Chapter 3

The Bessel Wavelet Convolution involving Hankel Transform

3.1 Introduction

In recent years, many properties of the wavelet convolution are studied by exploiting the theory of Fourier transform. The wavelet convolution product is an important tool to explore the various characterizations of the wavelet transform which is extensively given in the book [24]. This theory helps to define the wavelet convolution associated with the wavelet transform [27].

In [26], the properties of Bessel wavelet convolution product is studied and its certain estimates are obtained by using the theory of Bessel wavelet transform. In this chapter, our main focus is to expose the Bessel wavelet convolution associated with the Bessel wavelet transform.

In the present chapter, we discuss the various properties of the Bessel wavelet convolution by taking the Bessel wavelet transform and the Hankel transform tools. The boundedness on generalized Sobolev space $B^{\mu}_{p,k}(I), 1 \leq p < \infty$, associated with the normalized Bessel wavelet transform is obtained.

3.2 The Bessel Wavelet Convolution

In this section, using the Hirschmanian theory of Hankel transform[12] various results of the Bessel wavelet convolution are obtained.

Theorem 3.2.1. *If* $\overline{(h_\mu\psi)}(a\omega)(h_\mu f)(\omega) \in L^1_\sigma(I)$ *and* $\overline{(h_\mu\psi)}(a\omega)(h_\mu g)(\omega) \in L^1_\sigma(I)$, $(h_\mu\psi)(a\omega) \neq 0$ *for* $a \in I$ *and* $(B_\psi f)(b,a) = (B_\psi g)(b,a)\,\forall(b,a) \in I \times I$. *Then* $f = g$ *a.e.*

Proof. Given that

$$(B_\psi f)(b,a) = (B_\psi g)(b,a)\ \forall(b,a) \in I \times I. \tag{3.2.1}$$

Then from (1.2.6), we have

$$h_\mu^{-1}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\right](b) = h_\mu^{-1}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](b). \tag{3.2.2}$$

From [11, Corollary 2.9], we get

$$\overline{(h_\mu\psi)}(a\omega)(h_\mu f)(\omega) = \overline{(h_\mu\psi)}(a\omega)(h_\mu g)(\omega) \qquad a.e.$$

Since $(h_\mu\psi)(a\omega) \neq 0$, then we get

$$(h_\mu f)(\omega) = (h_\mu g)(\omega).$$

Again from [11, Corollary 2.9], we get

$$f = g \qquad a.e.$$

□

Theorem 3.2.2. *Let* $f, g \in L^1_\sigma(I)$ *and* $\psi \in L^1_\sigma(I)$, *then*

$$B_\psi(f \otimes g)(b,a) = (B_\psi f)(b,a)(B_\psi g)(b,a) \tag{3.2.3}$$

holds.

Proof. In view of [26, p.271], we have

$$(\overline{h_\mu \psi})(a\omega)\, h_\mu \left[(f \otimes g)\right](\omega) = \left[(\overline{h_\mu \psi})(a\cdot)(h_\mu f)(\cdot) \# (\overline{h_\mu \psi})(a\cdot)(h_\mu g)(\cdot)\right](\omega). \quad (3.2.4)$$

Multiplying both sides of the above equation by $j_\mu(b\omega)$ and integrating over I, we get

$$\begin{aligned}
&\int_0^\infty j_\mu(b\omega)(\overline{h_\mu \psi})(a\omega)\, h_\mu \left[(f \otimes g)\right](\omega) d\sigma(\omega) \\
&= \int_0^\infty j_\mu(b\omega) \left[(\overline{h_\mu \psi})(a\cdot)(h_\mu f)(\cdot) \# (\overline{h_\mu \psi})(a\cdot)(h_\mu g)(\cdot)\right](\omega) d\sigma(\omega) \\
&= \int_0^\infty j_\mu(b\omega) \Bigg[\int_0^\infty \int_0^\infty (\overline{h_\mu \psi})(az)(h_\mu f)(z).(\overline{h_\mu \psi})(ay)(h_\mu g)(y) \\
&\qquad\qquad D(\omega, y, z) d\sigma(y) d\sigma(z) \Bigg] d\sigma(\omega) \\
&= \int_0^\infty \int_0^\infty \left(\int_0^\infty j_\mu(b\omega) D(\omega, y, z) d\sigma(\omega) \right) (\overline{h_\mu \psi})(az)(h_\mu f)(z) \\
&\qquad\qquad (\overline{h_\mu \psi})(ay)(h_\mu g)(y) d\sigma(y) d\sigma(z).
\end{aligned}$$

From (1.1.5), we have

$$\begin{aligned}
&\int_0^\infty j_\mu(b\omega)(\overline{h_\mu \psi})(a\omega)\, h_\mu \left[(f \otimes g)\right](\omega) d\sigma(\omega) \\
&= \int_0^\infty \int_0^\infty j_\mu(yb) j_\mu(zb)(\overline{h_\mu \psi})(az)(h_\mu f)(z)(\overline{h_\mu \psi})(ay)(h_\mu g)(y) d\sigma(y) d\sigma(z) \\
&= \int_0^\infty j_\mu(zb)(\overline{h_\mu \psi})(az)(h_\mu f)(z) d\sigma(z) \int_0^\infty j_\mu(yb)(\overline{h_\mu \psi})(ay)(h_\mu g)(y) d\sigma(y) \\
&= h_\mu^{-1} \left[(\overline{h_\mu \psi})(a\cdot)(h_\mu f)(\cdot)\right](b) \;\; h_\mu^{-1} \left[(\overline{h_\mu \psi})(a\cdot)(h_\mu g)(\cdot)\right](b).
\end{aligned}$$

From (1.2.6), we get

$$B_\psi(f \otimes g)(b, a) = (B_\psi f)(b, a)\, (B_\psi g)(b, a).$$

□

Lemma 3.2.3. *Let $\psi \in L^1_\sigma(I)$. Then*

$$\int_0^\infty D(z,u,at)\psi(t)d\sigma(t) = \psi_{u,a}(z). \tag{3.2.5}$$

Proof. We have

$$\int_0^\infty D(z,u,at)\psi(t)d\sigma(t) = \int_0^\infty \left\{\int_0^\infty j_\mu(z\omega)j_\mu(u\omega)j_\mu(at\omega)d\sigma(\omega)\right\}\psi(t)d\sigma(t).$$

Putting $a\,\omega = s$, we get the following expression

$$\begin{aligned}\int_0^\infty D(z,u,at)\psi(t)d\sigma(t) &= \frac{1}{a^{2\mu+1}}\int_0^\infty \left\{\int_0^\infty j_\mu\left(\frac{zs}{a}\right) j_\mu\left(\frac{us}{a}\right) j_\mu(ts)\,d\sigma(s)\right\}\psi(t)d\sigma(t)\\ &= \frac{1}{a^{2\mu+1}}\int_0^\infty D\left(\frac{z}{a},\frac{u}{a},t\right)\psi(t)d\sigma(t)\\ &= \psi_{u,a}(z).\end{aligned}$$

□

Theorem 3.2.4. *Let $f, g \in L^1_\sigma(I)$ and assume that*

$$D_\psi(x,y,z) = \frac{1}{A_\psi}\int_0^\infty\int_0^\infty \overline{\psi}_{b,a}(x)\overline{\psi}_{b,a}(y)\psi_{b,a}(z)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}, \tag{3.2.6}$$

where $A_\psi = \int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}}d\sigma(a)$. Then the Bessel wavelet convolution will be in the following form :

$$(f\otimes g)(z) = \int_0^\infty\int_0^\infty D_\psi(x,y,z)\,f(x)\;g(y)\,d\sigma(x)d\sigma(y). \tag{3.2.7}$$

Proof. Since

$$D_\psi(x,y,z) = \frac{1}{A_\psi}\int_0^\infty\int_0^\infty \overline{\psi}_{b,a}(x)\overline{\psi}_{b,a}(y)\psi_{b,a}(z)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}},$$

from the inversion formula of Bessel wavelet transform, we have

$$D_\psi(x,y,z) = B_\psi^{-1}\left[\overline{\psi}_{b,a}(x)\overline{\psi}_{b,a}(y)\right](z).$$

The above expression implies that

$$\int_0^\infty D_\psi(x,y,z)\overline{\psi}_{b,a}(z)d\sigma(z) = \overline{\psi}_{b,a}(x)\overline{\psi}_{b,a}(y). \tag{3.2.8}$$

Multiplying both sides of (3.2.4) by $\frac{(h_\mu\psi)(a\omega)}{a^{2\mu+1}}$ and integrating 0 to ∞ with respect to a, we get

$$\begin{aligned}
&\int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}} d\sigma(a)\ h_\mu\left(f\otimes g\right)(\omega)\\
&= \int_0^\infty (h_\mu\psi)(a\omega)\frac{d\sigma(a)}{a^{2\mu+1}}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\\
&A_\psi\ h_\mu\left(f\otimes g\right)(\omega)\\
&= \int_0^\infty (h_\mu\psi)(a\omega)\frac{d\sigma(a)}{a^{2\mu+1}}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega).
\end{aligned}$$

From the inversion formula of Hankel transform, we get

$$\begin{aligned}
&A_\psi\ \left(f\otimes g\right)(z)\\
&= h_\mu^{-1}\left[\int_0^\infty (h_\mu\psi)(a\omega)\frac{d\sigma(a)}{a^{2\mu+1}}\right.\\
&\qquad \left.\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\right](z)\\
&= \int_0^\infty j_\mu(z\omega)\left[\int_0^\infty (h_\mu\psi)(a\omega)\frac{d\sigma(a)}{a^{2\mu+1}}\right.\\
&\qquad \left.\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\right]d\sigma(\omega)\\
&= \int_0^\infty j_\mu(z\omega)d\sigma(\omega)\int_0^\infty\left(\int_0^\infty j_\mu(a\omega t)\psi(t)d\sigma(t)\right)\frac{d\sigma(a)}{a^{2\mu+1}}\\
&\qquad \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}}\int_0^\infty d\sigma(\omega)\left(\int_0^\infty j_\mu(z\omega)j_\mu(a\omega t)\psi(t)d\sigma(t)\right)\\
&\qquad \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}}\int_0^\infty d\sigma(\omega)\int_0^\infty\left(\int_0^\infty D(z,at,u)j_\mu(u\omega)d\sigma(u)\right)\psi(t)d\sigma(t)\\
&\qquad \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)
\end{aligned}$$

$$= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty d\sigma(\omega) \int_0^\infty j_\mu(u\omega) \left(\int_0^\infty D(z,at,u)\psi(t)d\sigma(t)\right) d\sigma(u) \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega).$$

Using Lemma 3.2.3, we have

$$\begin{aligned}
&A_\psi\ (f\otimes g)(z)\\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty d\sigma(\omega) \int_0^\infty j_\mu(u\omega)\psi_{u,a}(z)d\sigma(u) \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty \psi_{u,a}(z)d\sigma(u) \int_0^\infty j_\mu(u\omega) \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)d\sigma(\omega)\\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty \psi_{u,a}(z)d\sigma(u)\ h_\mu^{-1}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](u)\\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty \psi_{u,a}(z)d\sigma(u)\ h_\mu^{-1}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\right](u)\ h_\mu^{-1}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](u).
\end{aligned}$$

From (1.2.6), we can write

$$\begin{aligned}
A_\psi\ (f\otimes g)(z) &= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty \psi_{u,a}(z)d\sigma(u)\,(B_\psi f)(u,a)\,(B_\psi g)(u,a)\\
&= \int_0^\infty\int_0^\infty \psi_{u,a}(z)\frac{d\sigma(a)d\sigma(u)}{a^{2\mu+1}} \left\{\int_0^\infty f(x)\overline{\psi}_{u,a}(x)d\sigma(x)\right\} \left\{\int_0^\infty g(y)\overline{\psi}_{u,a}(y)d\sigma(y)\right\}\\
&= \int_0^\infty\int_0^\infty f(x)g(y)d\sigma(x)d\sigma(y) \int_0^\infty\int_0^\infty \overline{\psi}_{u,a}(x)\ \overline{\psi}_{u,a}(y)\ \psi_{u,a}(z)\frac{d\sigma(a)d\sigma(u)}{a^{2\mu+1}}.
\end{aligned}$$

From (3.2.6), we get

$$(f \otimes g)(z) = \int_0^\infty \int_0^\infty D_\psi(x, y, z)\, f(x)\; g(y)\, d\sigma(x) d\sigma(y).$$

$\square$

Lemma 3.2.5. *If $\psi \in L^2_\sigma(I)$, then*

$$(h_\mu \psi_{b,a})(\omega) = j_\mu(b\omega)(h_\mu\psi)(a\omega). \tag{3.2.9}$$

Proof. We have

$$\begin{aligned}
(h_\mu \psi_{b,a})(\omega) &= \int_0^\infty j_\mu(\omega t)\psi_{b,a}(t) d\sigma(t) \\
&= \int_0^\infty j_\mu(\omega t) a^{-2\mu-1} \int_0^\infty \psi(z) D\left(\frac{t}{a}, \frac{b}{a}, z\right) d\sigma(z) d\sigma(t).
\end{aligned}$$

Putting $\frac{t}{a} = x$, we get

$$\begin{aligned}
(h_\mu \psi_{b,a})(\omega) &= \int_0^\infty j_\mu(\omega a x) \int_0^\infty \psi(z) D\left(x, \frac{b}{a}, z\right) d\sigma(z) d\sigma(x) \\
&= \int_0^\infty \psi(z) \left(\int_0^\infty j_\mu(\omega a x) D\left(x, \frac{b}{a}, z\right) d\sigma(x)\right) d\sigma(z) \\
&= \int_0^\infty \psi(z)\, j_\mu(b\omega)\, j_\mu(za\omega) d\sigma(z) \\
&= j_\mu(b\omega) \int_0^\infty \psi(z)\, j_\mu(za\omega) d\sigma(z) \\
&= j_\mu(b\omega)(h_\mu\psi)(a\omega).
\end{aligned}$$

$\square$

Theorem 3.2.6. *If $f \in L^2_\sigma(I)$, then f can be reconstructed by the formula*

$$f(t) = \frac{1}{A_\psi} \int_0^\infty \int_0^\infty (B_\psi f)(b, a)\psi_{b,a}(t) \frac{d\sigma(a) d\sigma(b)}{a^{2\mu+1}}, \tag{3.2.10}$$

where $\psi \in L^2_\sigma(I)$ be a basic wavelet satisfies admissibility condition A_ψ.

Proof. If $f \in L^2_\sigma(I)$, then we have

$$\frac{1}{A_\psi}\int_0^\infty\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}} = \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)d\sigma(b)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

Using Parseval's formula of the Hankel transform (1.1.14), we get

$$\frac{1}{A_\psi}\int_0^\infty\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}} = \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty h_\mu[(B_\psi f)(b,a)](\omega)(h_\mu\psi_{b,a})(\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

From (1.2.6) and Lemma 3.2.5, we have

$$\begin{aligned}
&\frac{1}{A_\psi}\int_0^\infty\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\\
&= \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty (\overline{h_\mu\psi})(a\omega)(h_\mu f)(\omega)j_\mu(b\omega)(h_\mu\psi)(a\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}}\\
&= \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}}d\sigma(a)\right)(h_\mu f)(\omega)j_\mu(b\omega)d\sigma(\omega)\\
&= \frac{1}{A_\psi}\int_0^\infty A_\psi\ (h_\mu f)(\omega)j_\mu(b\omega)d\sigma(\omega)\\
&= h_\mu^{-1}[(h_\mu f)](b)\\
&= f(t).
\end{aligned}$$

□

Theorem 3.2.7. *If $f \in L^2_\sigma(I)$, then the following Calderón's reproducing identity holds:*

$$f(t) = \frac{1}{A_\psi}\int_0^\infty \left(f\#\overline{\psi_a}\#\psi_a\right)\frac{d\sigma(a)}{a^{2\mu+1}}, \tag{3.2.11}$$

for a Bessel wavelet $\psi \in L^2_\sigma(I)$.

Proof. From (3.2.10), we have

$$\begin{aligned} f(t) &= \frac{1}{A_\psi}\int_0^\infty\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}} \\ &= \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)d\sigma(b)\right)\frac{d\sigma(a)}{a^{2\mu+1}}. \end{aligned}$$

Using Parseval's formula of the Hankel transform (1.1.14), we get

$$f(t) = \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty h_\mu[(B_\psi f)(b,a)](\omega)(h_\mu\psi_{b,a})(\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

From (1.2.6) and Lemma 3.2.5, we have

$$f(t) = \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty (\overline{h_\mu\psi})(a\omega)(h_\mu f)(\omega)j_\mu(b\omega)(h_\mu\psi)(a\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

Using (1.1.10), we get the following expression

$$\begin{aligned} f(t) &= \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty j_\mu(b\omega)h_\mu(f\#\overline{\psi_a})(\omega)(h_\mu\psi)(a\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}} \\ &= \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty j_\mu(b\omega)h_\mu(f\#\overline{\psi_a}\#\psi_a)(\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}} \\ &= \frac{1}{A_\psi}\int_0^\infty (f\#\overline{\psi_a}\#\psi_a)(t)\frac{d\sigma(a)}{a^{2\mu+1}}. \end{aligned}$$

□

Theorem 3.2.8. *Let* $f \in L^2_\sigma(I)$ *and* $\psi \in L^2_\sigma(I)$ *satisfying admissibility condition* $A_\psi = \int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}}d\sigma(a)$, *then the following reproducing identity holds:*

$$f(t) = \frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left(f\otimes\overline{\psi}_{b,a}\otimes\psi_{b,a}\right)(t)\,\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}}. \tag{3.2.12}$$

Proof. Assume that ϕ is an orthonormal wavelet in $L^2_\sigma(I)$. Taking the Bessel wavelet transform of right hand side of (3.2.12) with respect to ϕ, we have

$$
\begin{aligned}
& B_\phi\left[\frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left(f\otimes\overline{\psi}_{b,a}\otimes\psi_{b,a}\right)(t)\ \frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}}\right](b^{'},a^{'}) \\
&= \frac{1}{A_\psi}\int_0^\infty\int_0^\infty B_\phi\left\{\left(f\otimes\overline{\psi}_{b,a}\otimes\psi_{b,a}\right)(t)\right\}(b^{'},a^{'})\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}} \\
&= \left(B_\phi f\right)(b^{'},a^{'})\frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left(B_\phi\overline{\psi}_{b,a}\right)(b^{'},a^{'})\left(B_\phi\psi_{b,a}\right)(b^{'},a^{'})\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}} \\
&= \left(B_\phi f\right)(b^{'},a^{'})\frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left[\int_0^\infty \overline{\psi}_{b,a}(t)\overline{\phi}_{b',a'}(t)d\sigma(t)\right] \\
&\qquad \left[\int_0^\infty \psi_{b,a}(x)\overline{\phi}_{b',a'}(x)d\sigma(x)\right]\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}} \\
&= \left(B_\phi f\right)(b^{'},a^{'})\int_0^\infty \frac{1}{A_\psi}\left(\int_0^\infty\int_0^\infty \left[\left(B_\psi\overline{\phi}_{b',a'}\right)(b,a)\psi_{b,a}(x)\right]\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}}\right) \\
&\qquad \overline{\phi}_{b',a'}(x)d\sigma(x) \\
&= \left(B_\phi f\right)(b^{'},a^{'})\int_0^\infty \overline{\phi}_{b',a'}(x)\ \overline{\phi}_{b',a'}(x)d\sigma(x) \\
&= \left(B_\phi f\right)(b^{'},a^{'})\int_0^\infty \left[\overline{\phi}_{b',a'}(x)\right]^2 d\sigma(x) \\
&= \left(B_\phi f\right)(b^{'},a^{'}) \qquad\qquad (\text{ by orthogonality of } \phi).
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left(f\otimes\overline{\psi}_{b,a}\otimes\psi_{b,a}\right)(t)\ \frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}} &= B_\phi^{-1}\left[\left(B_\phi f\right)(b^{'},a^{'})\right](t) \\
&= f(t).
\end{aligned}
$$

□

3.3 Generalized Sobolev Space

Let $\psi \in L^2_\sigma(I)$ be an analysing Bessel wavelet which satisfies (1.2.8).
The integral

$$\begin{aligned}(L_\psi f)(b,a) &= \frac{1}{\sqrt{A_\psi}}(B_\psi f)(b,a) = \frac{1}{\sqrt{A_\psi}}\langle f, \psi_{b,a}\rangle \\ &= \frac{1}{\sqrt{A_\psi}}\int_0^\infty f(t)\overline{\psi}_{b,a}(t)d\sigma(t),\end{aligned}$$

defines an element of $L^2\left(I \times I, \frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right)$.
The Hankel transform of L_ψ is given as

$$h_\mu\left[(L_\psi f)(b,a)\right](\omega) = \frac{1}{\sqrt{A_\psi}}\overline{(h_\mu\psi)}(a\omega)\,(h_\mu f)(\omega). \qquad (3.3.1)$$

The operator L_ψ is also called a normalized form of the Bessel wavelet operator B_ψ and

$$L_\psi : L^2(I, d\sigma(t)) \to L^2\left(I \times I, \frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right),$$

is an isometry [21, p.245].
In this section, we are exploiting the results of [24] and study the normalized Bessel wavelet transform $L_\psi f$, which is defined on $L^2_\sigma(I, d\sigma(t))$ to generalized Sobolev space $B^\mu_{p,k}(I)$ and the space of its image set is denoted by $W^\mu_{p,k}$. The boundedness and other properties of $L_\psi f$ are given on $B^\mu_{p,k}(I)$ space.

Definition 3.3.1. The Zemanian space $H_\mu(I), I = (0,\infty)$ is the set of all infinitely differentiable functions ϕ on $(0,\infty)$ such that

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in(0,\infty)}\left|x^m\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-\frac{1}{2}}\phi(x)\right| < \infty, \qquad (3.3.2)$$

for all $m,k \in \mathbb{N}_0$. Then $f \in H'_\mu(I)$ is defined by the following way:

$$\langle f, \phi\rangle = \int_0^\infty f(x)\phi(x)d\sigma(x), \quad \phi \in H_\mu(I). \qquad (3.3.3)$$

Definition 3.3.2. Let $k(\xi)$ be an arbitrary weight function. The generalized Sobolev space $B^{\mu}_{p,k}(I), 1 \leq p < \infty$ is defined to be the space of all ultra-distributions $f \in H'_{\mu}(I), I = (0, \infty)$ such that

$$\|f\|_{p,k} = \left(\int_0^\infty |k(\xi) \, (h_\mu f) \, (\xi)|^p \, d\sigma(\xi) \right)^{1/p} < \infty \tag{3.3.4}$$

and

$$\|f\|_{\infty,k} = ess \, sup \, k(\xi) \, |(h_\mu f)(\xi)| \, . \tag{3.3.5}$$

Definition 3.3.3. Define the space $W^{\mu}_{p,k}$ of all measurable functions f on $I \times I$ such that

$$\|f(b,a)\|_{W^{\mu}_{p,k}} = \left(\int_I \|f(b,a)\|^p_{p,k} \, \frac{d\sigma(a)}{a^{2\mu+1}} \right)^{1/p} < \infty, \tag{3.3.6}$$

$1 \leq p < \infty$, $a \in (0, \infty)$.

Theorem 3.3.4. *Assume that analysing wavelet ψ satisfies the following admissibility condition:*

$$A_{\psi,p} = \int_0^\infty \frac{|(h_\mu \psi) \, (\xi)|^p}{\xi^{2\mu+1}} d\sigma(\xi) < \infty. \tag{3.3.7}$$

Let $(L_\psi f) \, (b,a)$ be the normalized Bessel wavelet transform of the function $f \in B^{\mu}_{p,k}(I)$, with respect to the analysing wavelet ψ satisfying (3.3.7). Then

$$\|(L_\psi f) \, (b,a)\|_{W^{\mu}_{p,k}} = C_p \, \|f\|_{p,k} \, , \tag{3.3.8}$$

where $C_p = (A_\psi)^{-p/2} \, A_{\psi,p}$.

Proof. Let $f \in H_\mu(I)$.

Then

$$\begin{aligned} \|(L_\psi f) \, (b,a)\|^p_{W^{\mu}_{p,k}} &= \int_0^\infty \|(L_\psi f) \, (b,a)\|^p_{k,p} \, \frac{d\sigma(a)}{a^{2\mu+1}} \\ &= \int_0^\infty \left(\int_0^\infty |k(\xi)|^p \, |h_\mu \, [(L_\psi f) \, (b,a)] \, (\xi)|^p \, d\sigma(\xi) \right) \frac{d\sigma(a)}{a^{2\mu+1}}. \end{aligned}$$

From (3.3.1), we have

$$\begin{aligned}
\|(L_\psi f)(b,a)\|^p_{W^\mu_{p,k}} &= \int_0^\infty \left(\int_0^\infty |k(\xi)|^p \frac{1}{A_\psi^{p/2}} |(h_\mu \psi)(a\xi)|^p |(h_\mu f)(\xi)|^p d\sigma(\xi) \right) \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= \frac{1}{A_\psi^{p/2}} \int_0^\infty \left(\int_0^\infty |k(\xi)|^p |(h_\mu f)(\xi)|^p d\sigma(\xi) \right) |(h_\mu \psi)(a\xi)|^p \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= \frac{1}{A_\psi^{p/2}} \int_0^\infty \|f\|^p_{p,k} |(h_\mu \psi)(a\xi)|^p \frac{d\sigma(a)}{a^{2\mu+1}}.
\end{aligned}$$

Putting $a\xi = u$, we have

$$\begin{aligned}
\|(L_\psi f)(b,a)\|^p_{W^\mu_{k,p}} &= \frac{1}{A_\psi^{p/2}} \int_0^\infty \frac{|(h_\mu \psi)(u)|^p}{u^{2\mu+1}} d\sigma(u) \|f\|^p_{p,k} \\
&= \frac{1}{A_\psi^{p/2}} A_{\psi,p} \, \|f\|^p_{p,k} \\
&= C_p \|f\|^p_{p,k} .
\end{aligned}$$

Since $H_\mu(I)$ is dense in $B^\mu_{p,k}(I)$, the above result can be extended to all $f \in B^\mu_{p,k}(I)$.

□

Theorem 3.3.5. *Let* $f \in B^\mu_{p,k}(I)$ *and* $\psi \in L^1_\sigma(I)$ *with* $\int_0^\infty \psi(t)d\sigma(t) = 1$. *Then* $(B_\psi f)(.,a) \to f(.)$ *in* $B^\mu_{p,k}(I)$ *as* $a \to 0$.

Proof. From (3.3.4), we have

$$\begin{aligned}
\|f\#\psi_a - f\|^p_{p,k} &= \int_0^\infty |h_\mu (f\#\psi_a - f)(\xi)|^p |k(\xi)|^p d\sigma(\xi) \\
&= \int_0^\infty |h_\mu (f\#\psi_a)(\xi) - (h_\mu f)(\xi)|^p |k(\xi)|^p d\sigma(\xi) \\
&= \int_0^\infty |(h_\mu f)(\xi)(h_\mu \psi)(a\xi) - (h_\mu f)(\xi)|^p |k(\xi)|^p d\sigma(\xi) \\
&= \int_0^\infty |(h_\mu f)(\xi) k(\xi)|^p |(h_\mu \psi)(a\xi) - 1|^p d\sigma(\xi) \\
&= \int_0^\infty |I(a,\xi)|^p d\sigma(\xi),
\end{aligned}$$

where $I(a,\xi) = (h_\mu f)(\xi)\, k(\xi) \left[(h_\mu \psi)(a\xi) - 1\right]$.
Under our assumption $\int_0^\infty \psi(t) d\sigma(t) = 1$, we have $\lim_{a\to 0} |I(a,\xi)| = 0$ a.e.
Set $M = \sup_{\xi \in I} |(h_\mu \psi)(a\xi) - 1|$, which is independent of a.
Then

$$|I(a,\xi)| \leq M \left|(h_\mu f)(\xi)\, k(\xi)\right|.$$

Now, applying the dominated convergence theorem, we have
$(B_\psi f)(.,a) = (f \# \psi_a)(.,a) \to f(.)$ in $B^\mu_{p,k}(I)$ as $a \to 0$. □