Chapter 1

Introduction

The Fourier analysis is a powerful tool to study those functions that may be represented by the sum of simpler trigonometric functions. This theory came into light from the work of Joseph Fourier, who showed that a periodic function can be expressed as the sum of trigonometric functions.

For $f \in L^1(\mathbb{R})$, the Fourier transform of $f(t)$ is denoted as $\hat{f}(\omega)$ and defined by

$$\hat{f}(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt. \tag{1.0.1}$$

The basic concept of the applicability of Fourier transform is that it decomposes a function into sinusoidal basis functions which are complex exponential of different frequencies. But there are some disadvantages of the Fourier transform. First, the Fourier transform of a signal does not provide local information in the sense that it does not reflect the change of wave number with space or of the frequency with time. Second, the Fourier transform enables us to investigate problems either in the time domain or frequency domain but not simultaneously in the both domain. To remove these difficulties, the concept of wavelet transform is necessary. The wavelet transform provides us the local and global information both.

A function ψ which satisfies

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \tag{1.0.2}$$

represents a wavelet because (1.0.2) implies that ψ changes sign from $-\infty$ to $+\infty$ and it vanishes at $-\infty$ and $+\infty$.

Let $\psi \in L^2(\mathbb{R})$, the admissibility condition for a wavelet is

$$0 < C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \tag{1.0.3}$$

where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$.

The continuous wavelet transform of $f \in L^2(\mathbb{R})$ with respect to a wavelet $\psi \in L^2(\mathbb{R})$ is defined as

$$(W_\psi f)(a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(t)\overline{\psi}_{a,b}(t)dt, \tag{1.0.4}$$

where

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R},\ a \neq 0$$

and a, b are scaling and translation parameters respectively.

Let $f, g \in L^2(\mathbb{R})$. Then Parseval's relation of the wavelet transform is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a,b)\overline{(W_\psi g)}(a,b) \frac{dbda}{a^2} = C_\psi \langle f, g \rangle, \tag{1.0.5}$$

where C_ψ is defined in (1.0.3).

The inversion formula of wavelet transform for $f \in L^2(\mathbb{R})$ is

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a,b)\psi_{a,b}(t) \frac{dbda}{a^2}. \tag{1.0.6}$$

The theory of Hankel transform was introduced by Hirschman[12], Haimo[11], Zemanian [38] and others. This theory can be suitably applied for solving axisymmetric boundary value problems in cylindrical and spherical coordinates. Hankel transform is also used in the study of partial differential equations, Sobolev spaces, wavelets and other problems in applied mathematics and mathematical physics.

Using the Hirschmanian theory of Hankel transform and Hankel convolution, Pathak et al.[21] and Upadhyay[30] introduced the concept of Bessel wavelet transform and Bessel wavelet convolution product and found Parseval's formula, inversion formula of the Bessel wavelet transform and various norm inequalities for the Bessel wavelet convolution product.

Using the Bessel wavelet transform, many results are obtained in the present thesis. Various results and properties which are related to our present research work are given below:

1.1 Hankel Transform of Hirschman Type

I.I.Hirschman[12], Haimo[11], Cholewinski[4] introduced the theory of Hankel transform on the space $L^p_\sigma(0,\infty)$.

We define $L^p_\sigma(0,\infty)$, $1 \le p \le \infty$, as the space of those real measurable functions ϕ on $(0,\infty)$ for which

$$\begin{aligned} \|\phi\|_{p,\sigma} &= \left(\int_0^\infty |\phi(x)|^p \, d\sigma(x)\right)^{1/p} < \infty, \quad 1 \le p < \infty, \\ \|\phi\|_{\infty,\sigma} &= ess \sup_{0<x<\infty} |\phi(x)| < \infty, \end{aligned}$$

where

$$d\sigma(x) = \frac{x^{2\mu}}{2^{\mu-1/2}\,\Gamma\left(\mu+1/2\right)}\, dx, \quad \mu > 0.$$

For $f \in L^1_\sigma(0,\infty)$, the Hankel transform is defined by

$$(h_\mu f)(t) = \int_0^\infty f(x) j_\mu(xt) d\sigma(x), \quad 0 < t < \infty, \tag{1.1.1}$$

where

$$j_\mu(x) = 2^{\mu-1/2}\,\Gamma(\mu+1/2)\,x^{1/2-\mu}\,J_{\mu-1/2}(x) \tag{1.1.2}$$

and $J_{\mu-1/2}(x)$ is the Bessel function of first kind of order $\mu - 1/2$.
The Hankel transform of $f \in L^1_\sigma(0,\infty)$ is bounded and continuous on $[0,\infty)$ with

$$\|(h_\mu f)(t)\|_{\infty,\sigma} \le \|f\|_{1,\sigma}\,. \tag{1.1.3}$$

If $f \in L^1_\sigma(0,\infty)$ and if $h_\mu f \in L^1_\sigma(0,\infty)$, then the inverse Hankel transform is given by

$$f(x) = (h_\mu^{-1} f)(x) = \int_0^\infty j_\mu(xt)(h_\mu f)(t)d\sigma(t), \quad 0 < x < \infty. \tag{1.1.4}$$

The basic function $D(x,y,z)$ is defined as

$$\begin{aligned} D(x,y,z) &:= \int_0^\infty j_\mu(xt)j_\mu(yt)j_\mu(zt)d\sigma(t) \\ &= \frac{2^{3\mu-5/2}\,[\Gamma(\mu+1/2)]^2}{\Gamma(\mu)\,\pi^{1/2}}\,(xyz)^{-2\mu+1}[\triangle(x,y,z)]^{2\mu-2}, \quad \mu > 0, \end{aligned}$$

where $\triangle(x,y,z)$ is the area of a triangle with sides x, y, z if such a triangle exists and zero otherwise.
It is clear that $D(x,y,z) \ge 0$ and symmetric in x, y, z. Applying inversion formula (1.1.4), we get

$$\int_0^\infty j_\mu(zt)D(x,y,z)d\sigma(z) = j_\mu(xt)j_\mu(yt), \tag{1.1.5}$$

valid for $0 < x < \infty$, $0 < y < \infty$ and $0 \le t < \infty$.

Setting $t = 0$ in the above equation, we obtain

$$\int_0^\infty D(x,y,z)d\sigma(z) = 1. \tag{1.1.6}$$

The Hankel translation τ_y of $f \in L^p_\sigma(0,\infty)$ for $1 \le p \le \infty$, is given by

$$(\tau_y f)(x) = f(x,y) := \int_0^\infty D(x,y,z)f(z)d\sigma(z), \; 0 < x, y < \infty. \tag{1.1.7}$$

If $f, g \in L^1_\sigma(0,\infty)$, then Hankel convolution is defined as

$$(f\#g)(x) := \int_0^\infty \int_0^\infty f(z)g(y)D(x,y,z)d\sigma(y)d\sigma(z). \tag{1.1.8}$$

Some properties of the Hankel convolution are given below:

(i). Let $f, g \in L^1_\sigma(0,\infty)$, then

$$\|f\#g\|_{1,\sigma} \leq \|f\|_{1,\sigma}\|g\|_{1,\sigma} \tag{1.1.9}$$

$$h_\mu(f\#g)(x) = (h_\mu f)(x)(h_\mu g)(x). \tag{1.1.10}$$

(ii). Let $f \in L^1_\sigma(0,\infty)$ and $g \in L^p_\sigma(0,\infty)$, $p \geq 1$. Then $(f\#g)$ exists, is continuous and

$$\|f\#g\|_{p,\sigma} \leq \|f\|_{1,\sigma}\|g\|_{p,\sigma}. \tag{1.1.11}$$

(iii). Let $f \in L^p_\sigma(0,\infty)$, $g \in L^q_\sigma(0,\infty)$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $(f\#g)$ exists, is continuous and

$$\|f\#g\|_{\infty,\sigma} \leq \|f\|_{p,\sigma}\|g\|_{q,\sigma}. \tag{1.1.12}$$

(iv). Let $f \in L^p_\sigma(0,\infty)$ and $g \in L^q_\sigma(0,\infty)$, $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$. Then $(f\#g)$ exists, is continuous and

$$\|f\#g\|_{r,\sigma} \leq \|f\|_{p,\sigma}\|g\|_{q,\sigma}. \tag{1.1.13}$$

Hankel transform is an isometric on $L^2_\sigma(0,\infty)$. Parseval's formula of the Hankel transform for $f, g \in L^2_\sigma(0,\infty)$ is given by

$$\int_0^\infty f(x)g(x)d\sigma(x) = \int_0^\infty (h_\mu f)(y)(h_\mu g)(y)d\sigma(y). \tag{1.1.14}$$

The above relation also holds for $f, g \in L^1_\sigma(0,\infty)$.

For $f \in L^p_\sigma(0,\infty)$ and $g \in L^q_\sigma(0,\infty)$, $1 < p, q < \infty$ with $\frac{1}{p}+\frac{1}{q}=1$, we have the

Hölder's inequality

$$\int_0^\infty |f(x)g(x)|\,d\sigma(x) \le \left(\int_0^\infty |f(x)|^p\,d\sigma(x)\right)^{1/p}\left(\int_0^\infty |g(x)|^q\,d\sigma(x)\right)^{1/q}. \quad (1.1.15)$$

If we take $p = q = 2$ in (1.1.15), we get

$$\int_0^\infty |f(x)g(x)|\,d\sigma(x) \le \left(\int_0^\infty |f(x)|^2\,d\sigma(x)\right)^{1/2}\left(\int_0^\infty |g(x)|^2\,d\sigma(x)\right)^{1/2}. \quad (1.1.16)$$

This inequality is called Cauchy- Schwarz inequality.

1.2 The Continuous Bessel Wavelet Transform

From [21], for $\psi \in L^p_\sigma(0,\infty), 1 \le p < \infty$ the Bessel wavelet defined as

$$\begin{aligned} \psi_{b,a}(x) &:= D_a\tau_b\psi(x) = D_a\psi(b,x) = a^{-2\mu-1}\psi\left(\frac{b}{a},\frac{x}{a}\right) && (1.2.1)\\ &= a^{-2\mu-1}\int_0^\infty D(b/a,x/a,z)\psi(z)d\sigma(z), && (1.2.2)\end{aligned}$$

where $b \ge 0$ and $a > 0$.

The continuous Bessel wavelet transform of $f \in L^2_\sigma(0,\infty)$ with respect to $\psi \in L^2_\sigma(0,\infty)$ is given as

$$\begin{aligned} (B_\psi f)(b,a) &:= \langle f(t), \psi_{b,a}(t)\rangle \\ &= \int_0^\infty f(t)\overline{\psi}_{b,a}(t)d\sigma(t) && (1.2.3)\\ &= a^{-2\mu-1}\int_0^\infty\int_0^\infty f(t)\overline{\psi}(z)D\left(\frac{b}{a},\frac{t}{a},z\right)d\sigma(z)d\sigma(t). && (1.2.4)\end{aligned}$$

Now, (1.2.4) can be easily expressed in the form of Hankel convolution

$$(B_\psi f)(b,a) = (f\#\overline{\psi_a})(b), \quad (1.2.5)$$

where $\psi_a(t) = \frac{1}{a^{2\mu+1}}\,\psi(\frac{t}{a})$.
From (1.1.10), the continuous Bessel wavelet transform of a function $f \in L^1_\sigma(0,\infty) \cap L^2_\sigma(0,\infty)$ can be written in the following form:

$$(B_\psi f)(b,a) = h_\mu^{-1}\left[\overline{(h_\mu\psi)}(a\cdot)(h_\mu f)(\cdot)\right](b) \tag{1.2.6}$$

$$= \int_0^\infty j_\mu(b\omega)(h_\mu f)(\omega)\overline{(h_\mu\psi)}(a\omega)d\sigma(\omega). \tag{1.2.7}$$

Let $\psi \in L^2_\sigma(0,\infty)$ be basic wavelet which satisfies the admissibility condition

$$0 < A_\psi := \int_0^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+1}} d\sigma(\omega) < \infty, \tag{1.2.8}$$

and defines Bessel wavelet transform (1.2.3), then we have

$$\int_0^\infty \int_0^\infty (B_\psi f)(b,a)\overline{(B_\psi g)(b,a)}a^{-2\mu-1}d\sigma(a)d\sigma(b) = A_\psi \langle f, g\rangle, \tag{1.2.9}$$

for all $f, g \in L^2_\sigma(0,\infty)$. This equation is known as Parseval's formula of the Bessel wavelet transform.

1.3 Hankel Transform of Zemanian Type

The Hankel transform [35] of a classical function $f \in L^1(0,\infty)$ is defined by

$$(h_\mu f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x)dx, \qquad \mu \geq -\frac{1}{2}, \tag{1.3.1}$$

where $0 < y < \infty$, and kernel J_μ is the Bessel function of first kind of order μ.
Let $f \in L^1(0,\infty)$ and $h_\mu f \in L^1(0,\infty)$. Then the inversion formula of Hankel transform is given by

$$f(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)(h_\mu f)(y)dy. \tag{1.3.2}$$

For $\mu \geq -\frac{1}{2}$. If $f(x)$ and $G(y)$ are in $L^1(0,\infty)$, if $F(y) = h_\mu[f(x)]$, and if $g(x) = h_\mu^{-1}[G(y)]$, then

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F(y)G(y)dy. \tag{1.3.3}$$

Let $f, g \in L^1(0,\infty)$. Then the Hankel convolution of two functions f and g is defined as follows:

$$(f\#g)(x) = \int_0^\infty f(x,y)g(y)dy \tag{1.3.4}$$

$$= \int_0^\infty (\tau_x f)(y)g(y)dy, \tag{1.3.5}$$

where Hankel translation τ_x is given by

$$(\tau_x f)(y) = f(x,y) = \int_0^\infty f(z)D_\mu(x,y,z)dz \tag{1.3.6}$$

and

$$D_\mu(x,y,z) = \int_0^\infty t^{-\mu-\frac{1}{2}}(xt)^{1/2}J_\mu(xt)(yt)^{1/2}J_\mu(yt)(zt)^{1/2}J_\mu(zt)dt, \tag{1.3.7}$$

for $x, y, z \in (0,\infty)$.

Applying the inversion formula of Hankel transform to (1.3.7), we find the following result

$$\int_0^\infty t^{\mu+1/2}J_\mu(zt)D_\mu(x,y,z)dz = (xt)^{1/2}J_\mu(xt)(yt)^{1/2}J_\mu(yt). \tag{1.3.8}$$

For f and $g \in L^1(0,\infty)$, the Hankel convolution satisfies the following relations:

$$\|f\#g\|_{L^1(0,\infty)} \leq \|f\|_{L^1(0,\infty)}\,\|g\|_{L^1(0,\infty)} \tag{1.3.9}$$

and

$$h_\mu(f\#g)(x) = x^{-\mu-1/2}(h_\mu f)(x)\,(h_\mu g)(x). \tag{1.3.10}$$

1.4 The Bessel Wavelet Transform based on Zemanian Theory

A function $\psi \in L^2(0,\infty)$ is called a Bessel wavelet if it satisfies the admissibility condition

$$C_{\mu,\psi} = \int_0^\infty x^{-2\mu-2} \left|(h_\mu \psi)(x)\right|^2 dx < \infty, \quad \mu \geq -\frac{1}{2}. \tag{1.4.1}$$

The continuous Bessel wavelet transform of a function $f \in L^2(0,\infty)$ with respect to a Bessel wavelet $\psi \in L^2(0,\infty)$ is defined as

$$(B_\psi f)(b,a) = a^{\mu-1/2} \int_0^\infty f(t)\bar{\psi}\left(\frac{t}{a}, \frac{b}{a}\right) dt, \tag{1.4.2}$$

where $a > 0$ and $b \geq 0$ and

$$\psi\left(\frac{t}{a}, \frac{b}{a}\right) = \int_0^\infty \psi(z) D_\mu \left(\frac{t}{a}, \frac{b}{a}, z\right) dz. \tag{1.4.3}$$

If $f, \psi \in L^2(0,\infty)$, then using the techniques of [30], we have

$$(B_\psi f)(b,a) = \int_0^\infty (bx)^{1/2} J_\mu(bx)(h_\mu f)(x) x^{-\mu-\frac{1}{2}} (h_\mu \psi)(ax) dx. \tag{1.4.4}$$

Parseval's relation of the continuous Bessel wavelet transform of two functions $f, g \in L^2(0,\infty)$ with respect to $\psi \in L^2(0,\infty)$ is

$$\int_0^\infty \int_0^\infty (B_\psi f)(b,a)(B_\psi g)(b,a) \frac{dadb}{a^2} = C_{\mu,\psi} \langle f, g \rangle, \tag{1.4.5}$$

where $C_{\mu,\psi}$ is given in (1.4.1).

If $f \in L^2(0,\infty)$ and $\psi \in L^2(0,\infty)$, then inversion formula of the Bessel wavelet transform is

$$f(t) = \frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty (B_\psi f)(b,a) \psi\left(\frac{t}{a}, \frac{b}{a}\right) \frac{dbda}{a^{5/2}}, \quad a > 0. \tag{1.4.6}$$

Thus, in the chapters 2, 3 and 4, our works are heavily based on the continuous Bessel wavelet transform of Hirshmanian theory. In the last two chapters, we study the continuous Bessel wavelet transform on $L^2(0, \infty)$ space by using the Zemanian theory of Hankel transform.