Chapter 4

Integrability of the Continuum Bessel Wavelet Kernel

4.1 Introduction

The kernel of any integral transform is important because it plays a decisive role to determine the nature of the solution of differential equation. The continuum wavelet was studied by Pinsky [28], by using the Fourier transform analysis. With the help of the aforesaid transform integrability conditions of the continuum wavelet kernel is investigated by the same author in [29] and many useful results are obtained.

The Bessel wavelet kernel is an important tool because it is a generalization of the Bessel function of first kind which is used as a kernel of the Hankel transform by Hirschman Jr. [12], Haimo [11] and others.

In the present chapter, our main objective is to explore the integrability conditions of the continuum Bessel wavelet kernel by using the theory of Hankel transform and Hankel convolution.

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4.2 Properties of the Continuous Bessel Wavelet Transform

In this section, we discuss some properties of the continuous Bessel wavelet transform which are helpful to give the sufficient conditions for the integrability of the Bessel wavelet kernel.

Definition 4.2.1. A function $\psi \in L^2_\sigma(I)$ is a normalized continuum Bessel wavelet if $\|\psi\|_{2,\sigma} = 1$ and its Hankel transform satisfies the admissibility condition

$$A_\psi := \int_0^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+1}} d\sigma(\omega) < \infty. \tag{4.2.1}$$

The admissibility condition (4.2.1) requires that $(h_\mu\psi)(0) = 0$ for existence of the integral. If $h_\mu\psi$ is continuous, then

$$(h_\mu\psi)(t) = \int_0^\infty j_\mu(xt)\psi(t)d\sigma(t) \tag{4.2.2}$$

implies that $(h_\mu\psi)(0) = 0 = \int_0^\infty \psi(t)d\sigma(t)$.
By rescaling the spatial coordinate, we may assume that both $\|\psi\|_{2,\sigma} = 1$ and A_ψ=1. Then, for any $\psi \in L^2_\sigma(I)$ with $A_\psi < \infty$, we can define

$$(h_\mu\psi_{new})(\xi) = \frac{1}{A_\psi^{\mu+1/2}}\ h_\mu\psi\left(\frac{\xi||\psi||_{2,\sigma}^{2/2\mu+1}}{A_\psi}\right). \tag{4.2.3}$$

It is clear that $\|h_\mu\psi_{new}\|_{2,\sigma} = \|\psi_{new}\|_{2,\sigma} = 1$ and $(A_\psi)^{2\mu} A_{\psi_{new}} = 1$. Using assumption $A_\psi = 1$, we have $A_{\psi_{new}} = 1$. This shows that (4.2.3) is a renormalized continuum Bessel wavelet.
We define L^2 norm

$$N(a) = \left(\int_0^\infty |(B_\psi f)(b,a)|^2 d\sigma(b)\right)^{1/2} = \|(B_\psi f)(a,\cdot)\|_{2,\sigma}. \tag{4.2.4}$$

Lemma 4.2.2. *Let $\psi \in L^2_\sigma(I)$ be a renormalized continuum Bessel wavelet and $f \in L^2_\sigma(I)$. Then*

(i)

$$|(B_\psi f)(b,a)| \leq a^{\frac{-2\mu-1}{2}} \|f\|_{2,\sigma} . \tag{4.2.5}$$

(ii) For $a > 0, b \to (B_\psi f)(b,a)$ is in $L^2_\sigma(I)$ and the norm $N(a)$ holds the following equality

$$\begin{aligned} \int_0^\infty \frac{[N(a)]^2}{a^{2\mu+1}} d\sigma(a) &= \int_0^\infty \left(\int_0^\infty |(B_\psi f)(b,a)|^2 \, d\sigma(b) \right) \frac{d\sigma(a)}{a^{2\mu+1}} \\ &= \|f\|^2_{2,\sigma} . \end{aligned} \tag{4.2.6}$$

Proof. From (1.2.5), we have

$$\begin{aligned} |(B_\psi f)(b,a)| &= |(f\#\psi_a)(b)| \\ &\leq \|f\#\psi_a\|_{\infty,\sigma} . \end{aligned}$$

From (1.1.12), we have

$$\begin{aligned} |(B_\psi f)(b,a)| &\leq \|f\|_{2,\sigma} \|\psi_a\|_{2,\sigma} \\ &= \|f\|_{2,\sigma} a^{\frac{-2\mu-1}{2}} \|\psi\|_{2,\sigma} \\ &= a^{\frac{-2\mu-1}{2}} \|f\|_{2,\sigma} , \end{aligned}$$

as $\|\psi\|_{2,\sigma} = 1$.

To prove (ii), we take $f \in L^1_\sigma(I) \cap L^2_\sigma(I)$. Then $(h_\mu f)(\xi) \in L^\infty_\sigma(I)$.

$$\begin{aligned} [N(a)]^2 &= \int_0^\infty |(B_\psi f)(b,a)|^2 \, d\sigma(b) \\ &= \int_0^\infty |(f\#\psi_a)(b)|^2 \, d\sigma(b). \end{aligned}$$

From Parseval's relation of the Hankel transform (1.1.14), we have further

$$
\begin{aligned}
[N(a)]^2 &= \int_0^\infty |h_\mu(f\#\psi_a)(\xi)|^2 \, d\sigma(\xi) \\
&= \int_0^\infty |(h_\mu f)(\xi)|^2 |(h_\mu \psi_a)(\xi)|^2 d\sigma(\xi) \\
&= \|h_\mu f\|_{\infty,\sigma} \int_0^\infty |(h_\mu \psi_a)(\xi)|^2 d\sigma(\xi) \\
&= \|h_\mu f\|_{\infty,\sigma} \, \|h_\mu \psi\|_{2,\sigma} \\
&< \infty.
\end{aligned}
$$

In particular, $b \to B_\psi f(b,a) \in L^2_\sigma(I)$ for every $a > 0$.

Now,

$$
\int_0^\infty [N(a)]^2 \frac{d\sigma(a)}{a^{2\mu+1}} = \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \left(\int_0^\infty |(h_\mu f)(\xi)|^2 |(h_\mu \psi_a)(\xi)|^2 d\sigma(\xi) \right).
$$

Using Fubini's theorem, we have

$$
\begin{aligned}
\int_0^\infty [N(a)]^2 \frac{d\sigma(a)}{a^{2\mu+1}} &= \int_0^\infty |(h_\mu f)(\xi)|^2 \left(\int_0^\infty \frac{|(h_\mu \psi)(a\xi)|^2}{a^{2\mu+1}} d\sigma(a) \right) d\sigma(\xi) \\
&= A_\psi \int_0^\infty |(h_\mu f)(\xi)|^2 \, d\sigma(\xi).
\end{aligned}
$$

Exploiting Parseval's relation (1.1.14), we get

$$
\int_0^\infty [N(a)]^2 \frac{d\sigma(a)}{a^{2\mu+1}} = \|f\|^2_{2,\sigma}, \qquad as \ A_\psi = 1.
$$

This proves (4.2.6), for $f \in L^1_\sigma(I) \cap L^2_\sigma(I)$. In particular, $a \to \|B_\psi f\|^2_{2,\sigma}$ is finite everywhere.

Now, if $f \in L^2_\sigma(I)$ and $f_n \in L^1_\sigma(I) \cap L^2_\sigma(I)$ with $\|f - f_n\|_{2,\sigma} \to 0$, then from (4.2.5), we have the pointwise bound

$$
|B_\psi f(b,a) - B_\psi f_n(b,a)| \leq \|f - f_n\|_{2,\sigma} \to 0. \tag{4.2.7}
$$

This shows that $B_\psi f_n$ converges uniformly to $B_\psi f$.

On the other hand, applying (4.2.6) to $f_m - f_n$, we see that $B_\psi f_n$ is a Cauchy sequence in the Hilbert space $L^2\left(I^2, \frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}\right)$.
Hence, there exists a limit F in this space for which

$$\begin{aligned}\int_0^\infty\int_0^\infty |F(b,a)|^2 \frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} &= \lim_{n\to\infty}\int_0^\infty\int_0^\infty |B_\psi f_n(b,a)|^2 \frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} \\ &= \lim_{n\to\infty}\|f_n\|_{2,\sigma}^2 \\ &= \|f\|_{2,\sigma}^2 .\end{aligned}$$

Now, take a subsequence that converges a.e. Along this subsequence we also have the uniform convergence to $B_\psi f$. Hence we conclude that $F = B_\psi f$ a.e. Taking $n \to \infty$, we get (4.2.6). Thus, (4.2.6) holds for $f \in L^2_\sigma(I)$.
In particular, $\int_0^\infty |(B_\psi f)(b,a)|^2\, d\sigma(b) < \infty$ for almost all $a > 0$, and hence the proof is complete. □

4.3 Kernel of the Inverse Transform

In this section, motivated from the results of Pinsky [29], we introduce the partial inverse transform associated with the Bessel wavelet transform and study their properties.

Definition 4.3.1. The partial inverse transform is defined as

$$S_\epsilon f(x) = \int_{a>\epsilon}\left(\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(x)d\sigma(b)\right)\frac{d\sigma(a)}{a^{2\mu+1}}, \tag{4.3.1}$$

for $\epsilon > 0$.

Theorem 4.3.2. *The partial inverse transform (4.3.1) can be expressed as*

$$S_\epsilon f(x) = \int_{a>\epsilon}(B_\psi f \# \psi_a)(x)\frac{d\sigma(a)}{a^{2\mu+1}}. \tag{4.3.2}$$

Proof. We write

$$\begin{aligned}
&\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(x)d\sigma(b) \\
&= \int_0^\infty (B_\psi f)(b,a)\left\{a^{-2\mu-1}\int_0^\infty \psi(z)D\left(\frac{b}{a},\frac{x}{a},z\right)d\sigma(z)\right\}d\sigma(b) \\
&= a^{-2\mu-1}\int_0^\infty (B_\psi f)(b,a)\left\{\int_0^\infty \psi(z)\right. \\
&\qquad \left.\left(\int_0^\infty j_\mu\left(\frac{b\xi}{a}\right)j_\mu\left(\frac{x\xi}{a}\right)j_\mu(z\xi)d\sigma(\xi)\right)d\sigma(z)\right\}d\sigma(b).
\end{aligned}$$

By applying Fubini's theorem, we have

$$\begin{aligned}
&\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(x)d\sigma(b) \\
&= a^{-2\mu-1}\int_0^\infty (B_\psi f)(b,a)\left\{\int_0^\infty\left(\int_0^\infty \psi(z)j_\mu(z\xi)d\sigma(z)\right)\right. \\
&\qquad \left. j_\mu\left(\frac{b\xi}{a}\right)j_\mu\left(\frac{x\xi}{a}\right)d\sigma(\xi)\right\}d\sigma(b) \\
&= a^{-2\mu-1}\int_0^\infty (B_\psi f)(b,a)\left\{\int_0^\infty (h_\mu\psi)(\xi)j_\mu\left(\frac{b\xi}{a}\right)j_\mu\left(\frac{x\xi}{a}\right)d\sigma(\xi)\right\}d\sigma(b) \\
&= a^{-2\mu-1}\left(\int_0^\infty (B_\psi f)(b,a)j_\mu\left(\frac{b\xi}{a}\right)d\sigma(b)\right)\left(\int_0^\infty (h_\mu\psi)(\xi)j_\mu\left(\frac{x\xi}{a}\right)d\sigma(\xi)\right) \\
&= a^{-2\mu-1}\int_0^\infty h_\mu[(B_\psi f)(b,a)]\left(\frac{\xi}{a}\right)(h_\mu\psi)(\xi)j_\mu\left(\frac{x\xi}{a}\right)d\sigma(\xi).
\end{aligned}$$

Putting $\frac{\xi}{a}=\omega$, we get

$$\begin{aligned}
\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(x)d\sigma(b) &= \int_0^\infty h_\mu[(B_\psi f)(b,a)\#\psi_a](\omega)j_\mu(x\omega)d\sigma(\omega) \\
&= h_\mu^{-1}\left\{h_\mu[(B_\psi f)(b,a)\#\psi_a]\right\}(x) \\
&= [(B_\psi f)(b,a)\#\psi_a](x).
\end{aligned}$$

Thus, from (4.3.1), we have

$$S_\epsilon f(x) = \int_{a>\epsilon}\left(B_\psi f\#\psi_a\right)(x)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

□

Theorem 4.3.3. *Let $\psi \in L^2_\sigma(I)$ be a renormalized continuum wavelet. Let $\epsilon > 0$, $f \in L^2_\sigma(I)$ and $x \in I$. Then the integral (4.3.1) converges absolutely and has the following pointwise bound*

$$|S_\epsilon f(x)| \leq A_\epsilon \, \|f\|_{2,\sigma}\,, \tag{4.3.3}$$

where $A_\epsilon = \left(\int_{a>\epsilon} \frac{1}{a^{2\mu+2}} da\right)^{1/2}$. Furthermore, $S_\epsilon f \in L^2_\sigma(I)$ and $\|S_\epsilon f - f\|_{2,\sigma} \to 0$ as $\epsilon \to 0$.

Proof. In the integrand of (4.3.2) by using the Cauchy-Schwarz inequality (1.1.16), we have the following pointwise bound

$$\begin{aligned} |B_\psi f \# \psi_{a,0}(x)| &\leq \|(B_\psi f)(a,.)\|_{2,\sigma} \|\psi_{a,0}\|_{2,\sigma} \\ &= N(a) \frac{\|\psi\|_{2,\sigma}}{(a^{2\mu+1})^{1/2}} \\ &= \frac{N(a)}{(a^{2\mu+1})^{1/2}} \qquad as \ \ \|\psi\|_{2,\sigma} = 1. \end{aligned}$$

Thus, we have estimate (4.3.2) by the following way

$$\begin{aligned} |S_\epsilon f(x)| &\leq \int_{a>\epsilon} \frac{N(a)}{(a^{2\mu+1})^{1/2}} \frac{d\sigma(a)}{a^{2\mu+1}} \\ &\leq \left(\int_{a>\epsilon} \frac{|N(a)|^2}{a^{2\mu+1}} d\sigma(a)\right)^{1/2} \left(\int_{a>\epsilon} \frac{1}{(a^{2\mu+1})^2} d\sigma(a)\right)^{1/2}. \end{aligned}$$

From (4.2.6), we have

$$\begin{aligned} |S_\epsilon f(x)| &\leq \|f\|_{2,\sigma} \left(\int_{a>\epsilon} \frac{1}{(a^{2\mu+1})^2} d\sigma(a)\right)^{1/2} \\ &\leq \|f\|_{2,\sigma} \left(\int_{a>\epsilon} \frac{1}{a^{2\mu+2}} da\right)^{1/2} \\ &= \|f\|_{2,\sigma} A_\epsilon, \end{aligned} \tag{4.3.4}$$

where $A_\epsilon = \left(\int_{a>\epsilon} \frac{1}{a^{2\mu+2}} da\right)^{1/2}$ and for each $\epsilon > 0$, A_ϵ is convergent for $\mu > -1/2$. To prove that $S_\epsilon f \in L^2_\sigma(I)$, we take $f \in L^1_\sigma(I) \cap L^2_\sigma(I)$. Then from the properties of

Hankel convolution [26], we have

$$\begin{aligned}\|B_\psi f\#\psi_a\|_{2,\sigma} &\leq \|B_\psi f\|_{1,\sigma}\,\|\psi_a\|_{2,\sigma}\\ &\leq \frac{\|B_\psi f\|_{1,\sigma}}{(a^{2\mu+1})^{1/2}} \qquad \Big(\because \|\psi\|_{2,\sigma}=1\Big)\\ &= \frac{\|f\#\psi_a\|_{1,\sigma}}{(a^{2\mu+1})^{1/2}}.\end{aligned}$$

Using the argument of [19, p. 1732], we have

$$\|B_\psi f\#\psi_a\|_{2,\sigma} \leq \frac{\|f\|_{1,\sigma}}{a^{2\mu+1}}. \tag{4.3.5}$$

Applying the generalized Minkowski inequality to (4.3.1), we get

$$\begin{aligned}\|S_\epsilon f\|_{2,\sigma} &\leq \|f\|_{1,\sigma}\int_{a>\epsilon}\frac{d\sigma(a)}{a^{4\mu+2}}\\ &\leq \|f\|_{1,\sigma}\int_{a>\epsilon}\frac{da}{a^{2\mu+2}}\\ &= \|f\|_{1,\sigma}\;A_\epsilon^2\\ &< \infty.\end{aligned}$$

Multiplying (4.3.1) by $g \in L^2_\sigma(I)$, we obtain

$$\begin{aligned}\langle S_\epsilon f, g\rangle &= \int_0^\infty S_\epsilon f(x)\overline{g(x)}d\sigma(x)\\ &= \int_0^\infty\left(\int_{a>\epsilon}\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}\right)\overline{g(x)}d\sigma(x)\\ &= \int_{a>\epsilon}\int_0^\infty (B_\psi f)(b,a)\overline{(B_\psi g)}(b,a)\,\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}\end{aligned}$$

and

$$\begin{aligned}\|S_\epsilon f\|_{2,\sigma} &= \sup_{g\neq 0}\frac{|\langle S_\epsilon f, g\rangle|}{\|g\|_{2,\sigma}} \leq \left(\int_{a>\epsilon}\int_0^\infty |B_\psi f(b,a)|^2\,\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}\right)^{1/2}\\ &\leq \|f\|_{2,\sigma} < \infty. \end{aligned}\tag{4.3.6}$$

Thus $S_\epsilon f \in L^2_\sigma(I)$ for $f \in L^1_\sigma(I) \cap L^2_\sigma(I)$. For $f \in L^2_\sigma(I)$, let $f_n \in L^1_\sigma(I) \cap L^2_\sigma(I)$ with $\|f - f_n\|_{2,\sigma} \to 0$. Then from (4.3.6), we find that

$$\|S_\epsilon f_n - S_\epsilon f_m\|_{2,\sigma} \leq \|f_n - f_m\|_{2,\sigma} \to 0. \tag{4.3.7}$$

Hence, (4.3.7) shows that $S_\epsilon f_n$ is a Cauchy sequence in $L^2_\sigma(I)$ and converges in $L^2_\sigma(I)$; in particular a subsequence converges pointwise a.e. Using (4.3.4) we have,

$$|S_\epsilon f(x) - S_\epsilon f_n(x)| \leq A_\epsilon \, \|f - f_n\|_{2,\sigma} \to 0$$

as $n \to \infty$ with fixed $\epsilon > 0$. This indicates that $S_\epsilon f_n(x)$ converges to $S_\epsilon f(x)$ uniformly in $L^2_\sigma(I)$ when $n \to \infty$ with fixed ϵ. Using the above arguments, we have $S_\epsilon f \in L^2_\sigma(I)$ with bounds $\|S_\epsilon f\|_{2,\sigma} \leq \|f\|_{2,\sigma}$.

Finally, to prove the L^2_σ convergence when $\epsilon \to 0$, we use the L^2_σ isometry (4.2.6) to write

$$\begin{aligned}
\langle f, g\rangle &= \int_0^\infty \int_0^\infty (B_\psi f)(b,a)\overline{(B_\psi g)}(b,a)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}, \\
\langle f - S_\epsilon f, g\rangle &= \int_{a<\epsilon} \int_0^\infty (B_\psi f)(b,a)\overline{(B_\psi g)}(b,a)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}, \\
|\langle f - S_\epsilon f, g\rangle| &\leq \left(\int_{a<\epsilon} \int_0^\infty |B_\psi f|^2 \, \frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}\right)^{1/2} \|g\|_{2,\sigma}, \\
\|f - S_\epsilon f\|_{2,\sigma} &= \sup_{g\neq 0} \frac{|\langle f - S_\epsilon f, g\rangle|}{\|g\|_{2,\sigma}} \\
&\leq \left(\int_{a<\epsilon} \int_0^\infty |B_\psi f|^2 \, \frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}\right)^{1/2},
\end{aligned}$$

which tends to zero by the dominated convergence theorem. This completes the proof of the theorem. □

Theorem 4.3.4. *For $\psi \in L^2_\sigma(I)$, if*

$$K_\epsilon(x,y) = \int_0^\infty \frac{1}{z^{2\mu+1}} \left\{\int_0^z (\psi \# \overline{\psi})(\xi) d\sigma(\xi)\right\} D(x,y,z) d\sigma(z), \tag{4.3.8}$$

then $K_\epsilon(x,y)$ is bounded and it represents the kernel of the partial inverse Bessel wavelet transform.

Proof. We can write (4.3.8) as

$$\begin{aligned} |K_\epsilon(x,y)| &\leq \int_0^\infty \frac{1}{z^{2\mu+1}} \left\{ \int_0^z \left|(\psi\#\overline{\psi})(\xi)\right| d\sigma(\xi) \right\} D(x,y,z)d\sigma(z) \\ &\leq \sup_\xi \left|(\psi\#\overline{\psi})(\xi)\right| \int_0^\infty \frac{1}{z^{2\mu+1}} \left\{ \int_0^z d\sigma(\xi) \right\} D(x,y,z)d\sigma(z). \end{aligned}$$

Since $(\psi\#\overline{\psi})(\xi) \in L^\infty_\sigma(I)$, then the above expression becomes

$$\begin{aligned} |K_\epsilon(x,y)| &\leq A \int_0^\infty \frac{1}{z^{2\mu+1}} \left(\int_0^z d\sigma(\xi) \right) D(x,y,z)d\sigma(z) \\ &= A \int_0^\infty \frac{1}{z^{2\mu+1}} \left(\int_0^z \frac{\xi^{2\mu}}{2^{\mu-1/2}\,\Gamma(\mu+1/2)} d\xi \right) D(x,y,z)d\sigma(z) \\ &= A \int_0^\infty \frac{1}{z^{2\mu+1}} \left(\frac{z^{2\mu+1}}{2^{\mu-1/2}\,\Gamma(\mu+1/2)\,(2\mu+1)} \right) D(x,y,z)d\sigma(z) \\ &= \frac{A}{2^{\mu-1/2}\,\Gamma(\mu+1/2)\,(2\mu+1)} \int_0^\infty D(x,y,z)d\sigma(z). \end{aligned}$$

From (1.1.6), we get

$$|K_\epsilon(x,y)| \leq \frac{A}{2^{\mu-1/2}\,\Gamma(\mu+1/2)\,(2\mu+1)}.$$

This implies that $K_\epsilon(x,y)$ is bounded. □

Theorem 4.3.5. *If $\psi \in L^2_\sigma(I)$, then we have*

$$\Psi(\xi) = h_\mu K(\xi) = \int_{a>1} \frac{|(h_\mu\psi)(a\xi)|^2}{a^{2\mu+1}} d\sigma(a). \tag{4.3.9}$$

Further, $S_\epsilon f(x)$ can be expressed in the following form

$$S_\epsilon f(x) = \int_0^\infty K_\epsilon(x,y) f(y) d\sigma(y), \tag{4.3.10}$$

where $K_\epsilon(x,y)$ is given by (4.3.8).

Proof. We have

$$\begin{aligned}\int_{a>1}\frac{|(h_\mu\psi)(a\xi)|^2}{a^{2\mu+1}}d\sigma(a) &= \int_{u>\xi}\frac{|(h_\mu\psi)(u)|^2}{u^{2\mu+1}}d\sigma(u)\\ &= \int_{u>\xi}\frac{(h_\mu\psi)(u)\overline{(h_\mu\psi)}(u)}{u^{2\mu+1}}d\sigma(u)\\ &= \int_{u>\xi}\frac{h_\mu(\psi\#\overline{\psi})(u)}{u^{2\mu+1}}d\sigma(u)\\ &= \int_{u>\xi}\frac{1}{u^{2\mu+1}}\left(\int_0^\infty j_\mu(u\xi)(\psi\#\overline{\psi})(\xi)d\sigma(\xi)\right)d\sigma(u).\end{aligned}$$

Changing the order of integration, the above expression yields

$$\int_{a>1}\frac{|(h_\mu\psi)(a\xi)|^2}{a^{2\mu+1}}d\sigma(a) = \int_0^\infty\left(\frac{1}{u^{2\mu+1}}\int_{u>\xi}(\psi\#\overline{\psi})(\xi)d\sigma(\xi)\right)j_\mu(u\xi)d\sigma(u),$$

where

$$K(u) = \frac{1}{u^{2\mu+1}}\int_0^u(\psi\#\overline{\psi})(\xi)d\sigma(\xi). \tag{4.3.11}$$

Then

$$\begin{aligned}\int_{a>1}\frac{|(h_\mu\psi)(a\xi)|^2}{a^{2\mu+1}}d\sigma(a) &= \int_0^\infty K(u)j_\mu(u\xi)d\sigma(u)\\ &= h_\mu K(\xi).\end{aligned}$$

Further, by the definition of the Hankel translation (1.1.7), we have

$$\begin{aligned}K_\epsilon(x,y) &= \int_0^\infty K(z)D(x,y,z)d\sigma(z)\\ &= \int_0^\infty\frac{1}{z^{2\mu+1}}\left\{\int_0^z(\psi\#\overline{\psi})(\xi)d\sigma(\xi)\right\}D(x,y,z)d\sigma(z).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^\infty K_\epsilon(x,y)f(y)d\sigma(y) &= \int_0^\infty\left\{\int_0^\infty\frac{1}{z^{2\mu+1}}\left(\int_0^z(\psi\#\overline{\psi})(\xi)d\sigma(\xi)\right)\right.\\ &\qquad \left. D\,(x,y,z)\ d\sigma(z)\right\}f(y)d\sigma(y)\\ &= S_\epsilon f(x).\end{aligned}$$

This completes the proof of (4.3.10). □

4.4 Properties of the Continuum Bessel Wavelet Kernel

The present section is devoted to the discussion of various properties of the Bessel wavelet kernel using (4.3.9).

Proposition 4.4.1. *Let $\psi \in L^2_\sigma(I)$ be a renormalized continuum wavelet and $\Psi(\xi) = h_\mu K(\xi)$ which is defined in (4.3.9). Then $\Psi \in L^1_\sigma(I)$, Ψ is continuous with $\Psi(0) = 1$ and $\Psi(\xi) \to 0$ when $\xi \to \infty$.*

Proof. From (4.3.9), we have

$$\begin{aligned}
\int_0^\infty |\Psi(\xi)|\, d\sigma(\xi) &= \int_0^\infty \left| \int_{\nu>\xi} \frac{|(h_\mu\psi)(\nu)|^2}{\nu^{2\mu+1}} d\sigma(\nu) \right| d\sigma(\xi) \\
&= \int_0^\infty \frac{|(h_\mu\psi)(\nu)|^2}{\nu^{2\mu+1}} \left(\int_{\xi<\nu} d\sigma(\xi) \right) d\sigma(\nu) \\
&= \int_0^\infty \frac{|(h_\mu\psi)(\nu)|^2}{\nu^{2\mu+1}} \left(\int_{\xi<\nu} \frac{\xi^{2\mu}}{2^{\mu-1/2}\,\Gamma(\mu+1/2)} d\xi \right) d\sigma(\nu) \\
&= \int_0^\infty \frac{|(h_\mu\psi)(\nu)|^2}{2^{\mu-1/2}\,\Gamma(\mu+1/2)} d\sigma(\nu) \\
&= \frac{1}{2^{\mu-1/2}\,\Gamma(\mu+1/2)} \|h_\mu\psi\|^2_{2,\sigma} \\
&= \frac{1}{2^{\mu-1/2}\,\Gamma(\mu+1/2)} \|\psi\|^2_{2,\sigma} \\
&< \infty.
\end{aligned}$$

The above expression implies that $\Psi \in L^1_\sigma(I)$. By (4.3.9), it is clear that Ψ is continuous with

$$\Psi(0) = h_\mu K(0) = \int_{u>0} \frac{|(h_\mu\psi)(u)|^2}{u^{2\mu+1}} d\sigma(u) = 1. \tag{4.4.1}$$

For $\xi \to \infty$, we get $|h_\mu \psi(u)|^2 \in L^1_\sigma(I)$. From the dominated convergence theorem we find that $\Psi(\xi) \to 0$ as $\xi \to \infty$. □

The following theorems give sufficient conditions for the integrability of the Bessel wavelet kernel.

Theorem 4.4.2. *Suppose that $\psi \in L^2_\sigma(I)$ is a renormalized continuum Bessel wavelet for which the associated wavelet kernel is non-negative: $K(x) \geq 0$ for $x \in I$. Then $\int_0^\infty K(x)d\sigma(x) = 1$; in particular, $K(x) \in L^1_\sigma(I)$. Hence, for any bounded uniformly continuous f, we have $\|S_\epsilon f - f\|_{\infty,\sigma} \to 0$ when $\epsilon \to 0$. If $f \in L^p_\sigma(I), 1 \leq p < \infty$, then $\|S_\epsilon f - f\|_{p,\sigma} \to 0$.*

Proof. Applying Parseval's identity to the Fejér kernel, we have

$$\int_0^M \left(1 - \frac{x}{M}\right) K(x)d\sigma(x) = \int_0^\infty h_\mu \left(1 - \frac{x}{M}\right)(\xi)\,(h_\mu K)(\xi)d\sigma(\xi).$$

From simple calculation, the right hand side of the above integral becomes

$$\int_0^M \left(1 - \frac{x}{M}\right) K(x)d\sigma(x) \leq M^{2\mu+1} \int_0^\infty \frac{1}{(2\mu+1)(2\mu+2)} \Psi(\xi)d\sigma(\xi).$$

The last integral is bounded for $\mu > -1/2$ because Ψ is bounded and continuous at $\xi = 0$ with $\Psi(0) = 1$. Fatou's Lemma and the Fejér kernel gives the following expression

$$\int_0^\infty K(x)d\sigma(x) \leq \lim_{M\to\infty} \int_0^M \left(1 - \frac{x}{M}\right) K(x)d\sigma(x) = 1.$$

With help of the dominated convergence theorem we can conclude that

$$\int_0^\infty K(x)d\sigma(x) = 1.$$

□

Theorem 4.4.3. *Let $\psi \in L^2_\sigma(I)$ be a renormalized continuum Bessel wavelet with $\int_{x>1} log(x)\left|\psi\#\overline{\psi}(x)\right| d\sigma(x) < \infty$. Then wavelet kernel $K \in L^1_\sigma(I)$ and $\int_0^\infty K(x)d\sigma(x) = 1$. Hence for any bounded uniformly continuous f, we have $\|S_\epsilon f - f\|_{\infty,\sigma} \to 0$ when $\epsilon \to 0$. If $f \in L^p_\sigma(I), 1 \le p < \infty$, then $\|S_\epsilon f - f\|_{p,\sigma} \to 0$.*

Proof. The integrability condition implies that $\psi\#\overline{\psi}(x) \in L^1_\sigma(I)$, in particular that $\psi \in L^1_\sigma(I)$ by Fubini's theorem. But $\psi \in L^1_\sigma(I)$ implies that $\int_0^\infty (\psi\#\overline{\psi})(x)d\sigma(x) = 0$. Thus, we use this to write the equivalent formula of (4.3.11)

$$\begin{aligned} K(x) &= \frac{1}{x^{2\mu+1}}\int_0^x (\psi\#\overline{\psi})(z)d\sigma(z) \\ &= \frac{1}{x^{2\mu+1}}\int_0^\infty (\psi\#\overline{\psi})(z)d\sigma(z) - \frac{1}{x^{2\mu+1}}\int_x^\infty (\psi\#\overline{\psi})(z)d\sigma(z) \\ &= -\frac{1}{x^{2\mu+1}}\int_x^\infty (\psi\#\overline{\psi})(z)d\sigma(z). \end{aligned} \tag{4.4.2}$$

Since K is continuous, we have that $\int_{x\le 1} |K(x)|\, d\sigma(x) < \infty$.

With the help of (4.4.2), we get

$$\begin{aligned} \int_{x\ge 1} |K(x)|\, d\sigma(x) &\le \int_{x\ge 1}\left(\frac{1}{x^{2\mu+1}}\int_x^\infty \left|\psi\#\overline{\psi}(z)\right| d\sigma(z)\right) d\sigma(x) \\ &= \int_{z\ge 1} \left|(\psi\#\overline{\psi})(z)\right| d\sigma(z) \int_1^z \frac{d\sigma(x)}{x^{2\mu+1}} \\ &= \int_{z\ge 1} \frac{log(z)}{2^{\mu-1/2}\,\Gamma(\mu+1/2)} \left|(\psi\#\overline{\psi})(z)\right| d\sigma(z) \\ &< \infty, \end{aligned}$$

which proves that $K \in L^1_\sigma(I)$.

From (4.4.1) it follows that

$$\int_0^\infty K(x)d\sigma(x) = \Psi(0) = 1.$$

□