1}(x))(a\xi)}\\ &= (h_\mu u)(\xi)\frac{1}{[1-(a\xi)^2]^\nu},\quad Re\,\nu>-1, Re\,\mu>-1.\end{aligned}$$

Taking $\nu=(\mu+\dfrac{1}{2})$

$$(h_\mu f)(\xi)=(h_\mu u)(\xi)[1-(a\xi)^2]^{-\mu-\frac{1}{2}}. \qquad (6.3.9)$$

Since [9, p.32], we have

$$h_\mu\left(\pi^{-1/2}2^{-\mu}\Gamma(1/2-\mu)x^{\mu-\frac{1}{2}}sinx\right)(a\xi)=\xi^{\mu+\frac{1}{2}}[1-(a\xi)^2]^{-\mu-\frac{1}{2}}, \qquad (6.3.10)$$

where $-1 < Re\,\mu < \frac{1}{2}$.

Using (6.3.10) in (6.3.9), we get

$$\begin{aligned}(h_\mu f)(\xi) &= \xi^{-\mu-\frac{1}{2}}(h_\mu u)(\xi)\; h_\mu\left(\pi^{-1/2}2^{-\mu}\Gamma(1/2-\mu)x^{\mu-\frac{1}{2}}sinx\right)(a\xi)\\ &= h_\mu\left(u\,\#\,\pi^{-1/2}2^{-\mu}\Gamma(1/2-\mu)\left(\frac{x}{a}\right)^{\mu-\frac{1}{2}}sin\frac{x}{a}\right)(\xi)\\ f(x) &= u\,\#\,\pi^{-1/2}2^{-\mu}\Gamma(1/2-\mu)\left(\frac{x}{a}\right)^{\mu-\frac{1}{2}}sin\frac{x}{a}.\end{aligned}$$