Chapter 2

Probabilistic contraction under a control function

2.1 Introduction

The purpose of this work is to give a generalization of Sehgal's contraction [101] by defining and utilizing a new class of control functions. One of the early uses of control functions in probabilistic fixed point theory is due to Choudhury et al. [18]. Other results involving control functions in fixed point theory on PM-spaces have appeared in works of [4, 26, 46, 53, 63, 100, 113]. Recently, several such control functions were generalized in a definition given by Fang [39].

Characteristics of the probabilistic metric spaces strongly depend on the t-norm associated with the space. Further, the notion of G-Cauchy sequence and G-completeness used in this chapter weaker than the ordinary Cauchy sequence and completeness of the probabilistic metric space. Here, we have established two theorems, in one of which it is shown that the contraction defined has a unique fixed point in a G-complete probabilistic metric space with any continuous t-norm while in the other theorem, we show that the same result is true in a complete probabilistic metric space provided we make a particular choice of minimum t-norm.

In the following, we mention some features of the present work:

(i) We introduce a new contraction mapping with the help of a control function.

(ii) The control function is new and independent of other such functions in literature given by different authors.

(iii) In one theorem, we use the weak version of completeness, namely G-completeness.

(iv) Probabilistic generalizations of the contraction mapping principle are established in G-complete and complete metric spaces.

(v) The results are further illustrated with suitable examples.

2.2 Preliminaries

Now, we describe the definitions and mathematical conventions which are required for the discussion of the results established in this chapter.

Definition 2.1. [63]

1. A sequence $\{x_n\}$ in (X, F, T) is called a G-Cauchy sequence if for any given $\varepsilon > 0$ and $\lambda \in (0, 1]$, there exists $N \in \mathbb{N}$ depending on ε and λ such that $F_{x_n,x_{n+k}}(\varepsilon) > 1 - \lambda$, for all $n \geq N$ and $k \in \mathbb{N}$.

2. A Menger space (X, F, T) is said to be G-complete if each G-Cauchy sequence $\{x_n\}$ in X is convergent to some point $x \in X$.

G-Cauchy property of a sequence is weaker than the Cauchy property of a sequence with the latter implying the former. We now introduce the class Φ of a function φ as follows:

Definition 2.2. Φ is the class all of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $t > 0$ there exists $r > t$ with $\varphi(r) \leq t$.

An example of this type of function is given below:

Example 2.1. $$\varphi(t) = \begin{cases} 0 & \textit{if} \quad t = 0 \\ \frac{1}{2^n} & \textit{if} \quad \frac{1}{2^n} \leq t < \frac{1}{2^{n-1}} \textit{ for } n \in \mathbb{N} \\ kt & \textit{if} \quad t \geq 1, \textit{ where } 0 < k < 1. \end{cases}$$

In the above example, it can be shown that the class Φ of Definition 2.2 is not covered by the class used by Fang [39].

2.3 Main results

Lemma 2.3. *Let (X, F, T) be a Menger space. Let $\{x_n\}$ be a sequence in X and $\{t_m\}$ be a sequence of non-negative real numbers. Suppose that the sequence $\left\{F_{x_n,x_{n+1}}(t_m)\right\}$ is non-decreasing in both variables m and n, for each $m, n \in \mathbb{N}$, that is , $F_{x_n,x_{n+1}}(t_m) \geq F_{x_{n-1},x_n}(t_m)$ and $F_{x_n,x_{n+1}}(t_{m+1}) \geq F_{x_n,x_{n+1}}(t_m)$ for each m and n. Then $\lim\limits_{n\to\infty}\left(\lim\limits_{m\to\infty} F_{x_n,x_{n+1}}(t_m)\right) = \lim\limits_{m\to\infty}\left(\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(t_m)\right)$.*

Proof. Denote $b_n = \underset{m\to\infty}{lim} F_{x_n,x_{n+1}}(t_m)$ and $c_m = \underset{n\to\infty}{lim} F_{x_n,x_{n+1}}(t_m)$.

Our assumption of monotonicity and boundedness of distribution function imply that these limits are finite.

Since $F_{x_{n+1},x_{n+2}}(t_m) \geq F_{x_n,x_{n+1}}(t_m)$, we have $\lim_{m\to\infty} F_{x_n,x_{n+1}}(t_m) \geq \lim_{m\to\infty} F_{x_{n-1},x_n}(t_m)$.

This gives that $b_{n+1} \geq b_n$, that is , the sequence $\{b_n\}$ is non-decreasing. Similarly, we find that the sequence $\{c_m\}$ is non-decreasing.

Let $b = \lim_{n\to\infty} b_n$ and $c = \lim_{m\to\infty} c_m$.

We have $F_{x_n,x_{n+1}}(t_m) \leq c_m \Rightarrow \lim_{m\to\infty} F_{x_n,x_{n+1}}(t_m) \leq \lim_{m\to\infty} c_m$ which implies that $b_n \leq c$, that is, $\lim_{n\to\infty} b_n \leq c$.

Thus, $b \leq c$. The proof of $c \leq b$ is analogous. Therefore, $b = c$, that is,

$$\underset{n\to\infty}{lim}\left(\underset{m\to\infty}{lim} F_{x_n,x_{n+1}}(t_m)\right) = \underset{m\to\infty}{lim}\left(\underset{n\to\infty}{lim} F_{x_n,x_{n+1}}(t_m)\right). \qquad \square$$

Lemma 2.4. *Let (X, F, T) be a Menger space with a t-norm T. Let $\{x_n\}$ be a sequence in (X, F, T). If there exists a function $\varphi \in \Phi$ such that*

$$F_{x_n,x_{n+1}}(\varphi(t)) \geq F_{x_{n-1},x_n}(t)\,, \textit{ for all } n \in \mathbb{N} \textit{ and } t > 0. \tag{2.1}$$

Then $\underset{n\to\infty}{lim} F_{x_n,x_{n+1}}(t) = 1$.

Proof. Let $t_0 > 0$ be arbitrary.

Since $\varphi \in \Phi$, there exists $t_1 > t_0$ such that

$$\varphi(t_1) \leq t_0. \tag{2.2}$$

Now, since $F_{x_n,x_{n+1}}(\varphi(t)) \geq F_{x_{n-1},x_n}(t)$, by the monotonic increasing property of the distribution function along with equations (2.1) and (2.2), we have

$$F_{x_n,x_{n+1}}(t_1) \geq F_{x_n,x_{n+1}}(t_0) \geq F_{x_n,x_{n+1}}(\varphi(t_1)) \geq F_{x_{n-1},x_n}(t_1) \geq F_{x_{n-1},x_n}(t_0). \quad (2.3)$$

Since the choice of $t_0 > 0$ is arbitrary, $\{F_{x_n,x_{n+1}}\}$ is monotonically increasing in n for all $t > 0$. In particular, the sequence $\left\{F_{x_n,x_{n+1}}(t_0)\right\}$ is monotonically increasing in n and being bounded above, is convergent. Let $\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(t_0) = l$. Let, if possible, $l < 1$. Then $\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(t_1) = l$ (by inequality (2.3)). Then, by sandwich theorem, $\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(t) = l < 1$, for all $t \in [t_0, t_1]$. Let

$$\bar{t} = sup\, A, \text{ where } A = \left\{t : \lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = l\right\}. \quad (2.4)$$

If $\bar{t}$ is finite then there exists monotonically increasing sequence $\{t_m\} \subset A$ such that for all $m \in \mathbb{N}$,

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t_m) = l \text{ and } t_m \to \bar{t} \text{ as } m \to \infty. \quad (2.5)$$

Since $F_{x_n,x_{n+1}}$ is left continuous, $F_{x_n,x_{n+1}}(\bar{t}) = \lim\limits_{m\to\infty} F_{x_n,x_{n+1}}(t_m)$.

Therefore, $\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(\bar{t}) = \lim\limits_{n\to\infty}\left(\lim\limits_{m\to\infty} F_{x_n,x_{n+1}}(t_m)\right) = \lim\limits_{m\to\infty}\left(\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(t_m)\right) = l$ (by Lemma 2.3 and (2.5)).

Therefore, $\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(\bar{t}) = l$. Then proceeding, as in the above case of t_0, there exists $\bar{t_1}$ such that $\lim\limits_{n\to\infty} F_{x_n,x_{n+1}}(\bar{t_1}) = l$ and $\bar{t_1} > \bar{t}$, which is a contradiction with (2.4). Thus, for all $t > t_0$

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = l. \quad (2.6)$$

Since $F_{x,y}(t) \to 1$ as $t \to \infty$, there exists $s > t_0$ such that $F_{x_k,x_{k+1}}(s) > l$ for some $k \in \mathbb{N}$. Also, we have established that $\left\{F_{x_n,x_{n+1}}(t)\right\}$ is monotonically increasing in n for all $t > 0$. Thus, the sequence $\left\{F_{x_n,x_{n+1}}(s)\right\}$ is monotonically increasing in

n, and that $\underset{n\to\infty}{lim} F_{x_n,x_{n+1}}(s) > l$. But this is a contradiction to (2.6) since $s > t_0$. Therefore, $\underset{n\to\infty}{lim} F_{x_n,x_{n+1}}(t) = 1$ for all $t > t_0$. Since $t_0 > 0$ is arbitrary, we conclude that

$$\underset{n\to\infty}{lim} F_{x_n,x_{n+1}}(t) = 1, \text{ for all } t > 0. \tag{2.7}$$

□

Lemma 2.5. *Let (X, F, T) be a Menger space with a t-norm T. Let $\{x_n\}$ be a sequence in (X, F, T). If there exists a function $\varphi \in \Phi$ such that*

$$F_{x_n,x_{n+1}}(\varphi(t)) \geq F_{x_{n-1},x_n}(t), \textit{ for all } n \in \mathbb{N} \textit{ and } t > 0. \tag{2.8}$$

Then $\{x_n\}$ is a G-Cauchy sequence in X.

Proof. In view of the definition of G-Cauchy sequence, we have to prove that

$$\underset{n\to\infty}{lim} F_{x_n,x_{n+k}}(t) = 1, \tag{2.9}$$

for all $t > 0$ and $k \in \mathbb{N}$. We use induction on k to prove this result. For $k = 1$, equation (2.9) becomes $\underset{n\to\infty}{lim} F_{x_n,x_{n+1}}(t) = 1$, which is true for all $t > 0$ by equation (2.7). Now, assume that equation (2.9) holds for some $k \in \mathbb{N}$ and for all $t > 0$, that is, for all $t > 0$,

$$\underset{n\to\infty}{lim} F_{x_n,x_{n+k}}(t) = 1. \tag{2.10}$$

Then,

$$F_{x_n,x_{n+k+1}}(t) \geq T\left(F_{x_n,x_{n+k}}\left(\frac{t}{2}\right), F_{x_{n+k},x_{n+k+1}}\left(\frac{t}{2}\right)\right).$$

Since, T is continuous,

$$\liminf_{n\to\infty} F_{x_n,x_{n+k+1}}(t) \geq T\left(\lim_{n\to\infty} F_{x_n,x_{n+k}}\left(\frac{t}{2}\right), \lim_{n\to\infty} F_{x_{n+k},x_{n+k+1}}\left(\frac{t}{2}\right)\right). \tag{2.11}$$

Using equations (2.7) and (2.10) in the inequality (2.11), we get

$$\liminf_{n\to\infty} F_{x_n,x_{n+k+1}}(t) \geq T(1,1) = 1,$$

which implies that $\lim_{n\to\infty} F_{x_n,x_{n+k+1}}(t) = 1$, for all $t > 0$. Thus, equation (2.9) is true if we replace k by $k+1$. Then, by induction, we conclude that $\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) = 1$, for all $t > 0$. Hence $\{x_n\}$ is a G-Cauchy sequence. □

In the next theorem, we establish the existence of a fixed point for the probabilistic Φ-contraction.

Theorem 2.6. *Let (X, F, T) be a G-complete Menger space with a continuous t-norm T. If $f : X \to X$ is a probabilistic φ-contraction , that is,*

$$F_{fx,fy}(\varphi(t)) \geq F_{x,y}(t), \text{ for all } x, y \in X \text{ and } t > 0, \tag{2.12}$$

where $\varphi \in \Phi$, then f has a fixed point $x_* \in X$ and $\{f^n(x_0)\}$ converges to x_* for arbitrary $x_0 \in X$.

Proof. Taking an arbitrary $x_0 \in X$, we define the sequence $\{x_n\}$ as follows: Let $x_0 \in X$ and $x_n = fx_{n-1}$, for all $n \in \mathbb{N}$. Then by (2.12),

$$\begin{aligned} F_{x_n,x_{n+1}}(\varphi(t)) &= F_{fx_{n-1},fx_n}(\varphi(t)) \\ &\geq F_{x_{n-1},x_n}(t), \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \end{aligned}$$

Then by Lemma 2.5, we conclude that $\{x_n\}$ is a G-Cauchy sequence in X. Since X is G-complete, we have $x_n \to x_*$ for some $x_* \in X$. Since $\varphi \in \Phi$, for each $t > 0$ there exists $r > t$ such that $\varphi(r) \leq t$. Now, $F_{fx_n,fx_*}(t) \geq F_{fx_n,fx_*}(\varphi(r)) \geq F_{x_n,x_*}(r)$. This gives, $F_{fx_n,fx_*}(t) \geq F_{x_n,x_*}(r)$. Taking limit $n \to \infty$ in this inequality and since $x_n \to x_*$, for each $t > 0$, we get

$$\lim_{n\to\infty} F_{fx_n,fx_*}(t) = 1. \tag{2.13}$$

Then,

$$\begin{aligned} F_{x_*,fx_*}(t) \geq T\left(F_{x_*,x_{n+1}}\left(\frac{t}{2}\right), F_{x_{n+1},fx_*}\left(\frac{t}{2}\right)\right) \\ = T\left(F_{x_*,x_{n+1}}\left(\frac{t}{2}\right), F_{fx_n,fx_*}\left(\frac{t}{2}\right)\right). \end{aligned}$$

Taking limit $n \to \infty$ in the above inequality, using equation (2.13) and the continuity of T, we get, for all $t > 0$,

$$F_{x_*,fx_*}(t) \geq T(1,1) = 1.$$

Hence, $fx_* = x_*$, which proves that x_* is a fixed point of f. □

In the next result we prove the uniqueness of the fixed point of f, for which we require the following lemma.

Lemma 2.7. *Let (X, F, T) be a Menger space and $x, y \in X$. If there exists a function $\varphi \in \Phi$ such that*

$$F_{x,y}(\varphi(t)) \geq F_{x,y}(t), \tag{2.14}$$

for all $t > 0$, then $x = y$.

Proof. In order to show $x = y$, we need to prove that $F_{x,y}(t) = 1$, for all $t > 0$. Suppose that there exists t_0 such that $F_{x,y}(t_0) < 1$. Since $\varphi \in \Phi$, there exists $t_1 > t_0$ such that $\varphi(t_1) \leq t_0$. Then equation (2.14) and monotonic increasing property of

$F_{x,y}$ together imply the following inequality

$$F_{x,y}(t_0) \geq F_{x,y}(\varphi(t_1)) \geq F_{x,y}(t_1) \geq F_{x,y}(t_0). \tag{2.15}$$

In case of strict inequality anywhere in (2.15), we have a contradiction. So we assume that equality holds, and in particular, $F_{x,y}(t_1) = F_{x,y}(t_0)$.

Clearly, the set $A = \{s : F_{x,y}(s) = F_{x,y}(t_0); s > t_0\}$ is non-empty by the above consideration. Let, if possible, $\bar{s}$ =sup$A$ be finite. Then there exists a monotonically increasing sequence $\{s_n\}$ with $s_n \in A$ for all $n \in \mathbb{N}$ such that $s_n \to \bar{s}$. Since $F_{x,y}$ is left continuous, it follows that $F_{x,y}(\bar{s}) = \lim_{n\to\infty} F_{x,y}(s_n) = F_{x,y}(t_0) < 1$. This implies that $\bar{s} \in A$. Then again treating $\bar{s}$ in the same way as t_0 we obtain some $\bar{s_1} > \bar{s}$ such that $F_{x,y}(\bar{s_1}) = F_{x,y}(\bar{s}) = F_{x,y}(t_0)$ which contradictions the fact that $\bar{s} = \sup A$.

Hence, A is unbounded above. Therefore, there exists $\{s_n\} \subset A$ which is monotone increasing and diverges to infinity. Hence, $\lim_{n\to\infty} F_{x,y}(s_n) = 1$. But $s_n \in A$. Therefore, $F_{x,y}(s_n) = F_{x,y}(t_0) < 1$, for all $n \in \mathbb{N}$, which is a contradiction.
Hence, $F_{x,y}(t) = 1$ for all $t > 0$, that is $x = y$. □

Theorem 2.8. *The fixed point in the Theorem 2.6 is unique.*

Proof. Suppose that y_* is another fixed point of f. Then using (2.12), we get

$$\begin{aligned} F_{x_*,y_*}(\varphi(t)) &= F_{fx_*,fy_*}(\varphi(t)) \\ &\geq F_{x_*,y_*}(t), \text{ for all } t > 0. \end{aligned}$$

Then by Lemma 2.7, we get $x_* = y_*$. □

In our next result, We show that Theorems 2.6 and 2.8 are valid if the space is complete Menger space as well if the t-norm is minimum t-norm.

Theorem 2.9. *Let (X, F, T) be a complete Menger space with minimum t-norm T. If $f : X \to X$ is a probabilistic φ-contraction, that is,*

$$F_{fx,fy}(\varphi(t)) \geq F_{x,y}(t), \text{ for all } x, y \in X \text{ and } t > 0, \tag{2.16}$$

where $\varphi \in \Phi$. Then f has a unique fixed point.

Proof. Starting with arbitrary $x_0 \in X$, we construct the sequence $x_n = fx_{n-1} = f^n x_0$ for all $n \geq 1$. Then by Lemma 2.4, we prove that

$$F_{x_n,x_{n+1}}(t) \to 1 \text{ as } n \to \infty \text{ for all } t > 0. \tag{2.17}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, then due to violation of Definition 1.19, there exist $\epsilon > 0$ and $0 < \lambda < 1$ and sequences of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that for all $k \geq 1$,

$$F_{x_{m(k)},x_{n(k)}}(\epsilon) \leq 1 - \lambda \tag{2.18}$$

and

$$F_{x_{m(k)},x_{n(k)-1}}(\epsilon) > 1 - \lambda. \tag{2.19}$$

Now, by a property of φ, there exists $r > \epsilon$ such that $\varphi(r) \leq \epsilon$. Let $r = \epsilon + \eta$, where $\eta > 0$. Then for all $k \geq 1$,

$$\begin{aligned} 1 - \lambda &\geq F_{x_{m(k)},x_{n(k)}}(\epsilon) && \text{(by (2.18))} \\ &\geq F_{fx_{m(k)-1},fx_{n(k)-1}}(\varphi(r)) \\ &\geq F_{x_{m(k)-1},x_{n(k)-1}}(r) && \text{(by (2.16))} \\ &= F_{x_{m(k)-1},x_{n(k)-1}}(\epsilon + \eta) && (\text{since } r = \epsilon + \eta) \\ &\geq T(F_{x_{m(k)-1},x_{m(k)}}(\eta), F_{x_{m(k)},x_{n(k)-1}}(\epsilon)). \end{aligned} \tag{2.20}$$

In view of (2.17), there exists an integer k_0 such that

$$F_{x_{m(k)-1},x_{m(k)}}(\eta) > 1-\lambda \text{ for all } k > k_0. \tag{2.21}$$

Then choosing $k > k_0$, by virtue of (2.19) and (2.21), since T is a minimum t-norm, we obtain,

$$1-\lambda \geq \min(F_{x_{m(k)-1},x_{m(k)}}(\eta), F_{x_{m(k)},x_{n(k)-1}}(\epsilon)) > 1-\lambda,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence and hence converges to some $x_* \in X$. The rest of the proof is analogous to Theorem 2.6 and the uniqueness of fixed point is by an application of Theorem 2.8. □

In the following, we have the corollary of Theorem 2.9 which is the Sehgal's contraction mapping theorem in the above mentioned space.

Corollary 2.10. *[101] Let (X, F, T) be a complete Menger space with $T(a, b) = \min\{a, b\}$. Let $f : X \to X$ be such that for some $0 < k < 1$, for all $x, y \in X$ and $t > 0$, the following inequality holds:*

$$F_{fx,fy}(kt) \geq F_{x,y}(t).$$

Then f has a unique fixed point.

Proof. The corollary follows by assuming $\varphi(t) = kt$, for all $t \geq 0$, in Theorem 2.9. □

2.4 Illustrations

In this section, we discuss two examples to illustrate the results obtained in the previous sections.

Example 2.2. *Let* $X = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\}$ *and* $T(a,b) = \min(a,b)$, *for all* $a, b \in X$. *Define a function* $F : X \times X \to \boldsymbol{D}^{+}$, *by* $F_{x,y}(t) = \frac{t}{t+|x-y|}$. *Then, clearly* (X, F, T) *is a G-complete Menger space. Define* $f : X \to X$ *by* $f(x) = \frac{x^2}{4}$, *for each* $x \in X$ *and* $\varphi : [0,\infty) \to [0,\infty)$ *by*

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{2^n} & \text{if } \frac{1}{2^n} \le t < \frac{1}{2^{n-1}} \\ kt & \text{if } t \ge 1, \text{where } \frac{1}{2} \le k < 1. \end{cases}$$

Obviously, $\varphi \in \Phi$. *Now, we want to show that* f *is* φ*-contraction.*

Case 1: $t = 0$.

Then $\varphi(t) = 0 = t$.

Case 2: $\frac{1}{2^n} \le t < \frac{1}{2^{n-1}}$.

Since $\frac{1}{2^{n-1}} > t$, *we have* $\frac{1}{2^n} > \frac{t}{2}$, *that is,* $\varphi(t) \ge \frac{t}{2}$.

Case 3: $t \ge 1$.

Since, $k \ge \frac{1}{2}$, *we have* $kt \ge \frac{t}{2}$, *that is,* $\varphi(t) \ge \frac{t}{2}$.

Thus, we get $\varphi(t) \ge \frac{t}{2}$, *for all* $t \in [0,\infty)$. *Now, since the function* $\frac{t}{t+1}$ *is strictly increasing on* $[0,\infty)$, *we have*

$$
\begin{aligned}
F_{fx,fy}\left(\varphi\left(t\right)\right) &= \frac{\varphi\left(t\right)}{\varphi\left(t\right)+|fx-fy|} \\
&= \frac{\varphi\left(t\right)}{\varphi\left(t\right)+\frac{1}{4}|x^2-y^2|} \\
&(since\ |x^2-y^2| = |x-y||x+y| \leq 2|x-y|\ \ as\ x,y\in X) \\
&\geq \frac{\frac{t}{2}}{\frac{t}{2}+\frac{1}{2}|x-y|} \\
&= \frac{t}{t+|x-y|} \\
&= F_{x,y}\left(t\right).
\end{aligned}
$$

Therefore, from the Theorem 2.6, f has a unique fixed point. In fact, the unique fixed point is $x = 0$.

Example 2.3. *We consider the complete Menger space (X, F, T), where $X = [0, \infty)$, $T(a, b) = \min\{a, b\}$ and F is defined as*

$$
F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & if\ \ |x-y| \geq t \\ 1 & if\ \ |x-y| < t. \end{cases}
$$

$$
Let\ \varphi(t) = \begin{cases} 0 & if\ \ t = 0 \\ \frac{1}{2^n} & if\ \ \frac{1}{2^n} \leq t < \frac{1}{2^{n-1}},\ \ for\ \ n \geq 1 \\ k & if\ k \leq t < k+1, \end{cases}
$$

and $fx = \frac{x}{x+1}$, for all $x \in X$. Then $\varphi(t) > \frac{t}{1+t}$. If $|fx-fy| < \varphi(t)$, then the inequality (2.16) is satisfied. So we assume that $|fx - fy| \geq \varphi(t)$. Now, $|fx - fy| \leq \frac{|x-y|}{1+|x-y|}$. Then, $\frac{t}{t+1} \leq \frac{|x-y|}{1+|x-y|}$. Since $\frac{t}{t+1}$ is monotone increasing in t, we have $|x - y| \geq t$, *and then it follows that*

$$\begin{aligned} F_{fx,fy}(\varphi(t)) &= \frac{\varphi(t)}{\varphi(t)+|fx-fy|} \\ &\geq \frac{\frac{t}{t+1}}{\frac{t}{t+1}+\frac{|x-y|}{1+|x-y|}} \\ &\geq \frac{\frac{t}{t+1}}{\frac{t}{t+1}+|x-y|} \\ &> \frac{t}{t+|x-y|} \\ &= F_{x,y}(t). \end{aligned}$$

Then by Theorem 2.9 there exist a unique fixed point in this example. Here, 0 *is the unique fixed point.*

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