Chapter 5

On Hicks' contraction using a control function

5.1 Introduction and Preliminaries

The structure of Menger spaces being inherently flexible, there have been extension of several results obtained on metric spaces in more than one nonequivalent ways. In particular, Hicks [50] has given an extension of Banach's contraction which is very different from the form considered by Sehgal et al. [101]. This contraction is called Hicks' contraction or C-contraction which, along with its several modifications and generalizations, have been considered in a good number of papers.

A new domain of study in probabilistic fixed point theory initiated after the introduction of control functions by Choudhury et al. [4]. This parallels a corresponding development in metric spaces which was initiated by Khan et al. [57] and elaborated through several works [45, 51, 55, 56, 72, 73, 88]. Other types of control functions have been used by several authors like Ciric [26], Fang [39] etc. Particularly, the result of Fang [39] is a culmination of a trend of development in this line.

In this chapter, we obtain a generalization of probabilistic contraction mapping principle by using control function introduced by Fang [39]. Specifically, we introduce a generalized (φ-C)-contraction and establish corresponding fixed point results in Menger spaces. Method of proof is different from existing one. In the following, we dwell upon some aspects of the probabilistic metric space and some other concepts which are required for our discussion in the rest of the chapter. In 2015, J. X. Fang [39] introduced the following class Φ of a functions φ:

Definition 5.1. The class Φ contains all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $r > 0 \ \exists \ s \geq r$ with $\lim_{n\to\infty} \varphi^n(s) = 0$. An example of this type of function is $\varphi : [0,\infty) \to [0,\infty)$, defined by

$$\varphi(t) = \begin{cases} \frac{t}{1+t}, & ; \quad 1 > t \geq 0 \\ \frac{4}{3} - \frac{t}{3}, & ; \quad 2 \geq t \geq 1, \\ -\frac{4}{3} + t, & ; \quad 2 < t < \infty. \end{cases}$$

We define the following convergence criteria for the Menger space.

Definition 5.2. L-convergence criteria is satisfied in a Menger space (X, F, T) if for a sequence $\{x_n\}$ in X and a sequence of positive real numbers $\{t_n\}$, $F_{x_n,x_{n+1}}(t_n) \to 1$ as $n \to \infty$ implies that $\{x_n\}$ is convergent.

Lemma 5.3. *Let $\varphi \in \Phi$. Then $\forall \ s > 0$, $\exists \ t \geq s$ such that $\varphi(t) < s$.*

5.2 Main results

Theorem 5.4. *Let L-convergence criteria is satisfied in a Menger space* (X, F, T) *with a continuous t-norm* T*. If* $f : X \to X$ *is a probabilistic (*φ*-C)-contraction , that is,*

$$F_{x,y}(t) > 1 - t \Rightarrow F_{fx,fy}(\varphi(t)) > 1 - kt, \;\; t > 0 \;\; and \;\forall\;\; x, y \in X, \tag{5.1}$$

where $\varphi \in \Phi$ *and* $0 < k < 1$*, then* f *has a fixed point* $x \in X$ *which is unique and* $\{f^n(x_0)\}$ *converges to* x *for any arbitrary* $x_0 \in X$*.*

Proof. Let $x_0 \in X$ be arbitrary. We write $x_n = f^n x_0 = f x_{n-1} \;\forall\; n \geq 1$. Let $\eta \in (0, 1)$ be given. By the property that $F_{x_0,x_1}(t) \to 1$ as $t \to \infty$, $\exists\; r > 0$ such that

$$F_{x_0,x_1}(r) > 1 - \eta.$$

Then from (5.1),

$$F_{fx_0,fx_1}(\varphi(r)) > 1 - k\eta,$$

that is,

$$F_{x_1,x_2}(\varphi(r)) > 1 - k\eta.$$

Continuing in this way, we obtain

$$F_{x_2,x_3}(\varphi^2(r)) > 1 - k^2\eta.$$

Moreover, $\forall\; n \in \mathbb{N}$,

$$F_{x_n,x_{n+1}}(\varphi^n(r)) > 1 - k^n\eta.$$

Again, since $F_{x_0,x_1}(t) \to 1$ as $t \to \infty$, for any $\varepsilon \in (0,1]$ there exists $t_1 > 0$ such that $F_{x_0,x_1}(t_1) > 1-\varepsilon$. Since $\varphi \in \Phi$, $\exists\ t_0 \geq t_1$ with

$$\varphi^n(t_0) \to 0 \ \text{ as } \ n \to \infty. \tag{5.2}$$

Now,

$$F_{x_0,x_1}(t_0) \geq F_{x_0,x_1}(t_1) > 1-\varepsilon.$$

This implies that

$$F_{fx_0,fx_1}(\varphi(t_0)) > 1-k\varepsilon > 1-\varepsilon,$$

that is,

$$F_{x_1,x_2}(\varphi(t_0)) > 1-k\varepsilon > 1-\varepsilon.$$

Continuing this process, we obtain

$$F_{x_n,x_{n+1}}(\varphi^n(t_0)) > 1-\varepsilon. \tag{5.3}$$

Then from (5.2) and (5.3), by L-convergence criteria, one can assure that $\{x_n\}$ is a convergent sequence. Let

$$x_n \to x \ \text{ as } \ n \to \infty. \tag{5.4}$$

Let $\varepsilon > 0$ be given. Since $\varphi \in \Phi$, there exists $r \geq \varepsilon$ with $\varphi(r) < \varepsilon$.

Now,

$$F_{x,fx}(\varepsilon) \geq T(F_{x,x_n}(\varepsilon - \varphi(r)), F_{x_n,fx}(\varphi(r))). \tag{5.5}$$

As $\{x_n\}$ converges to x, we have

$$F_{x_{n-1},x}(r) \to 1 \ \text{ as } \ n \to \infty.$$

That is, for arbitrary $\lambda \in (0,1)$, one can obtain N_1 such that $\forall\ n > N_1$,

$$F_{x_{n-1},x}(r) > 1 - \lambda,$$

which implies that

$$F_{fx_{n-1},fx}(\varphi(r)) = F_{x_n,fx}(\varphi(r)) > 1 - k\lambda > 1 - \lambda. \tag{5.6}$$

Again, as $\{x_n\}$ converges to x, it is possible to find N_2 such that $\forall\ n > N_2$, we get

$$F_{x_n,x}(\varepsilon - \varphi(r)) > 1 - \lambda. \tag{5.7}$$

We choose $N = \max\{N_2, N_1\}$. Then, $\forall\ n > N$, from (5.5), (5.6) and (5.7), we get

$$F_{x,fx}(\varepsilon) \geq T(1-\lambda, 1-\lambda).$$

Here, T is a continuous t-norm and λ is arbitrary , hence for any $\varepsilon > 0$, one can obtain that

$$F_{x,fx}(\varepsilon) = 1,$$

that is, $x = fx$.

Next, we show that the fixed point is unique. For this, suppose x and y are two fixed points of f, that is, $x = fx$ and $y = fy$.

Since $F_{x,y}(t) \to 1$ as $t \to \infty$, for any $\varepsilon \in (0,1]$ there exists $t_0 > 0$ such that $F_{x,y}(t_0) > 1 - \varepsilon$. Since $\varphi \in \Phi$, there exists $t_1 \geq t_0$ such that $\varphi^n(t_1) \to 0$ as $n \to \infty$. Let $t > 0$ be arbitrary, then there exists $n_0 \in \mathbb{N}$ such that $\varphi^n(t_1) < t$ for all $n \geq n_0$.

Thus, by monotonicity of the distribution function, we get

$$F_{x,y}(t) \geq F_{x,y}(\varphi^n(t_1)). \tag{5.8}$$

Now,

$$F_{x,y}(t_1) \geq F_{x,y}(t_0) > 1 - \varepsilon.$$

This implies that

$$F_{fx,fy}(\varphi(t_1)) = F_{x,y}(t_1) > 1 - k\varepsilon,$$

that is,

$$F_{x,y}(\varphi(t_1)) > 1 - k\varepsilon.$$

Continuing this process, we obtain

$$F_{x,y}(\varphi^n(t_1)) > 1 - k^n\varepsilon. \tag{5.9}$$

Now, (5.8) and (5.9) give

$$F_{x,y}(t) > 1 - k^n\varepsilon, \ \ \forall \ n \geq n_0. \tag{5.10}$$

Letting $n \to \infty$, $F_{x,y}(t) = 1 \ \forall \ t > 0$, i.e., $y = x$. □

If we take $\varphi(x) = kx$ where $k \in (0, 1)$, we obtain the following corollary, which is a probabilistic contraction mapping principle under L-convergence criteria.

Corollary 5.5. *Let (X, F, T) be a Menger space satisfying L-convergence criteria where T is a continuous t-norm. Let $k \in (0, 1)$ and f is self mapping on X s.t. for all $r > 0$, $x, y \in X$, and $\epsilon \in (0, 1)$*

$$F_{x,y}(r) > 1 - \epsilon \quad \Longrightarrow \quad F_{fx,fy}(kr) > 1 - k\epsilon. \tag{5.11}$$

Then f has a unique fixed point.

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