Chapter 3

Fixed points for φ-contraction in Menger PGM-spaces

3.1 Introduction

Contraction is one of the basic tools to prove existence and uniqueness of fixed point theorems in PM-spaces. Sehgal and Bharucha-Reid [101] introduced probabilistic k-contraction and proved probabilistic version of classical Banach fixed point principle. After that Ciric [26] generalizes the k-contraction and introduced the concept of φ-contraction in PM-space. In spite of the fact that probabilistic φ-contractions are natural generalizations of probabilistic k-contractions, the techniques used to prove the existence and uniqueness of fixed point results for probabilistic k-contractions are no longer usable for probabilistic φ-contractions. In 2010, Ciric [26] presented a fixed point theorem for probabilistic φ-contractions. Soon after the publication of Ciric's paper, Jachymski [53] found a counter example to the key lemma in [26], and established a modified version of Ciric's φ-function. Recently, Fang [39] further weakened the conditions on φ-function.

In 2006, Mustafa and Sims [70] introduced the notion of a generalized metric space. After that many authors obtained several fixed point theorems for mappings satisfying different contractive conditions in generalized metric spaces (see, [31, 68, 69, 89]). In 2014, Zhou et al. [116] introduced the concept of a generalized Menger probabilistic metric space. Further, Zhu et al. [117] obtained some fixed point theorems in PGM-spaces. For some recent results in PGM-spaces, we refer [4, 18, 24, 29, 34, 63, 107].

The purpose of this work is to introduce a new class of φ-contraction in PGM-spaces and to establish important fixed point results. We prove the existence and uniqueness of a fixed point for φ-contraction in PGM-spaces. The obtained results are illustrated by examples.

3.2 Preliminaries

We have already discussed the important definitions in first chapter. Here, we give an example of Menger PGM-space

Example 3.1 ([116]). *Let (X, F, T) be a PM-space. Define a function $G : X \times X \times X \to D^+$ by $G_{x,y,z}(t) = \min\{F_{x,y}(t), F_{y,z}(t), F_{x,z}(t)\}$, for all $x, y, z \in X$ and $t > 0$. Then (X, G, T) is a Menger PGM-space.*

Example 3.2. *Let (X, d) be a metric space. If we define*

$$G_{x,y,z}(t) = \left(\frac{t}{1+t}\right)^{d(x,y)+d(y,z)+d(z,x)}$$

and we choose t-norm as product t-norm defined by

$$T_p(a,b) = a.b \ \ \forall \ a,b \in [0,1].$$

Then (X, G, T_p) is a Menger PGM-space. In fact, $G_{x,y,z}(0) = 0$. Also, $\sup_{t>0} G_{x,y,z}(t) = 1$, $G_{x,y,z}(t)$ is non-decreasing and continuous in t. Therefore, $G_{x,y,z}(t)$ is a distribution function.

By the definition of $G_{x,y,z}(t)$, it is obvious that (i) and (iii) in Definition 1.12 hold. Next, we will show that (ii) and (iv) of Definition 1.12 also hold. Since $d(x,y) \le d(x,z) + d(z,y)$, $\forall x,y,z \in X$, with $y \neq z$, we have that

$$d(x,y) + d(x,y) \le d(x,y) + d(x,z) + d(z,y).$$

Then

$$\left(\frac{t}{1+t}\right)^{d(x,y)+d(x,y)} \ge \left(\frac{t}{1+t}\right)^{d(x,y)+d(y,z)+d(z,x)}.$$

Thus, $G_{x,x,y}(t) \ge G_{x,y,z}(t)$, for all $x,y,z \in X$ with $y \neq z$, and $t > 0$. By the definition of $G_{x,y,z}(t)$, we get

$$G_{x,y,z}(t+s) = \left(\frac{t+s}{1+t+s}\right)^{d(x,y)+d(y,z)+d(z,x)}.$$

Since, $\frac{t}{1+t}$ is strictly increasing on $[0,1)$, we have

$$\begin{aligned}
\left(\frac{t+s}{1+t+s}\right)^{d(x,y)+d(y,z)+d(z,x)} &\ge \left(\frac{t+s}{1+t+s}\right)^{d(x,a)+d(a,y)+d(y,z,)+d(z,a)+d(a,x)} \\
&= \left(\frac{t+s}{1+t+s}\right)^{d(x,a)+d(a,x)} \left(\frac{t+s}{1+t+s}\right)^{d(a,y)+d(y,z)+d(z,a)} \\
&\ge \left(\frac{t}{1+t}\right)^{d(x,a)+d(x,a)} \left(\frac{s}{1+s}\right)^{d(a,y)+d(y,z)+d(z,a)} \\
&= T_P\left(\left(\frac{t}{1+t}\right)^{d(x,a)+d(x,a)}, \left(\frac{s}{1+s}\right)^{d(a,y)+d(y,z)+d(z,a)}\right).
\end{aligned}$$

This implies that $G_{x,y,z}(t+s) \geq T_P(G_{x,a,a}(t), G_{a,y,z}(s))$. Thus, (X, G, T_P) is a Menger PGM-space.

Definition 3.1 ([4]). Let (X, G, T) be a Menger PGM-space with a continuous t-norm T. A mapping $f : X \to X$ is said to be a φ-contraction in Menger PGM-spaces if there exists a function $\varphi \in \Phi$ such that

$$G_{fx,fy,fz}(\varphi(t)) \geq G_{x,y,z}(t), \text{ for all } x, y, z \in X \text{ and } t > 0. \tag{3.1}$$

Definition 3.2 ([46]). A t-norm T is said to be of H-type if the family $\{T^p\}_{p\in\mathbb{N}}$ of its iterates defined for each $t \in (0,1)$ by $T^0(t) = 1$, $T^m(t) = T(t, T^{m-1}(t))$ for all $m \in \mathbb{N}$ is equicontinuous at $t = 1$.

Definition 3.3. In second chapter, we have defined the class Φ of functions φ as follows: Φ contains all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $t > 0$ there exists $r > t$ with $\varphi(r) \leq t$.

An example of this type of function is given as: Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{t}{4^n} & \text{if } \frac{1}{4^n} \leq t < \frac{1}{4^{n-1}} \\ kt & \text{if } t \geq 1, \text{where } 0 < k < 1. \end{cases}$$

3.3 Main Results

Lemma 3.4. *Suppose that the sequence $\left\{G_{x_n,x_{n+1},x_{n+1}}(t_m)\right\}$ is non-decreasing in both the variables m and n, i.e., $G_{x_n,x_{n+1},x_{n+1}}(t_m) \geq G_{x_{n-1},x_n,x_n}(t_m)$ and $G_{x_n,x_{n+1},x_{n+1}}(t_{m+1}) \geq G_{x_n,x_{n+1},x_{n+1}}(t_m)$ for each $m, n \in \mathbb{N}$. Then*
$\lim_{n\to\infty}\left(\lim_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)\right)$.

Proof. The proof follows the similar way as that of Lemma 2.3. □

Lemma 3.5. *Let (X, G, T) be a Menger PGM-space with a t-norm T. Let $\{x_n\}$ be a sequence in (X, G, T). If there exists a function $\varphi \in \Phi$ such that $G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq G_{x_{n-1},x_n,x_n}(t)$, for all $n \in \mathbb{N}$ and $t > 0$, then $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = 1$.*

Proof. Let $t_0 > 0$ be arbitrary. Since $\varphi \in \Phi$, there exists $t_1 > t_0$ such that $\varphi(t_1) \leq t_0$. Now, since $G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq G_{x_{n-1},x_n,x_n}(t)$, by the monotonic increasing property of distribution function, we have

$$\begin{aligned} G_{x_n,x_{n+1},x_{n+1}}(t_1) &\geq G_{x_n,x_{n+1},x_{n+1}}(t_0) \\ &\geq G_{x_n,x_{n+1},x_{n+1}}(\varphi(t_1)) \\ &\geq G_{x_{n-1},x_n,x_n}(t_1) \\ &\geq G_{x_{n-1},x_n,x_n}(t_0). \end{aligned}$$

Thus, the sequence $\{G_{x_n,x_{n+1},x_{n+1}}(t_0)\}$ is monotonically increasing in n and being bounded above, it is convergent. Let $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_0) = l$. We shall show that $l = 1$. On contrary, suppose $l < 1$. Then $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_1) = l$ (by the above inequality). By squeeze lemma,

$$\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = l < 1, \ \forall\, t \in [t_0,\, t_1].$$

Let $\bar{t} = \sup A$, where

$$A = \left\{ t : \lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = l \right\}. \tag{3.2}$$

If $\bar{t}$ is finite then there exists a monotonically increasing sequence $\{t_m\}$ such that for all $m \in \mathbb{N}$, $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) = l$ and $t_m \to \bar{t}$ as $m \to \infty$. Since $G_{x_n,x_{n+1},x_{n+1}}$ is left continuous,

$$G_{x_n,x_{n+1},x_{n+1}}(\bar{t}) = \lim_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m).$$

Therefore, by using Lemma 3.4 and $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) = l$, we have

$$\begin{aligned}\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(\bar{t}) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)\right)\\ &= \lim_{m\to\infty}\left(\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)\right) = l\end{aligned}$$

Therefore, $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(\bar{t}) = l$. Then proceeding as above, there exists $\bar{t_1}$ such that $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(\bar{t_1}) = l$ and $\bar{t_1} > \bar{t}$, which is a contradiction to (3.2). Thus, for all $t > t_0$,

$$\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = l. \tag{3.3}$$

Since $G_{x,y,y}(t) \to 1$ as $t \to \infty$, there exists $s > t_0$ such that $G_{x_k,x_{k+1},x_{k+1}}(s) > l$, for given $k \in \mathbb{N}$. Now, since $\left\{G_{x_n,x_{n+1},x_{n+1}}(t_0)\right\}$ is monotonically increasing in n and as $t_0 > 0$ is arbitrary, we get that $\left\{G_{x_n,x_{n+1},x_{n+1}}(t)\right\}$ is monotonically increasing in n for all $t > 0$. Thus, the sequence $\left\{G_{x_n,x_{n+1},x_{n+1}}(s)\right\}$ is monotonically increasing in n and we have $G_{x_n,x_{n+1},x_{n+1}}(s) > l$. But this is a contradiction as $l < 1$. Therefore, $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = 1$, for all $t > t_0$. Since $t_0 > 0$ is arbitrary, we conclude that $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = 1$, for all $t > 0$. □

Lemma 3.6. *Let (X, G, T) be a Menger PGM-space with a t-norm T of H-type. Let $\{x_n\}$ be a sequence in (X, G, T). If there exists a function $\varphi \in \Phi$ such that*

$$G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq G_{x_{n-1},x_n,x_n}(t), \tag{3.4}$$

for all $n \in \mathbb{N}$ and $t > 0$ then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $\beta > 0$ be arbitrary.

Since $\varphi \in \Phi$, then for each t_1 with $0 < t_1 < \beta$ there exists $r_1 > t_1$ such that $\varphi(r_1) \leq t_1$.

Now, if $\varphi(r_1) < t_1$, then we take $t = t_1$ and $r = r_1$.

If $\varphi(r_1) = t_1$, then choose t as $\min\{r_1, \beta\} > t > t_1$ and $r = r_1$.

Then in each case we have $\beta > t > \varphi(r)$ and $r > t$.

Let $n \geq 1$. Then for each t chosen in this way, we prove by induction that for any $k \in \mathbb{N}$,

$$G_{x_n,x_{n+k},x_{n+k}}(t) \geq T^{k-1}\left(G_{x_n,x_{n+1},x_{n+1}}\left(t - \varphi(r)\right)\right). \tag{3.5}$$

For $k = 1$, from (3.5) we have $G_{x_n,x_{n+1},x_{n+1}}(t) \geq T^0\left(G_{x_n,x_{n+1},x_{n+1}}\left(t - \varphi(r)\right)\right) = G_{x_n,x_{n+1},x_{n+1}}\left(t - \varphi(r)\right)$.

Therefore, (3.5) holds for $k = 1$. Assume that (3.5) holds for some k. Now, since T is monotone, from (iv) of Definition 1.12 and (3.4) we have

$$G_{x_n,x_{n+k+1},x_{n+k+1}}(t) = G_{x_n,x_{n+k+1},x_{n+k+1}}\left(t - \varphi(r) + \varphi(r)\right)$$

$$\geq T\left(G_{x_n,x_{n+1},x_{n+1}}\left(t - \varphi(r)\right), G_{x_{n+1},x_{n+k+1},x_{n+k+1}}\left(\varphi(r)\right)\right)$$

$$\geq T\left(G_{x_n,x_{n+1},x_{n+1}}\left(t - \varphi(r)\right), G_{x_n,x_{n+k},x_{n+k}}\left(r\right)\right)$$

$$\geq T\left(G_{x_n,x_{n+1},x_{n+1}}\left(t - \varphi(r)\right), G_{x_n,x_{n+k},x_{n+k}}\left(t\right)\right)$$

$$= T^k\left(G_{x_n,x_{n+1},x_{n+1}}\left(t - \varphi(r)\right)\right),$$

which completes the induction steps. Hence, (3.5) holds for all $k \in \mathbb{N}$ and for any $t < \beta$. To prove $\{x_n\}$ is a Cauchy sequence, we need to prove that $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$, for all $t > 0$. To this end, we first prove that $\lim_{m,n\to\infty} G_{x_n,x_m,x_m}(t) = 1$, for all $t > 0$.

Now, let $0 < \varepsilon < 1$. Since $\{T^n(t)\}$ is equicontinuous at $t = 1$ and $T^n(1) = 1$, so there exists $\delta > 0$ such that

$$T^n(s) > 1 - \varepsilon, \text{for all } s \in (1 - \delta, 1] \text{ and } n \geq 1. \tag{3.6}$$

From Lemma 3.5, it follows that $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t-\varphi(r)) = 1$. Hence, there exists $n_0 \in \mathbb{N}$ such that $G_{x_n,x_{n+1},x_{n+1}}(t-\varphi(r)) > 1-\delta$, for all $n \geq n_0$. Therefore, by (3.5) and (3.6), we have $G_{x_n,x_{n+k},x_{n+k}}(t) > 1-\varepsilon$, for all $k \geq 0$. Hence,

$$\lim_{n,m\to\infty} G_{x_n,x_m,x_m}(t) = 1, \tag{3.7}$$

for any $0 < t < \beta$. Now, by (iv) in Definition 1.12, we have, for all $t < \beta$,

$$\begin{aligned} G_{x_n,x_m,x_l}(t) &\geq T\left(G_{x_n,x_m,x_m}\left(\frac{t}{2}\right), G_{x_m,x_m,x_l}\left(\frac{t}{2}\right)\right) \\ &= T\left(G_{x_n,x_m,x_m}\left(\frac{t}{2}\right), G_{x_l,x_m,x_m}\left(\frac{t}{2}\right)\right). \end{aligned}$$

Taking limit $m, n, l \to \infty$ in this inequality and using the continuity of T, we get

$$\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) \geq T\left(\lim_{m,n\to\infty} G_{x_n,x_m,x_m}\left(\frac{t}{2}\right), \lim_{m,l\to\infty} G_{x_l,x_m,x_m}\left(\frac{t}{2}\right)\right). \tag{3.8}$$

From (3.7), for all $t < \beta$, we have

$$\lim_{n,m\to\infty} G_{x_n,x_m,x_m}\left(\frac{t}{2}\right) = 1,$$

$$\lim_{l,m\to\infty} G_{x_l,x_m,x_m}\left(\frac{t}{2}\right) = 1.$$

Using these two limits in inequality (3.8), we get

$$\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) \geq T(1,1) = 1.$$

That is, $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$, for all $0 < t < \beta$. Since, $\beta > 0$ is arbitrary, we have $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$, for all $t > 0$.

Hence $\{x_n\}$ is a Cauchy sequence. □

Lemma 3.7. *Let (X, G, T) be a Menger PGM-space and $x, y \in X$. If there exists a function $\varphi \in \Phi$ such that*

$$G_{x,y,y}(\varphi(t)) \geq G_{x,y,y}(t), \tag{3.9}$$

for all $t > 0$, then $x = y$.

Proof. In order to show $x = y$, we only need to prove that $G_{x,y,y}(t) = 1$, for all $t > 0$. On contrary, we suppose that $\exists\ t_0 \in \mathbb{R}^+$ such that $G_{x,y,y}(t_0) < 1$. Now, since $\varphi \in \Phi$, $\exists\, t_1 > t_0$ such that $\varphi(t_1) \leq t_0$. Then (3.9) and the monotonicity of $G_{x,y,y}$ give

$$G_{x,y,y}(t_0) \geq G_{x,y,y}(\varphi(t_1)) \geq G_{x,y,y}(t_1) \geq G_{x,y,y}(t_0). \tag{3.10}$$

If case of strict inequality in (3.10) we have a contradiction. So, we assume that equality holds. Then the set $A = \{s : G_{x,y,y}(s) = G_{x,y,y}(t_0);\ s > t_0\}$ is non-empty by the above inequality. Let $\bar{s}$ =sup$A$ be finite. Then there exists a monotonically increasing sequence $\{s_n\}$ with $s_n \in A$, for all $n \in \mathbb{N}$, such that $s_n \to \bar{s}$. Since $G_{x,y,y}$ is left continuous, it follows that

$G_{x,y,y}(\bar{s}) = \lim\limits_{n\to\infty} G_{x,y,y}(s_n) = G_{x,y,y}(t_0)$.

This implies that $\bar{s} \in A$. Then again treating $\bar{s}$ in the same way as t_0, we obtain either $G_{x,y,y}(\bar{s}) > G_{x,y,y}(\bar{s})$, which is a contradiction, or there exists $\bar{s_1} > \bar{s}$ such that $G_{x,y,y}(\bar{s_1}) = G_{x,y,y}(\bar{s}) = G_{x,y,y}(t_0)$, which is again a contradiction with $\bar{s} = \sup A$.

Hence $\bar{s}$ is not finite, i.e., $\lim\limits_{n\to\infty} G_{x,y,y}(s_n) = G_{x,y,y}(\bar{s}) = G_{x,y,y}(t_0) < 1$, which is also a contradiction as $\bar{s}$ is not finite. Therefore, $G_{x,y,y}(t) = 1$, for all $t > 0$, i.e., $x = y$. □

Theorem 3.8. *Let (X, G, T) be a complete Menger space with a t-norm T of H-type. If $f : X \to X$ is a probabilistic φ-contraction, i.e., $G_{fx,fy,fy}(\varphi(t)) \geq G_{x,y,y}(t)$, $\forall$ $x, y \in X$ and $t > 0$, where $\varphi \in \Phi$, then f has a unique fixed point $x \in X$.*

Proof. We define the sequence $\{x_n\}$ as follows: Let $x_0 \in X$ and $x_n = fx_{n-1}$, for all $n \in \mathbb{N}$. So, by the given contraction condition,

$$\begin{aligned} G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) &= G_{fx_{n-1},fx_n,fx_n}(\varphi(t)) \\ &\geq G_{x_{n-1},x_n,x_n}(t) \end{aligned}$$

for all $n \in \mathbb{N}$ and $t > 0$. Then by Lemma 3.6, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, G, T), and since X is complete, we have that $x_n \to x \in X$. Since $\varphi \in \Phi$, for each $t > 0$, there exists $r > t$ such that $\varphi(r) \leq t$. Now

$$\begin{aligned} G_{fx_n,fx,fx}(t) &\geq G_{fx_n,fx,fx}(\varphi(r)) \\ &\geq G_{x_n,x,x}(r) \\ &\geq G_{x_n,x,x}(t). \end{aligned}$$

Taking limit $n \to \infty$ in this inequality and keeping in mind that $x_n \to x$ for each $t > 0$, we get

$$\lim_{n\to\infty} G_{fx_n,fx,fx}(t) = 1. \tag{3.11}$$

Now, using (iv) of Definition 1.12 and the continuity of T, we get

$$\begin{aligned} G_{x,fx,fx}(t) &\geq T\left(G_{x,x_{n+1},x_{n+1}}\left(\frac{t}{2}\right), G_{x_{n+1},fx,fx}\left(\frac{t}{2}\right)\right) \\ &= T\left(G_{x,x_{n+1},x_{n+1}}\left(\frac{t}{2}\right), G_{fx_n,fx,fx}\left(\frac{t}{2}\right)\right). \end{aligned}$$

Taking limit $n \to \infty$ in this inequality, (3.11) and the continuity of T give

$$G_{x,fx,fx}(t) \geq T(1,1) = 1.$$

This implies that $G_{x,fx,fx}(t) = 1$, for all $t > 0$. Hence $fx = x$ which proves that x

is a fixed point of f. To show the uniqueness of fixed point of f, we suppose that y is another fixed point of f then by condition 3.1, we have

$G_{x,y,y}(\varphi(t)) = G_{fx,fy,fy}(\varphi(t)) \geq G_{x,y,y}(t)$, for all $t > 0$. Then by Lemma 3.7, we get $x = y$. □

Corollary 3.9. *Let (X, G, T) be a complete Menger PGM-space with a t-norm T of H-type. Let $f_0, f_1 : X \to X$ be two mappings such that $G_{f_0x,f_0y,f_0y}(\varphi(t)) \geq G_{x,y,y}(t)$ and $G_{f_1x,f_1y,f_1y}(\varphi(t)) \geq G_{x,y,y}(t)$ hold for all $x, y \in X$ and $t > 0$, where $\varphi \in \Phi$. If $f_0f_1 = f_1f_0$ then there exists a unique common fixed point of f_0 and f_1.*

Proof. Let $f = f_0f_1$. Since $\varphi \in \Phi$, for each $t > 0$, there exists $r > t$ such that $\varphi(r) \leq t$.

$$\begin{aligned} G_{fx,fy,fy}(\varphi(t)) &= G_{(f_0f_1)x,(f_0f_1)y,(f_0f_1)y}(\varphi(t)) \\ &= G_{f_0(f_1x),f_0(f_1y),f_0(f_1y)}(\varphi(t)) \\ &\geq G_{f_1x,f_1y,f_1y}(t) \\ &\geq G_{f_1x,f_1y,f_1y}(\varphi(r)) \\ &\geq G_{x,y,y}(r) \\ &\geq G_{x,y,y}(t). \end{aligned}$$

This implies that f is a probabilistic φ-contraction. Then by the Theorem 3.8, we conclude that f has a unique fixed point z in X. Since $f_0f_1 = f_1f_0$, we have $f(f_0z) = f_0f_1(f_0z) = f_0(f_1f_0z) = f_0z$ and $f(f_1z) = f_1f_0(f_1z) = f_1(f_0f_1z) = f_1z$. This gives that f_0z and f_1z are also fixed points of f. By the uniqueness of fixed point of f , we have $f_0z = f_1z = z$, i.e., z is a common fixed point of f_0 and f_1. It is clear that z is a unique common fixed point of f_0 and f_1. □

Theorem 3.10. *Let (X, G, T) be a complete Menger PGM-space with a t-norm T of H-type. Let $f : X \to X$ be a mapping satisfying*

$$G_{fx,fy,fz}(\varphi(t)) \geq \frac{1}{3}\left(G_{x,fx,fx}(t) + G_{y,fy,fy}(t) + G_{z,fz,fz}(t)\right) \tag{3.12}$$

for all $x, y, z \in X$, where $\varphi \in \Phi$. Then, for any $x_0 \in X$ the sequence $\{f^n(x_0)\}$ converges to a unique fixed point of f.

Proof. Take an arbitrary point $x_0 \in X$. Construct a sequence $\{x_n\}$ by $x_{n+1} = f^n(x_0)$ for all $n \geq 0$. Since $\varphi \in \Phi$, for each $t > 0$ there exists $r > t$ such that $\varphi(r) \leq t$. Then

$$\begin{aligned} G_{x_n,x_{n+1},x_{n+1}}(t) &\geq G_{x_n,x_{n+1},x_{n+1}}(\varphi(r)) \\ &= G_{fx_{n-1},fx_n,fx_n}(\varphi(r)) \\ &\geq \frac{1}{3}\left(G_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + 2G_{x_n,fx_n,fx_n}(r)\right) \\ &\geq \frac{1}{3}\left(G_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + 2G_{x_n,fx_n,fx_n}(t)\right) \\ &= \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + 2G_{x_n,x_{n+1},x_{n+1}}(t)\right). \end{aligned}$$

That is, for all $t > 0$,

$$G_{x_n,x_{n+1},x_{n+1}}(t) \geq G_{x_{n-1},x_n,x_n}(t). \tag{3.13}$$

Now, we prove that f is a φ-contraction in Menger PGM-space. For this, we have

$$\begin{aligned}
G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) &= G_{fx_{n-1},fx_n,fx_n}(\varphi(t)) \\
&\geq \frac{1}{3}\left(G_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + 2G_{x_n,fx_n,fx_n}(t)\right) \\
&= \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + 2G_{x_n,fx_n,fx_n}(t)\right) \\
&\geq \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + 2G_{fx_{n-1},fx_n,fx_n}(\varphi(r))\right) \\
&\geq \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + \frac{2}{3}\left(G_{x_{n-1},fx_{n-1},fx_{n-1}}(r) + 2G_{x_n,fx_n,fx_n}(r)\right)\right) \\
&= \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + \frac{2}{3}G_{x_{n-1},x_n,x_n}(r) + \frac{4}{3}G_{x_n,x_{n+1},x_{n+1}}(r)\right) \\
&\geq \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + \frac{2}{3}G_{x_{n-1},x_n,x_n}(t) + \frac{4}{3}G_{x_n,x_{n+1},x_{n+1}}(t)\right) \\
&= \frac{1}{3}\left(\frac{5}{3}G_{x_{n-1},x_n,x_n}(t) + \frac{4}{3}G_{x_n,x_{n+1},x_{n+1}}(t)\right) \\
&\geq \frac{1}{3}\left(\frac{5}{3}G_{x_{n-1},x_n,x_n}(t) + \frac{4}{3}G_{x_{n-1},x_n,x_n}(t)\right) \\
&= G_{x_{n-1},x_n,x_n}(t).
\end{aligned}$$

Here, first and third inequalities are due to (3.12), second and fourth are due to the monotonic increasing property of distribution function while the last one is due to (3.13). Therefore, f is φ-contraction, and Lemma 3.6 shows that $\{x_n\}$ is a Cauchy sequence. Since X is complete Menger PGM-space, there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now, since $\varphi(r) \leq t$ and $G_{fx_n,fx,fx}$ is monotonically increasing, from (3.12) we have

$$G_{fx_n,fx,fx}(t) \geq G_{fx_n,fx,fx}(\varphi(r)) \geq \tfrac{1}{3}\left(G_{x_n,fx_n,fx_n}(r) + 2G_{x,fx,fx}(r)\right).$$

Taking limit as $n \to \infty$ in this inequality, we get

$$G_{x,fx,fx}(t) \geq \tfrac{1}{3}\left(G_{x,x,x}(r) + 2G_{x,fx,fx}(r)\right) \geq \tfrac{1}{3}\left(G_{x,x,x}(t) + 2G_{x,fx,fx}(t)\right),$$

which gives $G_{x,fx,fx}(t) \geq G_{x,x,x}(t) = 1$, for all $t > 0$. Thus, we have proved that $fx = x$. To show the uniqueness of the fixed point of f, we suppose that y is another fixed point of f. Then, for all $t > 0$

$$\begin{aligned} G_{x,y,y}(t) &\geq G_{x,y,y}(\varphi(r)) \\ &= G_{fx,fy,fy}(\varphi(r)) \\ &\geq \frac{1}{3}\left(G_{x,fx,fx}(r) + 2G_{y,fy,fy}(r)\right) \\ &\geq \frac{1}{3}\left(G_{x,fx,fx}(t) + 2G_{y,fy,fy}(t)\right) \\ &= 1. \end{aligned}$$

Here, first and third inequality is due to the monotonic increasing property of distribution function and the second one is due to (3.12). This shows that $x = y$. Therefore, f has a unique fixed point. □

Example 3.3. *Let $X = [0, \infty)$ and $T(a, b) = \min\{a, b\}$, for all $a, b \in X$. Define a function $G : X \times X \times X \times [0, \infty) \to [0, \infty)$ by $G_{x,y,z}(t) = \frac{t}{t+(|x-y|+|y-z|+|z-x|)}$. Then (X, G, T) is a complete Menger PGM-space. Define $f : X \to X$ by $f(x) = \frac{x}{4}$, for each $x \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ by*

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{4^n} & \text{if } \frac{1}{4^n} \leq t < \frac{1}{4^{n-1}} \\ kt & \text{if } t \geq 1, \text{where } \frac{1}{4} \leq k < 1. \end{cases}$$

Obviously, $\varphi \in \Phi$. Now, we want to show that f is φ-contraction.

Case 1: $\frac{1}{4^n} \leq t < \frac{1}{4^{n-1}}$

Since $\frac{1}{4^{n-1}} > t$, we have $\frac{1}{4^n} > \frac{t}{4}$, i.e., $\varphi(t) \geq \frac{t}{4}$.

Case 2: $t \geq 1$

Since $k \geq \frac{1}{4}$, we have $kt \geq \frac{t}{4}$, i.e., $\varphi(t) \geq \frac{t}{4}$.

Thus, we have $\varphi(t) \geq \frac{t}{4}$ *for each* $t > 0$. *Now, since the function* $\frac{t}{t+1}$ *is strictly increasing on* $[0, \infty)$, *we have*

$$\begin{aligned}
G_{fx,fy,fz}(\varphi(t)) &= \frac{\varphi(t)}{\varphi(t) + (|fx - fy| + |fy - fz| + |fz - fx|)} \\
&= \frac{\varphi(t)}{\varphi(t) + \frac{1}{4}(|x - y| + |y - z| + |z - x|)} \\
&\geq \frac{\frac{t}{4}}{\frac{t}{4} + \frac{1}{4}(|x - y| + |y - z| + |z - x|)} \\
&= \frac{t}{t + (|x - y| + |y - z| + |z - x|)} \\
&= G_{x,y,z}(t).
\end{aligned}$$

Therefore, from the Theorem 3.8 f *has unique fixed point. In fact, the fixed point is* $x = 0$.

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