Chapter 5

Linear Time Invariant Filter associated with Bessel Wavelet Transform

5.1 Introduction

Fourier transform plays an important role to design the various type of filters. Filter can be represented in the form of " black box " that takes an input signal, processes it, and then returns an output signal that in some way modifies the input signal. In signal processing, filter is used to remove the unwanted frequency components from the signal to enhance the wanted ones.

The concept of linear time invariant filter is given in [3] and it is shown that the output signal from a linear time invariant filter of a sinusoidal input signal is also sinusoidal with the same frequency. Using the aforesaid result, linear time invariant filter is expressed in terms of convolution by exploiting the Fourier transform technique.

Now, our main aim in this chapter is to introduce the theory of linear time invariant filter associated with Hankel transform and Hankel convolution. Using aforesaid transform theory, linear time invariant filter can be expressed in terms of the Bessel wavelet transform [21].

From [3, p.180], we introduce the following concepts of filter which are useful for further discussion and investigation.

A filter L is said to be linear if it satisfy the following properties
(i)Additivity: $L[f+g] = L[f] + L[g]$.
(ii)Homogeneity: $L[cf] = cL[f]$, where c is a constant.

A filter L is said to be time invariant if for any signal f and any non-negative real number a

$$L[f_a](t) = (Lf)(t,a), \tag{5.1.1}$$

where $f_a(t) = f(t,a) = \int_0^\infty f(z)D(t,a,z)d\sigma(z)$.

Now, $(Lf)(t,a)$ is defined by

$$(Lf)(t,a) = \int_0^\infty (Lf)(z)D(t,a,z)d\sigma(z). \tag{5.1.2}$$

For the linear time invariant filter L, we can write the following representation

$$L\left[\int_0^\infty f(z)D(t,a,z)d\sigma(z)\right] = \int_0^\infty (Lf)(z)D(t,a,z)d\sigma(z). \tag{5.1.3}$$

5.2 Linear Time Invariant Filter

In this section, we are studying the properties of linear time invariant filter involving Hankel transform and Hankel convolution.

Lemma 5.2.1. *Let L be a linear time invariant transformation and λ be any fixed non-negative real number, then there exists a function h, such that*

$$L[j_\mu(\lambda t)] = (h_\mu h)(\lambda) j_\mu(\lambda t). \tag{5.2.1}$$

Proof. Let $h^\lambda(t) = L[j_\mu(\lambda t)]$. Since L is time invariant, then we have

$$L[j_\mu(\lambda t) j_\mu(\lambda a)] = h^\lambda(t, a), \tag{5.2.2}$$

for each non-negative real number a.

L is linear, then from the properties of homogeneity, we get the following expression

$$\begin{aligned} L[j_\mu(\lambda a) j_\mu(\lambda t)] &= j_\mu(\lambda a) L[j_\mu(\lambda t)] \\ &= j_\mu(\lambda a) h^\lambda(t). \end{aligned} \tag{5.2.3}$$

From (5.2.2) and (5.2.3), we get

$$h^\lambda(t, a) = j_\mu(\lambda a) h^\lambda(t). \tag{5.2.4}$$

Setting $t = 0$ in (5.2.4), we get

$$h^\lambda(0, a) = j_\mu(\lambda a) h^\lambda(0).$$

This implies

$$h^\lambda(a) = j_\mu(\lambda a) h^\lambda(0).$$

If we set $a = t$, then

$$h^\lambda(t) = j_\mu(\lambda t) h^\lambda(0).$$

Letting $h^\lambda(0) = (h_\mu h)(\lambda)$, then we have

$$L[j_\mu(\lambda t)] = h^\lambda(t) = (h_\mu h)(\lambda) j_\mu(\lambda t).$$ □

Theorem 5.2.2. *Let L be a linear time invariant transformation on the space of signals that are piecewise continuous function. Then there exits an integrable function h, such that*

$$L(f) = f\#h, \tag{5.2.5}$$

for all signals f.

Proof. With the help the Hankel inversion formula (1.1.4), we have

$$\begin{aligned} f(t) &= h_\mu^{-1}[h_\mu f](t) \\ &= \int_0^\infty (h_\mu f)(\lambda) j_\mu(\lambda t) d\sigma(\lambda). \end{aligned}$$

Now,

$$\begin{aligned} (Lf)(t) &= L\left[\int_0^\infty (h_\mu f)(\lambda) j_\mu(\lambda t) d\sigma(\lambda)\right] \\ &\approx L\left[\sum_j (h_\mu f)(\lambda_j) j_\mu(\lambda_j t) \triangle\lambda\right] \\ &= \sum_j (h_\mu f)(\lambda_j) L(j_\mu(\lambda_j t)) \triangle\lambda. \end{aligned}$$

The Riemann sum on right hand side of the above expression can be converted into an integral

$$(Lf)(t) = \int_0^\infty (h_\mu f)(\lambda) L(j_\mu(\lambda t)) d\sigma(\lambda).$$

Using Lemma 5.2.1, we have

$$(Lf)(t) = \int_0^\infty (h_\mu f)(\lambda)(h_\mu h)(\lambda) j_\mu(\lambda t) d\sigma(\lambda).$$

From (1.1.10), we get

$$(Lf)(t) = \int_0^\infty h_\mu (f\#h)(\lambda) j_\mu(\lambda t) d\sigma(\lambda).$$

Using (1.1.4), we write

$$\begin{aligned}(Lf)(t) &= h_\mu^{-1}\left[h_\mu(f\#h)\right](t)\\ &= (f\#h)(t).\end{aligned}$$

Therefore,

$$L(f) = f\#h.$$

□

Now we are giving an example in the support of Theorem 5.2.2.

Example 5.2.1. *Let l(t) be a function that has finite support. For a signal f, let*

$$\begin{aligned}(Lf)(t) &= (l\#f)(t)\\ &= \int_0^\infty l(t,x)f(x)d\sigma(x).\end{aligned} \tag{5.2.6}$$

This linear operator is time invariant because for any $a \geq 0$,

$$(Lf)(t,a) = \int_0^\infty (Lf)(z)D(t,a,z)d\sigma(z).$$

From (5.2.6), we get

$$\begin{aligned}(Lf)(t,a) &= \int_0^\infty \left(\int_0^\infty l(z,x)f(x)d\sigma(x)\right) D(t,a,z)d\sigma(z)\\ &= \int_0^\infty \left(\int_0^\infty \left\{\int_0^\infty l(y)D(z,x,y)d\sigma(y)\right\} f(x)d\sigma(x)\right) D(t,a,z)d\sigma(z)\\ &= \int_0^\infty \left(\int_0^\infty \left\{\int_0^\infty f(x)D(z,x,y)d\sigma(x)\right\} l(y)d\sigma(y)\right) D(t,a,z)d\sigma(z)\\ &= \int_0^\infty \left(\int_0^\infty f(z,y)l(y)d\sigma(y)\right) D(t,a,z)d\sigma(z)\\ &= \int_0^\infty \left(\int_0^\infty f(z,y)D(t,a,z)d\sigma(z)\right) l(y)d\sigma(y)\\ &= \int_0^\infty f(t,a,y)l(y)d\sigma(y)\end{aligned}$$

$$= \int_0^\infty f_a(t,y)l(y)d\sigma(y)$$
$$= (Lf_a)(t).$$

Physical Interpretation : Both $h(t)$ and $(h_\mu h)(\lambda)$ have physical interpretations. Assuming that $h(t)$ is continuous and integrable function on $I = (0,\infty)$ and δ is small positive real number.

Now, we define impulse signal

$$f_\delta(t) = \begin{cases} \frac{2^{\mu-1/2}\,\Gamma(\mu+1/2)(2\mu+1)}{\delta^{2\mu+1}} & \text{t} \leq \delta,\ \mu > -1/2, \\ 0, & \text{otherwise.} \end{cases}$$

If δ is small, then f_δ represents a strong signal but only lasts a short period of time (such as sound signal generated by hammer blow). Now, we can easily find $\int_0^\delta f_\delta(t)d\sigma(t) = 1$.

Applying L to f_δ and using (5.2.6), we obtain

$$\begin{aligned}(Lf_\delta)(t) &= \int_0^\infty f_\delta(\tau)h(t,\tau)d\sigma(\tau) \\ &= \int_0^\delta f_\delta(\tau)h(t,\tau)d\sigma(\tau).\end{aligned}$$

Since h is continuous, $h(t,\tau)$ is approximately equal to $h(t)$ for $\tau \leq \delta$.

Therefore,

$$(Lf_\delta)(t) \approx h(t)\int_0^\delta f_\delta(\tau)d\sigma(\tau) = h(t).$$

Thus, $h(t)$ is the approximate response obtained by applying L to an input signal that is an impulse on half plane. For that reason $h(t)$ is called impulse response function.

Since from Lemma 5.2.1, $(h_\mu h)(\lambda)$ is the amplitude of the response to a “pure frequency” signal $j_\mu(\lambda t)$ so $(h_\mu h)(\lambda)$ is called the system function.

Theorem 5.2.3. *Let $f \in L^2_\sigma(I)$ and $\psi_a \in L^2_\sigma(I)$. Then Bessel wavelet transform can be expressed as*

$$(B_\psi f)(t,a) = (Lf)(t), \tag{5.2.7}$$

where L is a time invariant linear filter.

Proof. From (1.2.5), we have

$$\begin{aligned}(B_\psi f)(t,a) &= (f\#\psi_a)(t)\\ &= h_\mu^{-1}\left[(h_\mu f)(\lambda)(h_\mu \psi_a)(\lambda)\right](t)\\ &= \int_0^\infty j_\mu(\lambda t)(h_\mu f)(\lambda)(h_\mu\psi_a)(\lambda)d\sigma(\lambda)\\ &= \int_0^\infty \left[j_\mu(\lambda t)(h_\mu\psi_a)(\lambda)\right](h_\mu f)(\lambda)d\sigma(\lambda).\end{aligned}$$

Using Lemma 5.2.1, we see

$$\begin{aligned}(B_\psi f)(t,a) &= \int_0^\infty L\left[j_\mu(\lambda t)\right](h_\mu f)(\lambda)d\sigma(\lambda)\\ &= \sum_j L\left[j_\mu(\lambda_j t)\right](h_\mu f)(\lambda_j)\triangle(\lambda)\\ &= L\sum_j \left[j_\mu(\lambda_j t)\right](h_\mu f)(\lambda_j)\triangle(\lambda)\\ &= L\left(\int_0^\infty j_\mu(\lambda t)(h_\mu f)(\lambda)d\sigma(\lambda)\right).\end{aligned}$$

From (1.1.4), we have

$$(B_\psi f)(t,a) = (Lf)(t).$$

□