Chapter 2

The Relation between Bessel Wavelet Convolution Product and Hankel Convolution Product involving Hankel Transform

2.1 Introduction

The wavelet convolution product and their properties were discussed by Pathak[24, 27] and got the relation between Fourier convolution and wavelet convolution by exploiting the theory of Fourier transform. Using the Bessel wavelet transform Pathak, Upadhyay and Pandey[26] formally defined the Bessel wavelet convolution product and studied their important properties. These concepts are useful to discuss the relation between Bessel wavelet convolution product and Hankel convolution product by using the Hankel transform.

In this chapter, the relation between Bessel wavelet convolution product and Hankel convolution product is exposed and find certain approximations of the Bessel wavelet transform.

Pinsky[28] introduced the concept of heuristic treatment of wavelet transform by exploiting the Fourier transform. From the results of [28], heuristic treatment of Bessel wavelet transform is investigated and its properties with the help of Hankel transform and Hankel convolution are studied.

2.2 The Bessel Wavelet Convolution Product

In this section, using the following relation of Bessel wavelet convolution product

$$B_\psi\left(f \otimes g\right)(b, a) = (B_\psi f)(b, a)(B_\psi g)(b, a), \tag{2.2.1}$$

we find the relation between Bessel wavelet convolution product and Hankel convolution product. Further, we obtain boundedness and approximation results of the Bessel wavelet convolution product.

Theorem 2.2.1. *Let $f, g \in L^1_\sigma(I) \cap L^2_\sigma(I)$ and $\psi \in L^2_\sigma(I)$. Then the Bessel wavelet convolution product can be written in the following form:*

$$\begin{aligned} E_\psi[h_\mu\left(f \otimes g\right)](\omega) &= \int_0^\infty \int_0^\infty (h_\mu f)(\eta)(h_\mu g)(\xi) D(\omega, \xi, \eta) \\ &\quad \left(\int_0^\infty \overline{(h_\mu \psi)}(a\eta)\ \overline{(h_\mu \psi)}(a\xi) \frac{d\sigma(a)}{a^{2\mu+1}}\right) d\sigma(\eta) d\sigma(\xi), \end{aligned} \tag{2.2.2}$$

where

$$E_\psi := \int_0^\infty \frac{\overline{(h_\mu \psi)}(a\omega)}{a^{2\mu+1}} d\sigma(a). \tag{2.2.3}$$

Proof. From (1.2.6), we have

$$h_\mu\left[(B_\psi f)(b, a)\right](\omega) = \overline{(h_\mu \psi)}(a\omega)\ (h_\mu f)(\omega).$$

Then

$$h_\mu\left[B_\psi(f\otimes g)(b,a)\right](\omega) = \overline{(h_\mu\psi)}(a\omega)\ [h_\mu(f\otimes g)](\omega),$$

so that using (2.2.1) and [26, pp.271], we have

$$\begin{aligned}
&\overline{(h_\mu\psi)}(a\omega)\ [h_\mu(f\otimes g)](\omega)\\
&= h_\mu\left[(B_\psi f)(b,a)(B_\psi g)(b,a)\right](\omega)\\
&= h_\mu\left[h_\mu^{-1}\left\{\overline{(h_\mu\psi)}(a.)(h_\mu f)(.)\right\}(b)\ h_\mu^{-1}\left\{\overline{(h_\mu\psi)}(a.)(h_\mu g)(.)\right\}(b)\right](\omega)\\
&= h_\mu\left[h_\mu^{-1}\left\{\overline{(h_\mu\psi)}(a.)(h_\mu f)(.)\#\overline{(h_\mu\psi)}(a.)(h_\mu g)(.)\right\}(b)\right](\omega)\\
&= \left[\overline{(h_\mu\psi)}(a.)(h_\mu f)(.)\#\overline{(h_\mu\psi)}(a.)(h_\mu g)(.)\right](\omega).
\end{aligned}$$

If we set $F_a = \overline{(h_\mu\psi)}(a.)\,(h_\mu f)(.)$ and $G_a = \overline{(h_\mu\psi)}(a.)(h_\mu g)(.)$, then we have

$$\begin{aligned}
&\overline{(h_\mu\psi)}(a\omega)\ [h_\mu(f\otimes g)](\omega)\\
&= (F_a\#G_a)(\omega)\\
&= \int_0^\infty\int_0^\infty F_a(\eta)G_a(\xi)D(\omega,\xi,\eta)\,d\sigma(\eta)\,d\sigma(\xi)\\
&= \int_0^\infty\int_0^\infty \overline{(h_\mu\psi)}(a\eta)(h_\mu f)(\eta)\overline{(h_\mu\psi)}(a\xi)(h_\mu g)(\xi)D(\omega,\xi,\eta)\,d\sigma(\eta)\,d\sigma(\xi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^\infty \frac{\overline{(h_\mu\psi)}(a\omega)}{a^{2\mu+1}}\ [h_\mu(f\otimes g)](\omega)d\sigma(a)\\
&= \int_0^\infty\left(\int_0^\infty\int_0^\infty \overline{(h_\mu\psi)}(a\eta)\,(h_\mu f)(\eta)\,\overline{(h_\mu\psi)}(a\xi)\right.\\
&\qquad\left.(h_\mu g)(\xi)D(\omega,\xi,\eta)\,d\sigma(\eta)\,d\sigma(\xi)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.
\end{aligned}$$

Thus, the above expression can be written as

$$\begin{aligned}
E_\psi\ [h_\mu(f\otimes g)](\omega) &= \int_0^\infty\int_0^\infty (h_\mu f)(\eta)(h_\mu g)(\xi)D(\omega,\xi,\eta)\\
&\left(\int_0^\infty \overline{(h_\mu\psi)}(a\eta)\ \overline{(h_\mu\psi)}(a\xi)\ \frac{d\sigma(a)}{a^{2\mu+1}}\right)d\sigma(\eta)\,d\sigma(\xi). \quad\square
\end{aligned}$$

Theorem 2.2.2. *Let $\psi \in L^2_\sigma(I)$ be a basic wavelet and it satisfies the admissibility condition*

$$A_\psi := \int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}} d\sigma(a). \tag{2.2.4}$$

Then

$$\int_0^\infty \frac{|(h_\mu\psi)(a\omega)(h_\mu\psi)(a\eta)|}{a^{2\mu+1}} d\sigma(a) \leq A_\psi. \tag{2.2.5}$$

Proof. We have

$$\int_0^\infty \frac{|(h_\mu\psi)(a\omega)(h_\mu\psi)(a\eta)|}{a^{2\mu+1}} d\sigma(a) = \int_0^\infty \frac{|(h_\mu\psi)(a\omega)(h_\mu\psi)(a\eta)|}{a^{\frac{2\mu+1}{2}}\, a^{\frac{2\mu+1}{2}}} d\sigma(a).$$

Using the Cauchy-Schwarz inequality (1.1.16), we have

$$\begin{aligned} &\int_0^\infty \frac{|(h_\mu\psi)(a\omega)(h_\mu\psi)(a\eta)|}{a^{2\mu+1}} d\sigma(a) \\ \leq\ & \left(\int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}} d\sigma(a)\right)^{\frac{1}{2}} \left(\int_0^\infty \frac{|(h_\mu\psi)(a\eta)|^2}{a^{2\mu+1}} d\sigma(a)\right)^{\frac{1}{2}}. \end{aligned}$$

From (2.2.4), we have

$$\int_0^\infty \frac{|(h_\mu\psi)(a\omega)(h_\mu\psi)(a\eta)|}{a^{2\mu+1}} d\sigma(a) \leq A_\psi^{\frac{1}{2}} \times A_\psi^{\frac{1}{2}} = A_\psi.$$

This implies that

$$\int_0^\infty \frac{|(h_\mu\psi)(a\omega)(h_\mu\psi)(a\eta)|}{a^{2\mu+1}} d\sigma(a) \leq A_\psi.$$

□

Theorem 2.2.3. *Let $f, g \in L^1_\sigma(I) \cap L^2_\sigma(I)$. Then the following relation holds*

$$h_\mu\left(f \otimes g\right)(\omega) = A'_\psi\ \left(h_\mu f \,\#\, h_\mu g\right)(\omega), \tag{2.2.6}$$

where $A'_\psi = \frac{D_\psi}{E_\psi}$ with $D_\psi = \int_0^\infty \frac{\left[\overline{(h_\mu\psi)(u)}\right]^2}{u^{2\mu+1}} d\sigma(u)$ and E_ψ defined in (2.2.3).

Proof. Firstly, we find the value of

$$\begin{aligned}
&\int_0^\infty \overline{(h_\mu\psi)}(a\eta)\ \overline{(h_\mu\psi)}(a\xi)\,\frac{d\sigma(a)}{a^{2\mu+1}} \\
&\qquad = \int_0^\infty \overline{(h_\mu\psi)}(a\eta)\ \overline{(h_\mu\psi)}(a\xi)\,\frac{da}{a\,2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})} \\
&\qquad = \frac{1}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}\int_0^\infty \frac{\overline{(h_\mu\psi)}(a\eta)}{a^{1/2}}(da)^{1/2}\ \ \frac{\overline{(h_\mu\psi)}(a\xi)}{a^{1/2}}(da)^{1/2} \\
&\qquad = \frac{1}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}\int_0^\infty \frac{\overline{(h_\mu\psi)}(u)}{u^{1/2}}(du)^{1/2}\ \ \frac{\overline{(h_\mu\psi)}(u)}{u^{1/2}}(du)^{1/2} \\
&\qquad = \int_0^\infty \frac{\left[\overline{(h_\mu\psi)}(u)\right]^2}{u}\,\frac{du}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})} \\
&\qquad = \int_0^\infty \frac{\left[\overline{(h_\mu\psi)}(u)\right]^2}{u^{2\mu+1}}d\sigma(u) = D_\psi.
\end{aligned}$$

Therefore, (2.2.3) becomes

$$E_\psi\, h_\mu(f\otimes g)(\omega) \;=\; D_\psi\int_0^\infty\int_0^\infty (h_\mu f)(\eta)\,(h_\mu g)(\xi)\,D(\omega,\xi,\eta)d\sigma(\eta)d\sigma(\xi).$$

So that

$$\begin{aligned}
h_\mu(f\otimes g)(\omega) &= \frac{D_\psi}{E_\psi}\,(h_\mu f\,\#\,h_\mu g)\,(\omega) \\
&= A'_\psi\,(h_\mu f\,\#\,h_\mu g)\,(\omega).
\end{aligned}$$

□

Theorem 2.2.4. ***(i)*** *Assume that* $f\in L^p_\sigma(I), g\in L^{p'}_\sigma(I)$, $1<p,p'<\infty$ *and* $\psi\in L^q_\sigma(I)\cap L^{q'}_\sigma(I)$ *such that* $\frac{1}{p}+\frac{1}{q}=1,\ \frac{1}{p'}+\frac{1}{q'}=1.$

Then

$$|B_\psi(f\otimes g)(b,a)| \;\le\; a^{-2\mu-1}\,||f||_{p,\sigma}\ ||g||_{p',\sigma}\ ||\psi||_{q,\sigma}\ ||\psi||_{q',\sigma}. \tag{2.2.7}$$

(ii) *Assume that* $f,g\in L^2_\sigma(I)$ *and* $\psi\in L^2_\sigma(I)$ *is a Bessel wavelet which satisfies admissibility condition*

$$A_\psi := \int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}}d\sigma(a),$$

then

$$\left|\int_0^\infty \int_0^\infty B_\psi\left(f\otimes g\right)(b,a)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right| \leq A_\psi \left\|f\right\|_{2,\sigma} \left\|g\right\|_{2,\sigma}. \tag{2.2.8}$$

Proof. (i) Using (2.2.1), we have

$$\left|B_\psi\left(f\otimes g\right)(b,a)\right| = \left|(B_\psi f)(b,a)(B_\psi g)(b,a)\right|.$$

From (1.2.5) and (1.1.12), we have

$$\begin{aligned}
&\left|B_\psi\left(f\otimes g\right)(b,a)\right| \\
&= \left|(f\#\overline{\psi_a})(b)\right|\left|(g\#\overline{\psi_a})(b)\right| \\
&\leq \left\|f\right\|_{p,\sigma}\left\|\bar{\psi}_a\right\|_{q,\sigma}\left\|g\right\|_{p',\sigma}\left\|\bar{\psi}_a\right\|_{q',\sigma} \\
&= a^{-4\mu-2}\left\|f\right\|_{p,\sigma}\left(a^{\frac{2\mu+1}{q}}\left\|\psi\right\|_{q,\sigma}\right)\left\|g\right\|_{p',\sigma}\left(a^{\frac{2\mu+1}{q'}}\left\|\psi\right\|_{q',\sigma}\right) \\
&= a^{-4\mu-2}a^{(2\mu+1)(\frac{1}{q}+\frac{1}{q'})}\left\|f\right\|_{p,\sigma}\left\|\psi\right\|_{q,\sigma}\left\|g\right\|_{p',\sigma}\left\|\psi\right\|_{q',\sigma} \\
&= a^{-2\mu-1}\left\|f\right\|_{p,\sigma}\left\|\psi\right\|_{q,\sigma}\left\|g\right\|_{p',\sigma}\left\|\psi\right\|_{q',\sigma},
\end{aligned}$$

for $\frac{1}{q}+\frac{1}{q'}=1$.

(ii) Using (2.2.1), we have

$$\begin{aligned}
&\left|\int_0^\infty \int_0^\infty B_\psi\left(f\otimes g\right)(b,a)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right| \\
&\qquad = \left|\int_0^\infty \int_0^\infty (B_\psi f)(b,a)(B_\psi g)(b,a)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right|.
\end{aligned}$$

From (1.2.9) and the Cauchy-Schwarz inequality (1.1.16), the above expression shows that

$$\begin{aligned}
\left|\int_0^\infty \int_0^\infty B_\psi\left(f\otimes g\right)(b,a)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right| &= A_\psi\left|\int_0^\infty f(x)\overline{g}(x)d\sigma(x)\right| \\
&\leq A_\psi\int_0^\infty \left|f(x)\overline{g}(x)\right|d\sigma(x) \\
&\leq A_\psi||f||_{2,\sigma}\,||g||_{2,\sigma}.
\end{aligned}$$

□

Theorem 2.2.5. *Let $h_\mu f \in L^p_\sigma(I)$ and $h_\mu g \in L^q_\sigma(I)$, then we have the following inequality*

(i)

$$\|h_\mu (f \otimes g)\|_{r,\sigma} \leq A_\psi^{'} \|h_\mu f\|_{p,\sigma} \|h_\mu g\|_{q,\sigma},$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

(ii) *For $p = 1$ and $q = 2$, we have the following relation*

$$\|h_\mu (f \otimes g)\|_{2,\sigma} \leq A_\psi^{'} \|f\|_{1,\sigma} \|g\|_{2,\sigma}. \tag{2.2.9}$$

Proof. **(i)** From (2.2.6), we have

$$\begin{aligned} \|h_\mu (f \otimes g)\|_{r,\sigma} &= A_\psi^{'} \|h_\mu f \,\#\, h_\mu g\|_{r,\sigma}. \\ \text{Using (1.1.13), we have} & \\ \|h_\mu (f \otimes g)\|_{r,\sigma} &\leq A_\psi^{'} \|h_\mu f\|_{p,\sigma} \|h_\mu g\|_{q,\sigma}. \end{aligned}$$

(ii) We have

$$\begin{aligned} \|h_\mu (f \otimes g)\|_{2,\sigma} &= A_\psi^{'} \|h_\mu f \,\#\, h_\mu g\|_{2,\sigma}. \\ \text{If we put } p = 1 \text{ and } q = 2, \text{ then} & \\ \|h_\mu (f \otimes g)\|_{2,\sigma} &\leq A_\psi^{'} \|h_\mu f\|_{1,\sigma} \|h_\mu g\|_{2,\sigma} \\ &\leq A_\psi^{'} \|f\|_{1,\sigma} \|g\|_{2,\sigma}. \end{aligned}$$

□

Theorem 2.2.6. *Let $k_n(w) = (h_\mu g_n)(w)$ for $n \in \mathbb{N}$ and $\phi(w) = (h_\mu f)(w)$ satisfy the following conditions :*

(i) $k_n(w) \geq 0, \qquad 0 < w < \infty,$

(ii) $\int_0^\infty k_n(w)\, d\sigma(w) = 1, \quad w = 0, 1, 2, 3, ...,$

(iii) $\lim_{n\to\infty} \int_\delta^\infty k_n(w)\, d\sigma(w) = 0, \quad for\, each\, \delta > 0,$

(iv) $\phi(w) \in L^\infty_\sigma(I)$,

(v) ϕ *is continuous at* w_0.

Then

$$\lim_{n\to\infty} |h_\mu(f \otimes g_n)(w_0)| \leq A'_\psi(h_\mu f)(w_0), \tag{2.2.10}$$

where A'_ψ *defined in Theorem 2.2.3.*

Proof. From (2.2.6), we have

$$\begin{aligned} h_\mu(f \otimes g_n)(w_0) &= A'_\psi \left(h_\mu f \,\#\, h_\mu g_n\right)(w_0) \\ &= A'_\psi \left(\phi \,\#\, k_n\right)(w_0). \end{aligned} \tag{2.2.11}$$

Let

$$\begin{aligned} I &= \left(\phi \,\#\, k_n\right)(w_0) - \phi(w_0) \\ &= \int_0^\infty \int_0^\infty \left[\phi(w) - \phi(w_0)\right] k_n(x) D(w_0, w, x) d\sigma(w) d\sigma(x). \end{aligned}$$

Since ϕ is continuous at w_0, for a given $\epsilon > 0$ we can choose $\delta > 0$ so small that $|\phi(w) - \phi(w_0)| < \epsilon$ for $|w - w_0| < \delta$.
Let $I_1 = \int_\delta^\infty \int_0^\infty \left[\phi(w) - \phi(w_0)\right] k_n(x) D(w_0, w, x) d\sigma(w) d\sigma(x)$
and
$I_2 = \int_0^\delta \int_0^\infty \left[\phi(w) - \phi(w_0)\right] k_n(x) D(w_0, w, x) d\sigma(w) d\sigma(x)$.
Then

$$\begin{aligned} |I_1| &\leq \int_\delta^\infty \int_0^\infty |\phi(w) - \phi(w_0)|\, k_n(x) D(w_0, w, x) d\sigma(w) d\sigma(x) \\ &\leq 2\, \|\phi\|_\infty \int_\delta^\infty \left(\int_0^\infty D(w_0, w, x) d\sigma(w) \right) k_n(x) d\sigma(x) \\ &= 2\, \|\phi\|_\infty \int_\delta^\infty k_n(x) d\sigma(x). \end{aligned}$$

Taking $n \to \infty$ in the last expression and using (iii), we get $\lim_{n\to\infty} I_1 = 0$.
Now, we have

$$|I_2| = \int_0^\delta \int_0^\infty |\phi(w) - \phi(w_0)| \, k_n(x) D(w_0, w, x) d\sigma(w) d\sigma(x).$$

Using the view of [12, Theorem 2(c)], we get

$$\begin{aligned} |I_2| &\leq \epsilon \int_0^\delta \int_0^\infty k_n(x) D(w_0, w, x) d\sigma(w) d\sigma(x) \\ &= \epsilon \int_0^\delta \left(\int_0^\infty D(w_0, w, x) d\sigma(w) \right) k_n(x) d\sigma(x) \\ &\leq \epsilon \int_0^\infty k_n(x) d\sigma(x) \leq \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} |I| \leq \epsilon$. Since ϵ is arbitrary, we have $\lim_{n\to\infty} I = 0$.
From (2.2.11), we have

$$\begin{aligned} \lim_{n\to\infty} h_\mu(f \otimes g_n)(w_0) &= \lim_{n\to\infty} A'_\psi (\phi \# k_n)(w_0) \\ &= A'_\psi \, \phi(w_0) = A'_\psi (h_\mu f)(w_0). \end{aligned}$$

□

Theorem 2.2.7. *Let $f \in L^1_\sigma(I), \phi(w) = (h_\mu f)(w)$ and $k_n(w)$ be same as in Theorem 2.2.6, satisfies all the three properties of Theorem 2.2.6. Then*

$$\lim_{n\to\infty} \left\| \frac{1}{A'_\psi} h_\mu(f \otimes g_n) - (h_\mu f) \right\|_{1,\sigma} = 0. \tag{2.2.12}$$

Proof. From (2.2.6), we have

$$\begin{aligned} \lim_{n\to\infty} \left\| \frac{1}{C'_\psi} h_\mu(f \otimes g_n) - (h_\mu f) \right\|_{1,\sigma} &= \lim_{n\to\infty} \|(h_\mu f \# h_\mu g_n) - (h_\mu f)\|_{1,\sigma} \\ &\leq \lim_{n\to\infty} \|(\phi \# k_n) - \phi\|_{1,\sigma}. \end{aligned}$$

Since $f \in L^1_\sigma(I)$, $(h_\mu f)(w) \in L^1_\sigma(I)$. Therefore, using Theorem 2.2.6, we have

$$\lim_{n\to\infty} \left\| \frac{1}{C'_\psi} h_\mu(f \otimes g_n) - (h_\mu f) \right\|_{1,\sigma} = 0.$$

□

2.3 Heuristic Treatment of the Bessel Wavelet Transform

In this section, we study the properties of heuristic treatment of the Bessel wavelet transform (1.2.4).

Theorem 2.3.1. *Let $(B_\psi f)(b,a)$ be the Bessel wavelet transform and $(B^*_\psi f)(b,a)$ be the adjoint Bessel wavelet operator on a function $f \in L^2_\sigma(I)$ with respect to wavelet $\psi \in L^2_\sigma(I)$. Then*

$$f = \int_0^\infty B^*_\psi \, B_\psi \, f \, \frac{d\sigma(a)}{a^{2\mu+1}}, \tag{2.3.1}$$

where $f(t) = j_\mu(\xi t)$ and normalization of admissibility condition

$$\int_0^\infty \frac{|(h_\mu\psi)(\nu)|^2}{\nu^{2\mu+1}} d\sigma(\nu) = 1. \tag{2.3.2}$$

Proof. Putting $f(t) = j_\mu(\xi t)$ in (1.2.4), we get

$$\begin{aligned}
(B_\psi f)(b,a) &= a^{-2\mu-1} \int_0^\infty j_\mu(\xi t)\, \bar{\psi}\left(\frac{t}{a}, \frac{b}{a}\right) d\sigma(t) \\
&= a^{-2\mu-1} \int_0^\infty \left(\int_0^\infty \bar{\psi}(z)\, D\left(\frac{t}{a}, \frac{b}{a}, z\right) d\sigma(z) \right) j_\mu(\xi t) d\sigma(t) \\
&= a^{-2\mu-1} \int_0^\infty \left(\int_0^\infty j_\mu(\xi t)\, D\left(\frac{t}{a}, \frac{b}{a}, z\right) d\sigma(t) \right) \bar{\psi}(z)\, d\sigma(z).
\end{aligned}$$

Substituting $\frac{t}{a} = u$, we obtain

$$(B_\psi f)(b,a) = a^{-2\mu-1} \int_0^\infty \left(\int_0^\infty j_\mu(ua\xi) D\left(u, \frac{b}{a}, z\right) a^{2\mu+1} d\sigma(u) \right) \bar{\psi}(z) d\sigma(z).$$

From (1.1.5), we have

$$\begin{aligned} (B_\psi f)(b,a) &= \int_0^\infty j_\mu(b\xi) j_\mu(za\xi) \bar{\psi}(z)\, d\sigma(z) \\ &= j_\mu(b\xi) \int_0^\infty j_\mu(za\xi) \bar{\psi}(z)\, d\sigma(z) \\ &= j_\mu(b\xi)\, \overline{(h_\mu \psi)}(a\xi). \end{aligned}$$

Now, we define the adjoint operator

$$\begin{aligned} B_\psi^* \, B_\psi f(t) &= a^{-2\mu-1} \int_0^\infty (B_\psi f)(b,a) \psi\left(\frac{t}{a}, \frac{b}{a}\right) d\sigma(b) \\ &= a^{-2\mu-1} \overline{(h_\mu \psi)}(a\xi) \int_0^\infty j_\mu(b\xi) \left(\int_0^\infty \psi(z)\ D\left(\frac{t}{a}, \frac{b}{a}, z\right) d\sigma(z) \right) d\sigma(b) \\ &= a^{-2\mu-1} \overline{(h_\mu \psi)}(a\xi) \int_0^\infty \left(\int_0^\infty D\left(\frac{t}{a}, \frac{b}{a}, z\right) j_\mu(b\xi) d\sigma(b) \right) \psi(z)\, d\sigma(z). \end{aligned}$$

Putting $\frac{b}{a} = v$, we get

$$\begin{aligned} & B_\psi^* \, B_\psi f(t) \\ &= a^{-2\mu-1} \overline{(h_\mu \psi)}(a\xi) \int_0^\infty \left(\int_0^\infty D\left(\frac{t}{a}, v, z\right) j_\mu(va\xi) a^{2\mu+1} d\sigma(v) \right) \psi(z)\, d\sigma(z) \\ &= \overline{(h_\mu \psi)}(a\xi) \int_0^\infty j_\mu(t\xi)\, j_\mu(za\xi)\, \psi(z) d\sigma(z) \\ &= \overline{(h_\mu \psi)}(a\xi)\ (h_\mu \psi)(a\xi)\, j_\mu(t\xi) \\ &= |(h_\mu \psi)(a\xi)|^2\, j_\mu(t\xi). \end{aligned}$$

Hence

$$\int_0^\infty B_\psi^* \, B_\psi \, f(t)\ \frac{d\sigma(a)}{a^{2\mu+1}} = j_\mu(t\xi) \int_0^\infty \frac{|(h_\mu \psi)(a\xi)|^2}{a^{2\mu+1}} d\sigma(a).$$

Therefore, from the above expression, we get

$$f(t) \;=\; j_\mu(t\xi) \;=\; \frac{\int_0^\infty B_\psi^*\, B_\psi\, f(t)\; \frac{d\sigma(a)}{a^{2\mu+1}}}{\int_0^\infty \frac{|(h_\mu\psi)(a\xi)|^2}{a^{2\mu+1}} d\sigma(a)}$$
$$= \frac{\int_0^\infty B_\psi^*\, B_\psi\, f(t)\; \frac{d\sigma(a)}{a^{2\mu+1}}}{\int_0^\infty \frac{|(h_\mu\psi)(\nu)|^2}{\nu^{2\mu+1}} d\sigma(\nu)}.$$

From (2.3.2), we can write the following representation

$$f(t) = \int_0^\infty B_\psi^*\, B_\psi\, f(t)\; \frac{d\sigma(a)}{a^{2\mu+1}},$$

for $f(t) = j_\mu(t\xi)$. □

Theorem 2.3.2. *Suppose that $\psi \in L^2_\sigma(I)$ is a continuum Bessel wavelet with*

$$A_\psi := \int_0^\infty \omega^{-2\mu-1}\, |(h_\mu\psi)(\omega)|^2\, d\sigma(\omega) = 1. \tag{2.3.3}$$

Then, for $f \in L^2_\sigma(I)$ following inversion formula holds

$$f(x) = \lim_{\epsilon\to 0, A, B\to\infty} \int_{\epsilon<a<A, b<B} (B_\psi f)(b,a)\psi_{b,a}(x)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}, \tag{2.3.4}$$

where $S(\epsilon, A, B)f = \int_{\epsilon<a<A,b<B}(B_\psi f)(b,a)\psi_{b,a}(x)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}$.

Proof. Let the integral in (2.3.4) belongs in $L^2_\sigma(\mathbb{R}^2_+, \frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}})$.

Now, we have

$$\|f - S(\epsilon, A, B)f\|_{2,\sigma} = \sup_{||g||_{2,\sigma}=1} |\langle f - S(\epsilon, A, B)f, g\rangle| .$$

Applying Fubini's theorem, we have

$$\langle S(\epsilon, A, B)f, g\rangle \;=\; \int_{\mathbb{R}_+} \bar{g}(x)\left(\int_{\epsilon<a<A,b<B}(B_\psi f)(b,a)\psi_{b,a}(x)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right) d\sigma(x)$$
$$= \int_{\epsilon<a<A,b<B}(B_\psi f)(b,a)\left(\int_{\mathbb{R}_+}\bar{g}(x)\psi_{b,a}(x)d\sigma(x)\right)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}$$

$$= \int_{\epsilon<a<A,b<B} (B_\psi f)(b,a)\,(\overline{B_\psi g})(b,a)\,\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}.$$

Thus, by Parseval's formula of the Hankel transform (1.1.14) and the Cauchy-Schwarz inequality (1.1.16), we have

$$\begin{aligned}
&|\langle f - S(\epsilon,A,B)f, g\rangle| \\
&= |\langle f,g\rangle - \langle S(\epsilon,A,B)f,g\rangle| \\
&= \Bigg| \int_{\mathbb{R}^2_+} (B_\psi f)(b,a)\,\overline{B_\psi g}(b,a)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}} \\
&\qquad - \int_{(\epsilon<a<A,b<B)} (B_\psi f)(b,a)\,\overline{B_\psi g}(b,a)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\Bigg| \\
&= \Bigg| \int_{(\epsilon<a<A,b<B)^c} (B_\psi f)(b,a)\,\overline{B_\psi g}(b,a)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\Bigg| \\
&\le \left(\int_{(\epsilon<a<A,b<B)^c} |(B_\psi f)(b,a)|^2\,\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right)^{1/2} \left(\int_{\mathbb{R}^2_+} |(B_\psi g)(b,a)|^2\,\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right)^{1/2} \\
&= \left(\int_{(\epsilon<a<A,b<B)^c} |(B_\psi f)(b,a)|^2\,\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right)^{1/2} A_\psi\,\|g\|_{2,\sigma}\,.
\end{aligned}$$

When $\epsilon \to 0$ and $A, B \to \infty$, the region of integration decreases to the empty set, hence the last integral tends to zero by the dominated convergence theorem.
This gives that

$$\|S(\epsilon,A,B)f - f\|_{2,\sigma} \to 0.$$

□

Theorem 2.3.3. *Suppose that $\psi \in L^2_\sigma(I)$ is a continuum Bessel wavelet which satisfies (2.3.3) and*

$$C_{\psi,\mu,s} := \int_0^\infty \frac{|(h_\mu\psi)(\xi)|^2}{\xi^{1+2\mu+2s}}d\sigma(\xi) < \infty,$$

for some $s > 0$.

Then

$$\int_0^\infty\int_0^\infty \frac{|(B_\psi f)(b,a)|^2}{a^{2\mu+2s+1}}d\sigma(a)d\sigma(b) = C_{\psi,\mu,s}\,\|f\|^2_{2,s}\,, \tag{2.3.5}$$

where $\|f\|_{2,s} = \int_0^\infty \xi^{2s}|(h_\mu f)(\xi)|^2 d\sigma(\xi)$ *is the Sobolev norm [28, p.290].*

Proof. From (1.16), we have

$$\begin{aligned}&\int_0^\infty [(B_\psi f)(b,a)\overline{(B_\psi g)}(b,a)]d\sigma(b)\\ &= \int_0^\infty h_\mu^{-1}\{(h_\mu f)(u)\,(\overline{h_\mu \psi})(au)\}(b)\,\overline{h_\mu^{-1}\{(h_\mu g)(u)(\overline{h_\mu \psi})(au)\}}(b)d\sigma(b).\end{aligned}$$

Now, using Parseval's formula of the Hankel transform(1.1.14), the above expression becomes

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{(B_\psi f)(b,a)\overline{(B_\psi g)}(b,a)}{a^{2\mu+2s+1}} d\sigma(a)d\sigma(b)\\ &= \int_0^\infty \int_0^\infty \frac{(h_\mu f)(u)(\overline{h_\mu \psi})(au)\overline{(h_\mu g)(u)(\overline{h_\mu \psi})(au)}}{a^{2\mu+2s+1}} d\sigma(a)d\sigma(u)\\ &= \int_0^\infty \int_0^\infty (h_\mu f)(u)(\overline{h_\mu g})(u)\ \frac{|(h_\mu \psi)(au)|^2}{a^{2\mu+2s+1}} d\sigma(a)d\sigma(u).\end{aligned}$$

If we take $f = g$, then we obtain

$$\int_0^\infty \int_0^\infty \frac{|(B_\psi f)(b,a)|^2}{a^{2\mu+2s+1}} d\sigma(a)d\sigma(b) = \int_0^\infty \int_0^\infty |(h_\mu f)(u)|^2\ \frac{|(h_\mu \psi)(au)|^2}{a^{2\mu+2s+1}} d\sigma(a)d\sigma(u).$$

Putting $au = \xi$, we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{|(B_\psi f)(b,a)|^2}{a^{2\mu+2s+1}} d\sigma(a)d\sigma(b)\\ &= \int_0^\infty |(h_\mu f)(u)|^2 \left(\int_0^\infty \frac{|(h_\mu \psi)(\xi)|^2}{\xi^{2\mu+2s+1}} d\sigma(\xi)\right) u^{2s} d\sigma(u)\\ &= A_{\psi,\mu,s} \int_0^\infty |(h_\mu f)(u)|^2\ u^{2s} d\sigma(u)\\ &= A_{\psi,\mu,s}\ ||f||_{2,s}^2.\end{aligned}$$

□

Chapter 3

The Bessel Wavelet Convolution involving Hankel Transform

3.1 Introduction

In recent years, many properties of the wavelet convolution are studied by exploiting the theory of Fourier transform. The wavelet convolution product is an important tool to explore the various characterizations of the wavelet transform which is extensively given in the book [24]. This theory helps to define the wavelet convolution associated with the wavelet transform [27].

In [26], the properties of Bessel wavelet convolution product is studied and its certain estimates are obtained by using the theory of Bessel wavelet transform. In this chapter, our main focus is to expose the Bessel wavelet convolution associated with the Bessel wavelet transform.

In the present chapter, we discuss the various properties of the Bessel wavelet convolution by taking the Bessel wavelet transform and the Hankel transform tools. The boundedness on generalized Sobolev space $B^{\mu}_{p,k}(I), 1 \leq p < \infty$, associated with the normalized Bessel wavelet transform is obtained.

3.2 The Bessel Wavelet Convolution

In this section, using the Hirschmanian theory of Hankel transform[12] various results of the Bessel wavelet convolution are obtained.

Theorem 3.2.1. *If* $\overline{(h_\mu\psi)}(a\omega)(h_\mu f)(\omega) \in L^1_\sigma(I)$ *and* $\overline{(h_\mu\psi)}(a\omega)(h_\mu g)(\omega) \in L^1_\sigma(I)$, $(h_\mu\psi)(a\omega) \neq 0$ *for* $a \in I$ *and* $(B_\psi f)(b,a) = (B_\psi g)(b,a)\,\forall(b,a) \in I \times I$*. Then* $f = g$ *a.e.*

Proof. Given that

$$(B_\psi f)(b,a) = (B_\psi g)(b,a)\ \forall(b,a) \in I \times I. \tag{3.2.1}$$

Then from (1.2.6), we have

$$h_\mu^{-1}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\right](b) = h_\mu^{-1}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](b). \tag{3.2.2}$$

From [11, Corollary 2.9], we get

$$\overline{(h_\mu\psi)}(a\omega)(h_\mu f)(\omega) = \overline{(h_\mu\psi)}(a\omega)(h_\mu g)(\omega) \qquad a.e.$$

Since $(h_\mu\psi)(a\omega) \neq 0$, then we get

$$(h_\mu f)(\omega) = (h_\mu g)(\omega).$$

Again from [11, Corollary 2.9], we get

$$f = g \qquad a.e.$$

□

Theorem 3.2.2. *Let* $f, g \in L^1_\sigma(I)$ *and* $\psi \in L^1_\sigma(I)$*, then*

$$B_\psi(f \otimes g)(b,a) = (B_\psi f)(b,a)(B_\psi g)(b,a) \tag{3.2.3}$$

holds.

Proof. In view of [26, p.271], we have

$$(\overline{h_\mu \psi})(a\omega)\, h_\mu \left[(f \otimes g)\right](\omega) = \left[(\overline{h_\mu \psi})(a\cdot)(h_\mu f)(\cdot)\#(\overline{h_\mu \psi})(a\cdot)(h_\mu g)(\cdot)\right](\omega). \quad (3.2.4)$$

Multiplying both sides of the above equation by $j_\mu(b\omega)$ and integrating over I, we get

$$\begin{aligned}
&\int_0^\infty j_\mu(b\omega)(\overline{h_\mu \psi})(a\omega)\, h_\mu \left[(f \otimes g)\right](\omega) d\sigma(\omega) \\
&= \int_0^\infty j_\mu(b\omega) \left[(\overline{h_\mu \psi})(a\cdot)(h_\mu f)(\cdot)\#(\overline{h_\mu \psi})(a\cdot)(h_\mu g)(\cdot)\right](\omega) d\sigma(\omega) \\
&= \int_0^\infty j_\mu(b\omega) \Bigg[\int_0^\infty \int_0^\infty (\overline{h_\mu \psi})(az)(h_\mu f)(z).(\overline{h_\mu \psi})(ay)(h_\mu g)(y) \\
&\qquad D(\omega, y, z) d\sigma(y) d\sigma(z)\Bigg] d\sigma(\omega) \\
&= \int_0^\infty \int_0^\infty \left(\int_0^\infty j_\mu(b\omega) D(\omega, y, z) d\sigma(\omega)\right) (\overline{h_\mu \psi})(az)(h_\mu f)(z) \\
&\qquad (\overline{h_\mu \psi})(ay)(h_\mu g)(y) d\sigma(y) d\sigma(z).
\end{aligned}$$

From (1.1.5), we have

$$\begin{aligned}
&\int_0^\infty j_\mu(b\omega)(\overline{h_\mu \psi})(a\omega)\, h_\mu \left[(f \otimes g)\right](\omega) d\sigma(\omega) \\
&= \int_0^\infty \int_0^\infty j_\mu(yb) j_\mu(zb)(\overline{h_\mu \psi})(az)(h_\mu f)(z)(\overline{h_\mu \psi})(ay)(h_\mu g)(y) d\sigma(y) d\sigma(z) \\
&= \int_0^\infty j_\mu(zb)(\overline{h_\mu \psi})(az)(h_\mu f)(z) d\sigma(z) \int_0^\infty j_\mu(yb)(\overline{h_\mu \psi})(ay)(h_\mu g)(y) d\sigma(y) \\
&= h_\mu^{-1}\left[(\overline{h_\mu \psi})(a\cdot)(h_\mu f)(\cdot)\right](b)\;\; h_\mu^{-1}\left[(\overline{h_\mu \psi})(a\cdot)(h_\mu g)(\cdot)\right](b).
\end{aligned}$$

From (1.2.6), we get

$$B_\psi(f \otimes g)(b, a) = (B_\psi f)(b, a)\,(B_\psi g)(b, a).$$

□

Lemma 3.2.3. *Let $\psi \in L^1_\sigma(I)$. Then*

$$\int_0^\infty D(z,u,at)\psi(t)d\sigma(t) = \psi_{u,a}(z). \tag{3.2.5}$$

Proof. We have

$$\int_0^\infty D(z,u,at)\psi(t)d\sigma(t) = \int_0^\infty \left\{\int_0^\infty j_\mu(z\omega)j_\mu(u\omega)j_\mu(at\omega)d\sigma(\omega)\right\}\psi(t)d\sigma(t).$$

Putting $a\,\omega = s$, we get the following expression

$$\begin{aligned}\int_0^\infty D(z,u,at)\psi(t)d\sigma(t) &= \frac{1}{a^{2\mu+1}}\int_0^\infty \left\{\int_0^\infty j_\mu\left(\frac{zs}{a}\right) j_\mu\left(\frac{us}{a}\right) j_\mu(ts)\, d\sigma(s)\right\}\psi(t)d\sigma(t)\\ &= \frac{1}{a^{2\mu+1}}\int_0^\infty D\left(\frac{z}{a},\frac{u}{a},t\right)\psi(t)d\sigma(t)\\ &= \psi_{u,a}(z).\end{aligned}$$

□

Theorem 3.2.4. *Let $f, g \in L^1_\sigma(I)$ and assume that*

$$D_\psi(x,y,z) = \frac{1}{A_\psi}\int_0^\infty\int_0^\infty \overline{\psi}_{b,a}(x)\overline{\psi}_{b,a}(y)\psi_{b,a}(z)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}, \tag{3.2.6}$$

where $A_\psi = \int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}}d\sigma(a)$. Then the Bessel wavelet convolution will be in the following form :

$$(f\otimes g)(z) = \int_0^\infty\int_0^\infty D_\psi(x,y,z)\, f(x)\; g(y)\, d\sigma(x)d\sigma(y). \tag{3.2.7}$$

Proof. Since

$$D_\psi(x,y,z) = \frac{1}{A_\psi}\int_0^\infty\int_0^\infty \overline{\psi}_{b,a}(x)\overline{\psi}_{b,a}(y)\psi_{b,a}(z)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}},$$

from the inversion formula of Bessel wavelet transform, we have

$$D_\psi(x,y,z) = B_\psi^{-1}\left[\overline{\psi}_{b,a}(x)\overline{\psi}_{b,a}(y)\right](z).$$

The above expression implies that

$$\int_0^\infty D_\psi(x,y,z)\overline{\psi}_{b,a}(z)d\sigma(z) = \overline{\psi}_{b,a}(x)\overline{\psi}_{b,a}(y). \tag{3.2.8}$$

Multiplying both sides of (3.2.4) by $\frac{(h_\mu\psi)(a\omega)}{a^{2\mu+1}}$ and integrating 0 to ∞ with respect to a, we get

$$\begin{aligned}
&\int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}} d\sigma(a)\ h_\mu\left(f\otimes g\right)(\omega)\\
&= \int_0^\infty (h_\mu\psi)(a\omega)\frac{d\sigma(a)}{a^{2\mu+1}}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\\
&A_\psi\ h_\mu\left(f\otimes g\right)(\omega)\\
&= \int_0^\infty (h_\mu\psi)(a\omega)\frac{d\sigma(a)}{a^{2\mu+1}}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega).
\end{aligned}$$

From the inversion formula of Hankel transform, we get

$$\begin{aligned}
&A_\psi\ \left(f\otimes g\right)(z)\\
&= h_\mu^{-1}\left[\int_0^\infty (h_\mu\psi)(a\omega)\frac{d\sigma(a)}{a^{2\mu+1}}\right.\\
&\qquad\left.\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\right](z)\\
&= \int_0^\infty j_\mu(z\omega)\left[\int_0^\infty (h_\mu\psi)(a\omega)\frac{d\sigma(a)}{a^{2\mu+1}}\right.\\
&\qquad\left.\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\right]d\sigma(\omega)\\
&= \int_0^\infty j_\mu(z\omega)d\sigma(\omega)\int_0^\infty\left(\int_0^\infty j_\mu(a\omega t)\psi(t)d\sigma(t)\right)\frac{d\sigma(a)}{a^{2\mu+1}}\\
&\qquad\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}}\int_0^\infty d\sigma(\omega)\left(\int_0^\infty j_\mu(z\omega)j_\mu(a\omega t)\psi(t)d\sigma(t)\right)\\
&\qquad\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)\\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}}\int_0^\infty d\sigma(\omega)\int_0^\infty\left(\int_0^\infty D(z,at,u)j_\mu(u\omega)d\sigma(u)\right)\psi(t)d\sigma(t)\\
&\qquad\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot)\right](\omega)
\end{aligned}$$

$$= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty d\sigma(\omega) \int_0^\infty j_\mu(u\omega) \left(\int_0^\infty D(z, at, u)\psi(t)d\sigma(t) \right) d\sigma(u)$$
$$\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot) \right](\omega).$$

Using Lemma 3.2.3, we have

$$\begin{aligned}
&A_\psi \ (f \otimes g)(z) \\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty d\sigma(\omega) \int_0^\infty j_\mu(u\omega)\psi_{u,a}(z)d\sigma(u) \\
&\qquad \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot) \right](\omega) \\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty \psi_{u,a}(z)d\sigma(u) \int_0^\infty j_\mu(u\omega) \\
&\qquad \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot) \right](\omega)d\sigma(\omega) \\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty \psi_{u,a}(z)d\sigma(u) \ h_\mu^{-1} \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\#\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot) \right](u) \\
&= \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty \psi_{u,a}(z)d\sigma(u) \ h_\mu^{-1} \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot) \right](u) \\
&\qquad h_\mu^{-1} \left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu g)(\cdot) \right](u).
\end{aligned}$$

From (1.2.6), we can write

$$A_\psi \ (f \otimes g)(z) = \int_0^\infty \frac{d\sigma(a)}{a^{2\mu+1}} \int_0^\infty \psi_{u,a}(z)d\sigma(u) \left(B_\psi f\right)(u, a) \left(B_\psi g\right)(u, a)$$

$$\begin{aligned}
&= \int_0^\infty \int_0^\infty \psi_{u,a}(z)\frac{d\sigma(a)d\sigma(u)}{a^{2\mu+1}} \left\{ \int_0^\infty f(x)\overline{\psi}_{u,a}(x)d\sigma(x) \right\} \\
&\qquad \left\{ \int_0^\infty g(y)\overline{\psi}_{u,a}(y)d\sigma(y) \right\} \\
&= \int_0^\infty \int_0^\infty f(x)g(y)d\sigma(x)d\sigma(y) \\
&\qquad \int_0^\infty \int_0^\infty \overline{\psi}_{u,a}(x) \ \overline{\psi}_{u,a}(y) \ \psi_{u,a}(z)\frac{d\sigma(a)d\sigma(u)}{a^{2\mu+1}}.
\end{aligned}$$

From (3.2.6), we get

$$(f \otimes g)(z) = \int_0^\infty \int_0^\infty D_\psi(x,y,z)\, f(x)\; g(y)\, d\sigma(x) d\sigma(y).$$

□

Lemma 3.2.5. *If $\psi \in L^2_\sigma(I)$, then*

$$(h_\mu \psi_{b,a})(\omega) = j_\mu(b\omega)(h_\mu\psi)(a\omega). \tag{3.2.9}$$

Proof. We have

$$\begin{aligned}(h_\mu \psi_{b,a})(\omega) &= \int_0^\infty j_\mu(\omega t)\psi_{b,a}(t) d\sigma(t)\\ &= \int_0^\infty j_\mu(\omega t) a^{-2\mu-1} \int_0^\infty \psi(z) D\left(\frac{t}{a}, \frac{b}{a}, z\right) d\sigma(z) d\sigma(t).\end{aligned}$$

Putting $\frac{t}{a} = x$, we get

$$\begin{aligned}(h_\mu \psi_{b,a})(\omega) &= \int_0^\infty j_\mu(\omega a x) \int_0^\infty \psi(z) D\left(x, \frac{b}{a}, z\right) d\sigma(z) d\sigma(x)\\ &= \int_0^\infty \psi(z) \left(\int_0^\infty j_\mu(\omega a x) D\left(x, \frac{b}{a}, z\right) d\sigma(x)\right) d\sigma(z)\\ &= \int_0^\infty \psi(z)\, j_\mu(b\omega)\, j_\mu(za\omega) d\sigma(z)\\ &= j_\mu(b\omega) \int_0^\infty \psi(z)\, j_\mu(za\omega) d\sigma(z)\\ &= j_\mu(b\omega)(h_\mu\psi)(a\omega).\end{aligned}$$

□

Theorem 3.2.6. *If $f \in L^2_\sigma(I)$, then f can be reconstructed by the formula*

$$f(t) = \frac{1}{A_\psi} \int_0^\infty \int_0^\infty (B_\psi f)(b,a) \psi_{b,a}(t) \frac{d\sigma(a) d\sigma(b)}{a^{2\mu+1}}, \tag{3.2.10}$$

where $\psi \in L^2_\sigma(I)$ be a basic wavelet satisfies admissibility condition A_ψ.

Proof. If $f \in L^2_\sigma(I)$, then we have

$$\frac{1}{A_\psi}\int_0^\infty\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}} = \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)d\sigma(b)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

Using Parseval's formula of the Hankel transform (1.1.14), we get

$$\frac{1}{A_\psi}\int_0^\infty\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}} = \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty h_\mu[(B_\psi f)(b,a)](\omega)(h_\mu\psi_{b,a})(\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

From (1.2.6) and Lemma 3.2.5, we have

$$\begin{aligned}
&\frac{1}{A_\psi}\int_0^\infty\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\\
&= \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty (\overline{h_\mu\psi})(a\omega)(h_\mu f)(\omega)j_\mu(b\omega)(h_\mu\psi)(a\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}}\\
&= \frac{1}{A_\psi}\int_0^\infty\left(\int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}}d\sigma(a)\right)(h_\mu f)(\omega)j_\mu(b\omega)d\sigma(\omega)\\
&= \frac{1}{A_\psi}\int_0^\infty A_\psi\ (h_\mu f)(\omega)j_\mu(b\omega)d\sigma(\omega)\\
&= h_\mu^{-1}[(h_\mu f)](b)\\
&= f(t).
\end{aligned}$$

□

Theorem 3.2.7. *If $f \in L^2_\sigma(I)$, then the following Calderón's reproducing identity holds:*

$$f(t) = \frac{1}{A_\psi}\int_0^\infty\left(f\#\overline{\psi_a}\#\psi_a\right)\frac{d\sigma(a)}{a^{2\mu+1}}, \tag{3.2.11}$$

for a Bessel wavelet $\psi \in L^2_\sigma(I)$.

Proof. From (3.2.10), we have

$$\begin{aligned} f(t) &= \frac{1}{A_\psi}\int_0^\infty \int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)\frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}} \\ &= \frac{1}{A_\psi}\int_0^\infty \left(\int_0^\infty (B_\psi f)(b,a)\psi_{b,a}(t)d\sigma(b)\right)\frac{d\sigma(a)}{a^{2\mu+1}}. \end{aligned}$$

Using Parseval's formula of the Hankel transform (1.1.14), we get

$$f(t) = \frac{1}{A_\psi}\int_0^\infty \left(\int_0^\infty h_\mu[(B_\psi f)(b,a)](\omega)(h_\mu\psi_{b,a})(\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

From (1.2.6) and Lemma 3.2.5, we have

$$f(t) = \frac{1}{A_\psi}\int_0^\infty \left(\int_0^\infty (\overline{h_\mu\psi})(a\omega)(h_\mu f)(\omega)j_\mu(b\omega)(h_\mu\psi)(a\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}}.$$

Using (1.1.10), we get the following expression

$$\begin{aligned} f(t) &= \frac{1}{A_\psi}\int_0^\infty \left(\int_0^\infty j_\mu(b\omega)h_\mu(f\#\overline{\psi_a})(\omega)(h_\mu\psi)(a\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}} \\ &= \frac{1}{A_\psi}\int_0^\infty \left(\int_0^\infty j_\mu(b\omega)h_\mu(f\#\overline{\psi_a}\#\psi_a)(\omega)d\sigma(\omega)\right)\frac{d\sigma(a)}{a^{2\mu+1}} \\ &= \frac{1}{A_\psi}\int_0^\infty (f\#\overline{\psi_a}\#\psi_a)(t)\frac{d\sigma(a)}{a^{2\mu+1}}. \end{aligned}$$

□

Theorem 3.2.8. *Let* $f \in L^2_\sigma(I)$ *and* $\psi \in L^2_\sigma(I)$ *satisfying admissibility condition* $A_\psi = \int_0^\infty \frac{|(h_\mu\psi)(a\omega)|^2}{a^{2\mu+1}}d\sigma(a)$, *then the following reproducing identity holds:*

$$f(t) = \frac{1}{A_\psi}\int_0^\infty \int_0^\infty \left(f \otimes \overline{\psi}_{b,a} \otimes \psi_{b,a}\right)(t)\,\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}}. \tag{3.2.12}$$

Proof. Assume that ϕ is an orthonormal wavelet in $L^2_\sigma(I)$. Taking the Bessel wavelet transform of right hand side of (3.2.12) with respect to ϕ, we have

$$\begin{aligned}
& B_\phi\left[\frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left(f\otimes\overline{\psi}_{b,a}\otimes\psi_{b,a}\right)(t)\ \frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}}\right](b^{'},a^{'}) \\
&= \frac{1}{A_\psi}\int_0^\infty\int_0^\infty B_\phi\left\{\left(f\otimes\overline{\psi}_{b,a}\otimes\psi_{b,a}\right)(t)\right\}(b^{'},a^{'})\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}} \\
&= (B_\phi f)(b^{'},a^{'})\frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left(B_\phi\overline{\psi}_{b,a}\right)(b^{'},a^{'})\left(B_\phi\psi_{b,a}\right)(b^{'},a^{'})\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}} \\
&= (B_\phi f)(b^{'},a^{'})\frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left[\int_0^\infty \overline{\psi}_{b,a}(t)\overline{\phi}_{b',a'}(t)d\sigma(t)\right] \\
&\qquad \left[\int_0^\infty \psi_{b,a}(x)\overline{\phi}_{b',a'}(x)d\sigma(x)\right]\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}} \\
&= (B_\phi f)(b^{'},a^{'})\int_0^\infty \frac{1}{A_\psi}\left(\int_0^\infty\int_0^\infty \left[\left(B_\psi\overline{\phi}_{b',a'}\right)(b,a)\psi_{b,a}(x)\right]\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}}\right) \\
&\qquad \overline{\phi}_{b',a'}(x)d\sigma(x) \\
&= (B_\phi f)(b^{'},a^{'})\int_0^\infty \overline{\phi}_{b',a'}(x)\ \overline{\phi}_{b',a'}(x)d\sigma(x) \\
&= (B_\phi f)(b^{'},a^{'})\int_0^\infty \left[\overline{\phi}_{b',a'}(x)\right]^2 d\sigma(x) \\
&= (B_\phi f)(b^{'},a^{'}) \qquad \text{(by orthogonality of } \phi).
\end{aligned}$$

$$\begin{aligned}
\frac{1}{A_\psi}\int_0^\infty\int_0^\infty \left(f\otimes\overline{\psi}_{b,a}\otimes\psi_{b,a}\right)(t)\ \frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}} &= B_\phi^{-1}\left[(B_\phi f)(b^{'},a^{'})\right](t) \\
&= f(t).
\end{aligned}$$

□

3.3 Generalized Sobolev Space

Let $\psi \in L^2_\sigma(I)$ be an analysing Bessel wavelet which satisfies (1.2.8).
The integral

$$\begin{aligned}(L_\psi f)(b,a) &= \frac{1}{\sqrt{A_\psi}}(B_\psi f)(b,a) = \frac{1}{\sqrt{A_\psi}}\langle f, \psi_{b,a}\rangle \\ &= \frac{1}{\sqrt{A_\psi}}\int_0^\infty f(t)\overline{\psi}_{b,a}(t)d\sigma(t),\end{aligned}$$

defines an element of $L^2\left(I \times I, \frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right)$.
The Hankel transform of L_ψ is given as

$$h_\mu\left[(L_\psi f)(b,a)\right](\omega) = \frac{1}{\sqrt{A_\psi}}\overline{(h_\mu\psi)}(a\omega)\,(h_\mu f)(\omega). \tag{3.3.1}$$

The operator L_ψ is also called a normalized form of the Bessel wavelet operator B_ψ and

$$L_\psi : L^2(I, d\sigma(t)) \to L^2\left(I \times I, \frac{d\sigma(a)d\sigma(b)}{a^{2\mu+1}}\right),$$

is an isometry [21, p.245].
In this section, we are exploiting the results of [24] and study the normalized Bessel wavelet transform $L_\psi f$, which is defined on $L^2_\sigma(I, d\sigma(t))$ to generalized Sobolev space $B^\mu_{p,k}(I)$ and the space of its image set is denoted by $W^\mu_{p,k}$. The boundedness and other properties of $L_\psi f$ are given on $B^\mu_{p,k}(I)$ space.

Definition 3.3.1. The Zemanian space $H_\mu(I), I = (0,\infty)$ is the set of all infinitely differentiable functions ϕ on $(0,\infty)$ such that

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in(0,\infty)}\left|x^m\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-\frac{1}{2}}\phi(x)\right| < \infty, \tag{3.3.2}$$

for all $m, k \in \mathbb{N}_0$. Then $f \in H'_\mu(I)$ is defined by the following way:

$$\langle f, \phi\rangle = \int_0^\infty f(x)\phi(x)d\sigma(x), \quad \phi \in H_\mu(I). \tag{3.3.3}$$

Definition 3.3.2. Let $k(\xi)$ be an arbitrary weight function. The generalized Sobolev space $B^{\mu}_{p,k}(I), 1 \leq p < \infty$ is defined to be the space of all ultra-distributions $f \in H^{'}_{\mu}(I), I = (0, \infty)$ such that

$$\|f\|_{p,k} = \left(\int_0^{\infty} |k(\xi) \, (h_\mu f) \, (\xi)|^p \, d\sigma(\xi) \right)^{1/p} < \infty \tag{3.3.4}$$

and

$$\|f\|_{\infty,k} = ess\,sup\,k(\xi) \, |(h_\mu f)(\xi)| \, . \tag{3.3.5}$$

Definition 3.3.3. Define the space $W^{\mu}_{p,k}$ of all measurable functions f on $I \times I$ such that

$$\|f(b,a)\|_{W^{\mu}_{p,k}} = \left(\int_I \|f(b,a)\|^p_{p,k} \, \frac{d\sigma(a)}{a^{2\mu+1}} \right)^{1/p} < \infty, \tag{3.3.6}$$

$1 \leq p < \infty$, $a \in (0, \infty)$.

Theorem 3.3.4. *Assume that analysing wavelet ψ satisfies the following admissibility condition:*

$$A_{\psi,p} = \int_0^{\infty} \frac{|(h_\mu \psi) \, (\xi)|^p}{\xi^{2\mu+1}} d\sigma(\xi) < \infty. \tag{3.3.7}$$

Let $(L_\psi f) \, (b,a)$ be the normalized Bessel wavelet transform of the function $f \in B^{\mu}_{p,k}(I)$, with respect to the analysing wavelet ψ satisfying (3.3.7). Then

$$\|(L_\psi f) \, (b,a)\|_{W^{\mu}_{p,k}} = C_p \, \|f\|_{p,k} \, , \tag{3.3.8}$$

where $C_p = (A_\psi)^{-p/2} \, A_{\psi,p}$.

Proof. Let $f \in H_\mu(I)$.

Then

$$\begin{aligned} \|(L_\psi f) \, (b,a)\|^p_{W^{\mu}_{p,k}} &= \int_0^{\infty} \|(L_\psi f) \, (b,a)\|^p_{k,p} \, \frac{d\sigma(a)}{a^{2\mu+1}} \\ &= \int_0^{\infty} \left(\int_0^{\infty} |k(\xi)|^p \, |h_\mu \, [(L_\psi f) \, (b,a)] \, (\xi)|^p \, d\sigma(\xi) \right) \frac{d\sigma(a)}{a^{2\mu+1}}. \end{aligned}$$

From (3.3.1), we have

$$\begin{aligned}
\|(L_\psi f)(b,a)\|^p_{W^\mu_{p,k}} &= \int_0^\infty \left(\int_0^\infty |k(\xi)|^p \frac{1}{A_\psi^{p/2}} |(h_\mu \psi)(a\xi)|^p |(h_\mu f)(\xi)|^p \, d\sigma(\xi) \right) \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= \frac{1}{A_\psi^{p/2}} \int_0^\infty \left(\int_0^\infty |k(\xi)|^p |(h_\mu f)(\xi)|^p \, d\sigma(\xi) \right) |(h_\mu \psi)(a\xi)|^p \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= \frac{1}{A_\psi^{p/2}} \int_0^\infty \|f\|^p_{p,k} |(h_\mu \psi)(a\xi)|^p \frac{d\sigma(a)}{a^{2\mu+1}}.
\end{aligned}$$

Putting $a\xi = u$, we have

$$\begin{aligned}
\|(L_\psi f)(b,a)\|^p_{W^\mu_{k,p}} &= \frac{1}{A_\psi^{p/2}} \int_0^\infty \frac{|(h_\mu \psi)(u)|^p}{u^{2\mu+1}} d\sigma(u) \, \|f\|^p_{p,k} \\
&= \frac{1}{A_\psi^{p/2}} A_{\psi,p} \, \|f\|^p_{p,k} \\
&= C_p \, \|f\|^p_{p,k} \, .
\end{aligned}$$

Since $H_\mu(I)$ is dense in $B^\mu_{p,k}(I)$, the above result can be extended to all $f \in B^\mu_{p,k}(I)$.

□

Theorem 3.3.5. *Let $f \in B^\mu_{p,k}(I)$ and $\psi \in L^1_\sigma(I)$ with $\int_0^\infty \psi(t) d\sigma(t) = 1$. Then $(B_\psi f)(.,a) \to f(.)$ in $B^\mu_{p,k}(I)$ as $a \to 0$.*

Proof. From (3.3.4), we have

$$\begin{aligned}
\|f \# \psi_a - f\|^p_{p,k} &= \int_0^\infty |h_\mu (f \# \psi_a - f)(\xi)|^p |k(\xi)|^p \, d\sigma(\xi) \\
&= \int_0^\infty |h_\mu (f \# \psi_a)(\xi) - (h_\mu f)(\xi)|^p |k(\xi)|^p \, d\sigma(\xi) \\
&= \int_0^\infty |(h_\mu f)(\xi) (h_\mu \psi)(a\xi) - (h_\mu f)(\xi)|^p |k(\xi)|^p \, d\sigma(\xi) \\
&= \int_0^\infty |(h_\mu f)(\xi) k(\xi)|^p |(h_\mu \psi)(a\xi) - 1|^p \, d\sigma(\xi) \\
&= \int_0^\infty |I(a,\xi)|^p \, d\sigma(\xi),
\end{aligned}$$

where $I(a,\xi) = (h_\mu f)(\xi)\, k(\xi)\, [(h_\mu \psi)(a\xi) - 1]$.
Under our assumption $\int_0^\infty \psi(t) d\sigma(t) = 1$, we have $\lim_{a\to 0} |I(a,\xi)| = 0$ a.e.
Set $M = \sup_{\xi \in I} |(h_\mu \psi)(a\xi) - 1|$, which is independent of a.
Then

$$|I(a,\xi)| \leq M\, |(h_\mu f)(\xi)\, k(\xi)|\,.$$

Now, applying the dominated convergence theorem, we have
$(B_\psi f)(.,a) = (f \# \psi_a)(.,a) \to f(.)$ in $B^\mu_{p,k}(I)$ as $a \to 0$. □