Chapter 4

Fixed points for (φ-ψ)-weak contractions in fuzzy and Menger spaces

4.1 Introduction

In metric fixed point theory, there are several generalizations of the Banach contraction theorem. One such generalization from large existing literature in this line of research is weak contraction principal which establishes a new contraction in between Banach contraction and the non-expansive mapping. Alber et al. [1] first introduced the weak contraction in Hilbert spaces after that, it was adopted to metric space by Rhoades [87]. Following the similar ideas of such weak contraction, several results were established, not all of which are generalization of Banach contraction theorem. Actually, they contribute a much larger class of contraction known as weak contraction in metric fixed point theory. To establish existence and uniqueness of fixed point theorem in Menger PM-spaces, contraction is one of the basic tools. Sehgal and Bharucha-Reid [101] introduced probabilistic k-contraction and proved probabilistic version of classical Banach fixed point principle. Efforts have been made over the years to generalize and extend the k-contraction, which let to the concepts of φ-contraction, weak-contraction and generalized weak contraction etc. in Menger and fuzzy metric spaces. Few references from the large exiting literature are [4, 32, 39, 87]. In other spaces, which are generalizations of usual metric spaces, such ideas and results are also addressed by several authors (see [5, 6, 7, 11, 30, 54, 64, 66, 81, 84, 93, 115]). Motivated by the recent results in [25, 32, 92], in the present work we prove a fixed point theorem for weak contraction mappings in fuzzy metric spaces, and another theorem is proved in Menger space. The results in this chapter are established in fuzzy metric spaces in the sense of George and Veeramani.

The following lemmas are required to prove our results.

Lemma 4.1. *[92] If T is a continuous t-norm, and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that $a_n \to a, c_n \to c$ as $n \to \infty$, then $\overline{\lim_{k\to\infty}} T(a_k, T(b_k, c_k)) = T(a, T(\overline{\lim_{k\to\infty}} b_k, c))$ and $\underline{\lim_{k\to\infty}} T(a_k, T(b_k, c_k)) = T(a, T(\underline{\lim_{k\to\infty}} b_k, c))$.*

Lemma 4.2. *[92] Let $\{f(k, .) : (0, \infty) \to (0, 1], k = 0, 1, 2, \dots\}$ be a sequence of functions such that $f(k, .)$ is continuous and monotone increasing for each $k \geq 0$. Then $\overline{\lim_{k\to\infty}} f(k, t)$ is a left continuous function in t and $\underline{\lim_{k\to\infty}} f(k, t)$ is a right continuous function in t.*

4.2 A theorem in Menger spaces

Theorem 4.3. *Let (X, F, T) be a complete PM-space such that 'T' is an arbitrary continuous t-norm and let $f : X \to X$ be a self mapping satisfying the following* *condition:*

$$\psi(F_{fx,fy}(t)) \leq \psi(F_{x,y}(t)) - \varphi(F_{x,y}(t)), \tag{4.1}$$

where $\psi, \varphi : (0,1] \to [0,\infty)$ *are two functions such that:*

(i) ψ *is monotone decreasing and continuous function with* $\psi(s) = 0$ *if and only if* $s = 1$,

(ii) φ *is lower semi-continuous function with* $\varphi(s) = 0$ *if and only if* $s = 1$.

Then f *has a unique fixed point in* X.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\} \subset X$ such that $x_{n+1} = fx_n$, for each $n \geq 0$. If there exists a positive integer k such that $x_k = x_{k+1}$ then x_k is a fixed point of f. Hence, we shall assume that $x_n \neq x_{n+1}$, for all $n \geq 0$. Now, from (4.1)

$$\psi(F_{x_n,x_{n+1}}(t)) = \psi(F_{fx_{n-1},fx_n}(t)) \leq \psi(F_{x_{n-1},x_n}(t)) - \varphi(F_{x_{n-1},x_n}(t)). \tag{4.2}$$

Since ψ is monotone decreasing, we have

$$F_{x_{n-1},x_n}(t) \leq F_{x_n,x_{n+1}}(t).$$

Therefore, $\{F_{x_n,x_{n+1}}(t)\}$ is a monotone increasing sequence of non-negative real numbers. Hence, there exists $r > 0$ such that

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = r.$$

Taking the limit as $n \to \infty$ in (4.2), we obtain

$$\psi(r) \leq \psi(r) - \varphi(r),$$

which is a contradiction unless $r = 1$.

Hence,

$$\lim_{n\to\infty} F_{x_n, x_{n+1}}(t) = 1. \tag{4.3}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. If not so, there exist λ, $\epsilon > 0$ with $\lambda \in (0, 1)$ such that for each integer k, there are two integers $l(k)$ and $m(k)$ such that

$m(k) > l(k) \geq k,$

$F_{x_{l(k)}, x_{m(k)}}(\epsilon) \leq 1 - \lambda$ and

$F_{x_{l(k)}, x_{m(k)-1}}(\epsilon) > 1 - \lambda.$

Now, by triangle inequality, for any s with $\frac{\epsilon}{2} > s > 0$ and for all $k > 0$, we have

$$\begin{aligned} 1 - \lambda &\geq F_{x_{l(k)}, x_{m(k)}}(\epsilon) \\ &\geq T(F_{x_{l(k)}, x_{l(k)+1}}(s), T(F_{x_{l(k)+1}, x_{m(k)+1}}(\epsilon - 2s), F_{x_{m(k)+1}, x_{m(k)}}(s))). \end{aligned} \tag{4.4}$$

For $t > 0$, we define the function

$$h_1(t) = \overline{\lim_{k\to\infty}} F_{x_{l(k)+1}, x_{m(k)+1}}(t).$$

Taking limit supremum as $k \to \infty$ on both the sides of (4.4), using (4.3), the continuity of T and Lemma 4.1, we get that

$$\begin{aligned} 1-\lambda &\geq T(1, T(\overline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon-2s),1)) \\ &= T(1, \overline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon-2s)) \\ &= \overline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon-2s) \\ &= h_1(\epsilon-2s). \end{aligned}$$

By an application of Lemma 4.2, h_1 is left continuous. Taking limit as $s \to 0$ in the above ineqaulity, we obtain,

$$h_1(\epsilon) = \overline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon) \leq 1-\lambda. \tag{4.5}$$

Next, for all $t > 0$, we define the function

$$h_2(t) = \underline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(t).$$

In the similar process, we can prove that

$$h_2(\epsilon) = \underline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon) \geq 1-\lambda. \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\overline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon) \leq 1-\lambda \leq \underline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon).$$

This implies that

$$\lim_{k\to\infty} F_{x_{l(k)+1},x_{m(k)+1}}(t) = 1-\lambda. \tag{4.7}$$

Again, by (4.5)

$$\overline{\lim_{k\to\infty}} F_{x_{l(k)},x_{m(k)}}(\epsilon) \leq 1-\lambda. \tag{4.8}$$

For $t > 0$, we define the function

$$h_3(t) = \underline{\lim_{k\to\infty}} F_{x_{l(k)},x_{m(k)}}(t).$$

Now for $s > 0$,

$$F_{x_{l(k)},x_{m(k)}}(\epsilon + 2s) \geq T(F_{x_{l(k)},x_{l(k)+1}}(s), T(F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon), F_{x_{m(k)+1},x_{m(k)}}(s))).$$

Taking limit infimum as $k \to \infty$ on both the sides, we have

$$\underline{\lim_{k\to\infty}} F_{x_{l(k)},x_{m(k)}}(\epsilon + 2s) \geq T(1, T(\underline{\lim_{k\to\infty}} F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon), 1)) = 1-\lambda.$$

Thus, $h_3(\epsilon + 2s) \geq 1-\lambda$.

Taking limit as $s \to 0$, we obtain

$$h_3(\epsilon) = \underline{\lim_{k\to\infty}} F_{x_{l(k)},x_{m(k)}}(\epsilon) \geq 1-\lambda. \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$\lim_{k\to\infty} F_{x_{l(k)},x_{m(k)}}(t) = 1-\lambda. \tag{4.10}$$

Now,

$$\psi(F_{x_{l(k)+1},x_{m(k)+1}}(\epsilon)) \leq \psi(F_{x_{l(k)},x_{m(k)}}(\epsilon)) - \varphi(F_{x_{l(k)},x_{m(k)}}(\epsilon)).$$

Taking limit as $k \to \infty$, and using (4.7) and (4.10) we obtain

$\psi(1-\lambda) \leq \psi(1-\lambda) - \varphi(1-\lambda)$, which is a contradiction.

Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Now,

$$\begin{aligned} \psi(F_{x_{n+1},fp}(t)) &= \psi(F_{fx_n,fp}(t)) \\ &\leq \psi(F_{x_n,p}(t)) - \varphi(F_{x_n,p}(t)). \end{aligned}$$

Taking limit as $n \to \infty$, we get

$\psi(F_{p,fp}(t)) \leq \psi(F_{p,fp}(t)) - \varphi(F_{p,fp}(t)) = 0,$

which implies that $\varphi(F_{p,fp}(t)) = 1$, that is,

$F_{p,fp}(t) = 1$ or $p = fp$.

We next establish that fixed point is unique. Let p and q be two fixed points of f.

Putting $x = p$ and $y = q$ in (4.1), we get

$\psi(F_{fp,fq}(t)) \leq \psi(F_{p,q}(t)) - \varphi(F_{p,q}(t))$
or, $\psi(F_{p,q}(t)) \leq \psi(F_{p,q}(t) - \varphi(F_{p,q}(t)$
or, $\varphi(F_{p,q}(t)) \leq 0,$
or, equivalently, $\psi(F_{p,q}(t)) = 1$, that is, $p = q$. □

The following example is in support of Theorem 4.3.

Example 4.1. *Let* $X = [0, 1]$. *Define a function* $F : X \times X \to D^+$ *by*

$$F_{x,y}(t) = \begin{cases} 1, & \text{if } t \leq 0 \\ e^{-\frac{|x-y|}{t}}, & \text{if } t > 0. \end{cases}$$

for all $x, y \in X$. Then (X, F, T) is a complete Menger probabilistic metric space, where $'T'$ is product t-norm. Let $\psi, \varphi : (0, 1] \to [0, \infty)$ be defined by

$$\psi(s) = \frac{1}{s} - 1, \quad \varphi(s) = \frac{1}{s} - \frac{1}{\sqrt{s}}, \quad \forall s \in (0, 1]. \tag{4.11}$$

Then ψ and φ satisfy all the conditions of Theorem 4.3. Let the mapping $f : X \to X$ be defined by $f(x) = \frac{x^2}{4}$, for all $x \in X$.

Now, we shall show that f satisfy (4.1).

With the choices of φ and ψ as in (4.11), the inequality (4.1) takes the form

$$\frac{1}{F_{fx,fy}(t)} - 1 \leq \frac{1}{F_{x,y}(t)} - 1 - \frac{1}{F_{x,y}(t)} + \frac{1}{\sqrt{F_{x,y}(t)}},$$

that is,

$$F_{fx,fy}(t) \geq \sqrt{F_{x,y}(t)}.$$

Now,

$$\begin{aligned} F_{fx,fy}(t) &= e^{-\frac{|fx-fy|}{t}} \\ &= e^{-\frac{|x^2-y^2|}{4t}} \\ &= e^{-(\frac{|x-y|}{2t})(\frac{|x+y|}{2})} \\ &\geq e^{-(\frac{|x-y|}{2t})} \\ &= \sqrt{F_{x,y}(t)}. \end{aligned}$$

Hence, all the conditions of Theorem (4.3) are satisfied.

Thus, 0 is the unique fixed point of f.

4.3 A theorem in fuzzy metric spaces

Theorem 4.4. *Let (X, M, T) be a complete fuzzy metric space with an arbitrary continuus t-norm $'T'$ and let $f : X \to X$ be a self mapping satisfying the following condition:*

$$\psi(M(fx, fy, t)) \leq \psi(\min(M(x, y, t), M(x, fx, t), M(y, fy, t))) \\ -\varphi(\min(M(x, y, t), M(y, fy, t))), \tag{4.12}$$

where $\psi, \varphi : (0, 1] \to [0, \infty)$ are two functions such that:

(i) *ψ is continuous and monotone decreasing function with $\psi(t) = 0$ if and only if $t = 1$,*

(ii) *φ is lower semi continuous function with $\varphi(t) = 0$ if and only if $t = 1$.*

Then f has a unique fixed point.

Proof. Let $x_0 \in X$. We define the sequence $\{x_n\}$ as $x_{n+1} = fx_n$, for each $n \geq 0$. If there exists a positive integer k such that $x_k = x_{k+1}$, then x_k is a fixed point of f. Hence, we shall assume that $x_n \neq x_{n+1}$, for all $n \geq 0$. Now, from (4.12)

$$\begin{aligned} \psi(M(x_{n+1}, x_{n+2}, t)) &= \psi(M(fx_n, fx_{n+1}, t)) \\ &\leq \psi(\min\{M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t)\}) \\ &\quad -\varphi(\min\{M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t)\}). \end{aligned} \tag{4.13}$$

Suppose that $M(x_n, x_{n+1}, t) > M(x_{n+1}, x_{n+2}, t)$, for some positive integer n. Then from (4.13), we have

$\psi(M(x_{n+1}, x_{n+2}, t)) \leq \psi(M(x_{n+1}, x_{n+2}, t)) - \varphi(M(x_{n+1}, x_{n+2}, t))$,

that is, $\varphi(M(x_{n+1}, x_{n+2}, t)) \leq 0$,

which implies that $M(x_{n+1}, x_{n+2}, t) = 1$.

This gives that $x_{n+1} = x_{n+2}$, which is a contradiction.

Therefore, $M(x_{n+1}, x_{n+2}, t) \leq M(x_n, x_{n+1}, t)$ for all $n \geq 0$, and $\{M(x_n, x_{n+1}, t)\}$ is a monotone increasing sequence of non-negative real numbers. Hence, there exists $r > 0$ such that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = r$.

In view of the above facts, from (4.13), we have

$\psi(M(x_{n+1}, x_{n+2}, t)) \leq \psi(M(x_n, x_{n+1}, t)) - \varphi(M(x_n, x_{n+1}, t))$, for all $n \geq 0$.

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of φ and ψ we have $\psi(r) \leq \psi(r) - \varphi(r)$, which is a contradiction unless $r = 1$. Hence

$$M(x_n, x_{n+1}, t) \to 1 \quad \text{as } n \to \infty. \tag{4.14}$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. If not so, there exist λ, $\epsilon > 0$ with $\lambda \in (0, 1)$ such that for each integer k, there exists integers $l(k)$ and $m(k)$ such that $m(k) > l(k) \geq k$ and

$$M(x_{l(k)}, x_{m(k)}, \epsilon) \leq 1 - \lambda, \quad \text{for all } k > 0. \tag{4.15}$$

By choosing $m(k)$ to be the smallest integer exceeding $l(k)$ for which (4.15) holds, then for all $k > 0$, we have

$$M(x_{l(k)}, x_{m(k)-1}, \epsilon) > 1 - \lambda.$$

Now, by triangle inequality, for any s with $0 < s < \frac{\epsilon}{2}$, for all $k > 0$, we have

$$\begin{aligned} 1-\lambda &\geq M(x_{l(k)}, x_{m(k)}, \epsilon) \\ &\geq T(M(x_{l(k)}, x_{l(k)+1}, s), T(M(x_{l(k)+1}, x_{m(k)+1}, \epsilon - 2s), M(x_{m(k)+1}, x_{m(k)}, s))). \end{aligned} \tag{4.16}$$

For $t > 0$, we define the function $h_1(t) = \overline{\lim_{n\to\infty}} M\left(x_{l(k)+1}, x_{m(k)+1}, t\right)$.

Taking limit supremum as $k \to \infty$ on both the sides of (4.16), using (4.14), the continuity property of T and Lemma 4.1, we conclude that

$$\begin{aligned} 1-\lambda &\geq T(1, T(\overline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon - 2s), 1)) \\ &= \overline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon - 2s) \\ &= h_1(\epsilon - 2s). \end{aligned}$$

By an application of Lemma 4.2, h_1 is left continuous.

Letting limit as $s \to 0$ in the above inequality, we obtain

$$h_1(\epsilon) = \overline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon) \leq 1 - \lambda. \tag{4.17}$$

Next, for all $t > 0$, we define the function

$$h_2(t) = \underline{\lim_{k\to\infty}} M\left(x_{l(k)+1}, x_{m(k)+1}, t\right).$$

In above a process, we can prove that

$$h_2(\epsilon) = \underline{\lim_{k\to\infty}} M\left(x_{l(k)+1}, x_{m(k)+1}, \epsilon\right) \geq 1 - \lambda. \tag{4.18}$$

Combining (4.17) and (4.18), we get

$$\overline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon) \leq 1-\lambda \leq \underline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon).$$

This implies that

$$\lim_{n\to\infty} M(x_{l(k)+1}, x_{m(k)+1}, t) = 1-\lambda. \tag{4.19}$$

Now by (4.17),

$$\overline{\lim_{k\to\infty}} M(x_{l(k)}, x_{m(k)}, \epsilon) \leq 1-\lambda.$$

For $t > 0$, we define the function

$$h_3(t) = \underline{\lim_{k\to\infty}} M(x_{l(k)}, x_{m(k)}, \epsilon). \tag{4.20}$$

Now for $s > 0$,

$$M(x_{l(k)}, x_{m(k)}, \epsilon{+}2s) \geq T(M(x_{l(k)}, x_{l(k)+1}, s), T(M(x_{l(k)+1}, x_{m(k)+1}, \epsilon), M(x_{m(k)+1}, x_{m(k)}, s))).$$

Taking limit infimum as $k \to \infty$ on both the sides, we have

$$\underline{\lim_{k\to\infty}} M(x_{l(k)}, x_{m(k)}, \epsilon + 2s) \geq T(1, T(\underline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon), 1)) = 1-\lambda.$$

Thus,

$$h_3(\epsilon + 2s) \geq 1-\lambda. \tag{4.21}$$

Taking limit as $s \to 0$, we get $h_3(\epsilon) \geq 1-\lambda$. Combining (4.20) and (4.21), we obtain

$$\lim_{n\to\infty} M(x_{l(k)}, x_{m(k)}, \epsilon) = 1.$$

Now,

$$\begin{aligned}\psi(M(x_{l(k)+1}, x_{m(k)+1}, \epsilon)) \quad &\leq \quad \psi(\min\{M(x_{l(k)}, x_{m(k)}, \epsilon), M(x_{l(k)}, x_{l(k)+1}, \epsilon), M(x_{m(k)}, x_{m(k)+1}, \epsilon)\}) \\ &\quad -\varphi(\min\{M(x_{l(k)}, x_{m(k)}, \epsilon), M(x_{m(k)}, x_{m(k)+1}, \epsilon)\}).\end{aligned}$$

Taking limit as $k \to \infty$, we get

$\psi(1-\lambda) \leq \psi(1-\lambda) - \varphi(1-\lambda)$, which is a contradiction.

Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Now,

$$\begin{aligned}\psi(M(x_{n+1}, fp, t)) &= \psi(M(fx_n, fp, t)) \\ &\leq \psi(\min\{M(x_n, p, t), M(x_n, x_{n+1}, t), M(p, fp, t)\}) \\ &\quad -\varphi(\min\{M(x_n, p, t), M(p, fp, t)\}).\end{aligned}$$

Taking limit as $n \to \infty$, we get

$\psi(M(p, fp, t)) \leq \psi(M(p, fp, t)) - \varphi(M(p, fp, t))$,

which implies that $\varphi(M(p, fp, t)) = 0$, that is,

$M(p, fp, t) = 1$ or $p = fp$.

We next establish that fixed point is unique. Let p and q be two fixed points of f.

Putting $x = p$ and $y = q$ in (4.12),

$\psi(M(fp, fq, t)) \leq \psi(\min\{M(p, q, t), M(p, fp, t), M(q, fq, t)\}) - \varphi(\min\{M(p, q, t), M(q, fq, t)\})$

or, $\psi(M(p, q, t)) \leq \psi(\min\{M(p, q, t), M(p, p, t), M(q, q, t)\}) - \varphi(\min\{M(p, q, t), M(q, q, t)\})$

or, $\psi(M(p, q, t)) \leq \psi(M(p, q, t)) - \varphi(M(p, q, t))$

or, $\varphi(M(p, q, t)) \leq 0$

or, equivalently, $M(p, q, t) = 1$, that is, $p = q$. □

The following example is in support of Theorem 4.4.

Example 4.2. *Let $X = [0,1]$ and define $M : X \times X \times (0,\infty] \to [0,1]$ by*

$$M(x,y,t) = e^{-\frac{|x-y|}{t}},$$

for all $x, y \in X$ and $t > 0$. Then (X, M, T) is a complete fuzzy metric space, where 'T' is a product t-norm. Let $\psi, \varphi : (0,1] \to [0,\infty)$ be defined by $\psi(s) = \frac{1}{s} - 1$ and $\varphi(s) = \frac{1}{s} - \frac{1}{\sqrt{s}}$. Then ψ and φ satisfy all the conditions of Theorem (4.4). Let the mapping $f : X \to X$ be defined by $fx = \frac{x}{2}$, for all $x \in X$.

Now, we will show that

$$\psi(M(fx,fy,t)) \leq \psi(M(x,y)) - \varphi(N(x,y)), \tag{4.22}$$

where $M(x,y) = \min\{M(x,y,t),\ M(x,fx,t),\ M(y,fy,t)\}$

and $N(x,y) = \min\{M(x,y,t),\ M(y,fy,t)\}$.

Now,

$$\max\left\{|x-y|, \frac{x}{2}, \frac{y}{2}\right\} = \begin{cases} x-y & 0 \leq y \leq \frac{x}{2} \\ \frac{x}{2} & \frac{x}{2} < y \leq x \\ \frac{y}{2} & x < y \leq 2x \\ y-x & 2x < y \leq 1 \end{cases}$$

and

$$\max\left\{|x-y|, \frac{y}{2}\right\} = \begin{cases} x-y & 0 \leq y \leq \frac{2x}{3} \\ \frac{y}{2} & \frac{2x}{3} < y \leq 2x \\ y-x & 2x < y \leq 1. \end{cases}$$

Case (1): When $0 \leq y \leq \frac{x}{2}$ or $2x < y \leq 1$, then

$$\psi(M(fx,fy,t)) = \psi(e^{-|\frac{x-y}{2t}|}) = e^{|\frac{x-y}{2t}|} - 1$$

and

$$\psi(M(x,y)) - \varphi(N(x,y)) = \psi(e^{-\frac{|x-y|}{t}}) - \varphi(e^{-\frac{|x-y|}{t}}) = e^{|\frac{x-y}{2t}|} - 1.$$

Obviously, in this case, (4.22) is satisfied.

Case (2): When $\frac{x}{2} < y \leq \frac{2x}{3}$, *then*

$$\psi(M(fx, fy, t)) = \psi(e^{-\frac{x-y}{2t}}) = e^{\frac{x-y}{2t}} - 1$$

and

$$\psi(M(x,y)) - \varphi(N(x,y)) = \psi(e^{-\frac{x}{2t}}) - \varphi(e^{-\frac{x-y}{t}}) = e^{\frac{x}{2t}} - 1 - e^{\frac{x-y}{t}} + e^{\frac{x-y}{2t}}.$$

In this case, $\frac{x}{2} \geq x - y$ *and exponetial function is an increasing function. Therefore,* $e^{\frac{x-y}{2t}} \leq e^{\frac{x}{2t}} - e^{\frac{x-y}{t}} + e^{\frac{x-y}{2t}}$ *and hence (4.22) is satisfied.*

Case (3): When $\frac{2x}{3} < y \leq x$, *then*

$$\psi(M(fx, fy, t)) = \psi(e^{-\frac{x-y}{2t}}) = e^{\frac{x-y}{2t}} - 1$$

and

$$\psi(M(x,y)) - \varphi(N(x,y)) = \psi(e^{-\frac{x}{2t}}) - \varphi(e^{-\frac{y}{2t}}) = e^{\frac{x}{2t}} - 1 - e^{\frac{y}{2t}} + e^{\frac{y}{4t}}.$$

Since, in this case, $\frac{x-y}{2} \leq \frac{y}{4}$ *and* $\frac{x}{2} \geq \frac{y}{2}$, *(4.22) is satisfied.*

Case (4): $x < y \leq 2x$, *then*

$$\psi(M(fx, fy, t)) = \psi(e^{-\frac{y-x}{2t}}) = e^{\frac{y-x}{2t}} - 1$$

and

$$\psi(M(x,y)) - \varphi(N(x,y)) = \psi(e^{-\frac{y}{2t}}) - \varphi(e^{-\frac{y}{2t}}) = e^{\frac{x}{4t}} - 1.$$

Since, in this case, $\frac{y}{2} \geq y - x$, *(4.22) is satisfied. Hence, all the conditions of Theorem (4.4) are satisfied. Thus,* 0 *is the unique fixed point of* f.

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