Chapter 1

Introduction

The fixed point theory has been originated to establish the solutions of differential equations via the method of successive approximations discussed independently by Picard [83] and Liouville [65]. But formally it was initiated in the starting of twentieth century as a significant part of analysis. In 1906, Frechet introduced the concept of metric spaces. The distance between two points in these spaces is a non-negative real number. Following this idea, in 1922, the great Polish mathematician Stefan Banach [9] established the first result on metric fixed point theory. This result is known as Banach contraction mapping principle which states that "a contraction mapping on a complete metric space has a unique fixed point". Banach used the idea of contraction map to obtain this celebrated result. Another important fixed point theorem is due to Brouwer stating that "a continuous map on a closed unit ball in $\mathbb{R}^n$ has a fixed point". Brouwer fixed point theorem is of an important tool for the numerical treatment of various types of equations. Schauder's fixed point theorem [97] is a generalization of Brouwer fixed point theorem which exactly states that "a continuous map on a convex compact subspace of a Banach space has a fixed point". These celebrated results have been used and extended in different ways by several authors (see, [10, 13]).

In 1942, K. Menger [67] introduced the idea of probabilistic metric spaces as a generalization of metric spaces. Probabilistic metric spaces are mathematical structures in which the distance between any two objects is given by a probability distribution function rather than by a non-negative number. After that many authors have extended this concept in various directions. Menger space is a particular type of such spaces where the probabilistic triangular inequalities obtained by the use of a t-norm. The theory of this structure was developed mainly after 1960 through the works of different mathematicians. A comprehensive account of this is found in the book of Schweizer and Sklar [98] published in 1983. The theoretical aspects of this structure are considered important as a study in probabilistic analysis which is an extension of metric space concepts into the domain of theoretical probability. It is also important in applications due to the inherent probabilistic nature of its geometry. A recent example of such an application in nuclear fusion related problem can be found in [41]. There are also several studies in analysis which have been extended to probabilistic metric spaces. For such extension, one can refer to [14] where a probabilistic fixed point result is used to prove the existence of the solution of the differential equation in probabilistic metric spaces.

The probabilistic metric space has been generalized in many directions. These extensions are motivated by the same needs as that of generalizing ordinary metric spaces. Fixed point results in those generalizations have also been developed in recent years. In 1963, Gahler [40] introduced the idea of 2-metric spaces as a natural generalization of an ordinary metric space. But some authors pointed out that 2-metric need not be continuous function of its variables. Later, Dhage [31] defined D-metric spaces as another class of generalized metric spaces. However, it is found that the D-metric need not be continuous in its variables as mentioned in [70].

Further, in 2006 this concept is extended by Mustafa and Sims [70] to define generalized metric spaces (G-metric spaces). In the sequel to such developments, different probabilistic generalization had been proposed (see [95, 96, 98, 112]) and the study of the triangle inequalities played a central role to the development of generalized probabilistic metric spaces. In 2014, Zhou et al. [116] introduced a Menger probabilistic G-metric space (Menger PGM-space) which generalizes the Menger space. The detailed development of this space can be seen in [4, 24, 29, 34, 63, 107]. For more details about the major steps in this development, we refer [99] to the readers. In 1965, Zadeh [114] introduced the idea of fuzzy set to tackle uncertainty in day to day life. Afterwards, it was explored by several authors towards its various applications. This theory played as a powerful tool in handling uncertainty in nature, industry and engineering applications etc. In the lack of complete and precise information in various situations, this theory has lead to the idea of approximate reasoning to tackle decision making problems. This has helped in dealing with complex situations for which other traditional mathematical tools fail to give a proper solution. Moreover, quantum particle physics in particular string theory has been explained using fuzzy topology by El Naschie [35]. To apply this concept in analysis and topology, fuzzy metric spaces have been defined in different ways by several researchers [42, 100]. Kramosil and Michalek [100] developed the idea of fuzzy metric space. A fixed point result in such space was given by Grabiec in [44] by extending the contraction mapping principle due to Banach. Another concept of Fuzzy Metric spaces was introduced by George and Veeramani [42] by modifying the idea of fuzzy metric spaces and constructed a Hausdorff topology there. They proved that a fuzzy metric can be induced in Hausdorff topology for any given metric.

We are interested in fixed point theory in probabilistic and fuzzy metric spaces. Probabilistic fixed point theory is widely recognized to have initiated by the work of Sehgal et al. [101] where the authors developed probabilistic metric spaces by extending the Banach's contraction mapping principle. The importance of the type of space we consider is that the metric in a probabilistic metric space takes values in a set of distribution functions rather than non-negative real numbers. In this way these spaces have randomness within their structures. Several aspects of mathematical theory of these spaces have developed over the years (for details see [2, 21, 52, 98, 110]). During last four decades, the theory of fixed point has developed enormously to solve many interesting problems in these spaces. Many aspects of such developments are described in the book of Hadzic et al.[47]. Some other results on these spaces are noted in [3, 14, 17, 18, 19, 20, 27, 28, 32, 33, 36, 37, 38, 43, 46, 47, 48, 50, 58, 60, 61, 62, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 82, 85, 86, 102, 103]

This chapter is divided into five sections.

In section 1.1, we give a description of probabilistic metric spaces.

In section 1.2, we discuss on probabilistic generalized metric spaces.

In section 1.3, we describe the concept of fuzzy metric spaces.

In section 1.4 and 1.5 we give contraction and some fixed point results on metric and probabilistic metric spaces.

1.1 A description of probabilistic metric spaces

In this section, we present the following important definitions on probabilistic metric spaces, which are essential for the discussion of the thesis. Just for the purpose of comparison and reference we first give the following definition of a metric space due to Frechet (1906).

Definition 1.1. Let X be an abstract set and $d : X \times X \to \mathbb{R}$ be a mapping that associate a real number $d(a, b)$ with every pair (a, b) of elements of X. If the mapping d satisfy the following conditions:

(i) $d(a, b) = 0$ if and only if $a = b$,

(ii) $d(a, b) > 0$,

(iii) $d(a, b) = d(b, a)$,

(iv) $d(a, d) \leq d(a, c) + d(c, a)$ for all $a, b, c \in X$.

Then d is called a metric and the pair (X, d) is called a metric space.

Definition 1.2. Distribution Function [67]

A mapping $F : \mathbb{R} \to [0, \infty)$ is called a distribution function if it is non-decreasing and left-continuous with $\inf\limits_{x\in\mathbb{R}} F(x) = 0$ and $\sup\limits_{x\in\mathbb{R}} F(x) = 1$. The set of all distribution functions is denoted by D^+.

An example of distribution function is:

Example 1.1. $H(t) = \begin{cases} 0 & \textit{if} \quad t \leq 0 \\ 1 & \textit{if} \quad t > 0. \end{cases}$

Definition 1.3. Probabilistic metric space [47, 99]

Let X be a non empty set and $F : X \times X \to D^+$ be a mapping We denote $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(t)$ represents the value of $F_{x,y}$ at $t \in \mathbb{R}$. If the function $F_{x,y}$ satisfies the following conditions for all $x, y \in X$,

(i) $F_{x,y}(0) = 0$,

(ii) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $F_{x,y}(t) = F_{y,x}(t)$ for all $t > 0$,

(iv) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$ then $F_{x,z}(t_1 + t_2) = 1$ for $t_1, t_2 > 0$,

then ordered pair (X, F) is said to be probabilistic metric space (PM-space).

In view of the condition (i) of 1.3 which states that $F_{x,y}(0) = 0$,, the condition (ii) of 1.3 is equivalent to the condition: $x = y$ if and only if $F_{x,y}(t) = H(t)$. Further, with the description of $F_{x,y}(t)$ as the probability that the distance between x and y is at most x, we know that the first three conditions of 1.3 are the generalization of the first three conditions of 1.1. The condition (iv) of 1.3 is a 'minimal' generalization of the triangle inequality (iv) of 1.1 which is interpreted as follows: If it is true that the distance between x and y is less than t_1, and likewise true that the distance between y and z is less than t_2, then it is true that the distance between x and z is less than $t_1 + t_2$.

The condition (iv) of 1.3 is always satisfied in metric spaces, where it reduces to the ordinary triangle inequality. However, in those PM-spaces in which the equality $F_{x,z}(t_1) = 1$ does not hold for any finite t_1, the condition (iv) of 1.3 will be satisfied only vacuously. It is therefore of interest to have 'stronger' generalization of triangle inequality.

Definition 1.4. t-norm [67, 99]

A t-norm is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying:

(i) $T(1, a) = a$,

(ii) $T(a, b) = T(b, a)$,

(iii) $T(c,d) \geq T(a,b)$ whenever $c \geq a$ and $d \geq b$,

(iv) $T(T(a,b),c) = T(a,T(b,c))$, for all $a,b,c,d \in [0,1]$.

The three basic continuous t-norms can be given as:

(i) minimum t-norm, say T_m, defined by $T_m(a,b) = \min\{a,b\}$,

(ii) product t-norm, say T_p, defined by $T_p(a,b) = a.b$,

(iii) The Lukasiewicz t-norm, say T_L, defined by $T_L(a,b) = \max\{a+b-1,0\}$.

These t-norms are related as: $T_L \leq T_p \leq T_m$.

Another important t-norm, introduced by Hadzic and Pap is known as Hadzic type t-norm.

Definition 1.5. Hadzic type t-norm [47]

A t-norm T is said to be Hadzic type t-norm if the family $\{T^n\}_{n\in N}$ of the iterates, defined for each $s \in (0,1)$ as $T^0(s) = 1$, $T^{n+1}(s) = T(T^n(s),s)$ for all integer $n \geq 0$, is equi-continuous at $s = 1$, that is, given $\epsilon > 0$ there exists $\eta(\epsilon) \in (0,1)$ such that

$1 \geq s > \eta(\epsilon) \Rightarrow T^n(s) \geq 1-\epsilon$ for all $n \geq 0$.

Using t-norm probabilistic metric spaces have been generalized to Menger probabilistic metric spaces.

Definition 1.6. Menger probabilistic metric space [47, 99]

A Menger probabilistic metric space or simply Menger space is a triplet (X,F,T) where X is a non empty set, $F : X \times X \to D^+$ be a function and T is a t-norm such that the following hold:

1. $F_{x,y}(0) = 0$ for all $x, y \in X$
2. $F_{x,y}(t) = 1$ for all $t > 0$ and $x, y \in X$ if and only if $x = y$,
3. $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$,
4. $F_{x,y}(t+s) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $s, t \geq 0$ and $x, y, z \in X$.

Let (X, F, T) be a Menger PM-space. If $p \in X$ and for positive real numbers t and r such that $r < 1$, the open ball $N_p(r, t)$ with center at $p \in X$ is defined by $N_p(r, t) = \{q \in X : F_{p,q}(t) > 1 - r\}$. The interpretation is that, $N_p(r, t)$ is the set of all points q in X for which the probability of the distance between p and q being less than t is greater than $1 - r$.

A subset $O \subset X$ is open if for each $p \in O$, there exist $t > 0$ and $0 < r < 1$ such that $B_p(r, t) \subset O$. Let β denote the family of all open subsets of X. Then β is a topology on X induced by the Menger probabilistic metric F. This topology is Hausdorff topology.

Definition 1.7. Convergent sequence in Menger space [47, 99]

A sequence $\{x_n\} \subset X$ is said to converge to some point $x \in X$ if given $\epsilon > 0$, $\lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that for all $n > N_{\epsilon,\lambda}$, $F_{x_n,x}(\epsilon) > 1 - \lambda$ holds.

Definition 1.8. Cauchy sequence in Menger space [47, 99]

A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if given $\epsilon > 0$, $\lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $m, n > N_{\epsilon,\lambda}$.

Definition 1.9. Complete Menger space [47, 99]

A Menger space (X, F, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

1.2 A description of generalized Menger spaces

Branciari [12] introduced the idea of generalized metric spaces to generalize the definition of metric spaces. He replaced the triangle inequality by a quadrangular inequality. The definition of generalized metric space is as follows:

Definition 1.10. Generalized metric spaces [12]

Let X be a non empty set. $\mathbb{R}^+$ be the set of all non negative real numbers and d is a mapping from $X \times X$ into $\mathbb{R}^+$. If for all $x, y \in X$ and for all points $p, q \in X$, each of them different from x and y, the following holds.

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ and
3. $d(x, y) \leq d(x, p) + d(p, q) + d(q, y)$.

Then d is called generalized metric and (X, d) is generalized metric space.

Banach contraction mapping theorem in generalized metric spaces was established by Branciari [12] in the same work. By an example, he showed that there exist generalized metric spaces which are not the metric spaces. After that some other type of fixed point theorem are established by various authors. Some of these results may be seen in [89, 90, 91]. Apart from the way 1.10, many others have introduced generalized metric spaces in different ways. Recently, Mustafa and Sims [70] introduced a new class of generalized metric spaces, called G-metric spaces, as follows:

Definition 1.11. G-Metric space [70]

Let X be a non-empty set and G be a function defined on $X \times X \times X$ to $\mathbb{R}^+$. Then (X, G) is a G-metric space if for all $x, y, z, a \in X$, G satisfy the following conditions:

(i) $G(x, y, z) = 0$ if $x = y = z$,

(ii) $G(x, x, z) > 0$ for $x \neq y$,

(iii) $G(x, y, z) \geq G(x, x, y)$ for $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = G(y, x, z) = G(z, x, y) = G(z, y, x)$,

(v) $G(x, a, a) + G(a, y, z) \geq G(x, y, z)$.

It was shown in [70] that ordinary metric can be generalized as the G-metric. In the same vein, Zhou et al. [116] introduced the probabilistic generalization of G-metric spaces with obtaining some fixed point results. This metric space is known as Menger probabilistic generalized metric space or Menger PGM-space.

Definition 1.12. Menger PGM-space [116]

The 3-tuple (X, G, T) is called a Menger PGM-space if X is a non-empty set, T is a continuous t-norm and $G : X \times X \times X \to D^+$ be a mapping satisfying the following conditions:

(i) $G_{x,y,z}(t) = 1$, for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$,

(ii) $G_{x,x,y}(t) \geq G_{x,y,z}(t)$, for all $x, y \in X$ with $z \neq y$ and $t > 0$,

(iii) $G_{x,y,z}(t) = G_{p(x,y,z)}(t)$, where p is a permutation function,

(iv) $G_{x,y,z}(t + s) \geq T(G_{x,a,a}(s), G_{a,y,z}(t))$, for all $x, y, z, a \in X$ and $s, t > 0$.

Example 1.2. *[116]*

Let (X, F, T) *be a PM-space. Define a function* $G : X \times X \times X \to D^+$ *by* $G_{x,y,z}(t) = \min\{F_{x,y}(t), F_{y,z}(t), F_{x,z}(t)\}$ $\forall\ x, y, z \in X$ *and* $t > 0$*. Then* (X, G, T) *is a Menger PGM-space.*

Definition 1.13. Cauchy sequence in Menger PGM-space [116]

Let (X, G, T) be a Menger PGM-space. Then $\{x_n\} \subset X$ is said to be convergent to $x \in X$ if given $\lambda > 0$, $\epsilon > 0$ we can find $n_0 \in \mathbb{N}$ depending upon ϵ and $\lambda > 0$ such that for all $n \geq n_0$, $G_{x,x_n,x_n}(\epsilon) \geq 1 - \lambda$ holds.

Definition 1.14. Convergent sequence in Menger PGM-space [116]

Let (X, G, T) be a Menger PGM-space. A $\{x_n\} \subset X$ is called a Cauchy sequence if for any given $\varepsilon > 0$ and $\lambda \in (0, 1]$ there exists $n_0 \in \mathbb{N}$ depending upon ϵ and $\lambda > 0$ such that $G_{x_n,x_m,x_l}(\varepsilon) \geq 1 - \lambda$, whenever $m, n, l \geq n_0$.

Definition 1.15. Complete Menger PGM-space [116]

A Menger PGM-space (X, G, T) is said to be complete if every Cauchy sequence $\{x_n\}$ in X is convergent to some point $x \in X$.

1.3 A description of fuzzy metric spaces

Definition 1.16. KM-fuzzy metric spaces[59]

The 3-tuple (X, M, T) is a fuzzy metric space in the sense of Kramosil and Michalek if X is a non-empty set, T is a t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following axioms:

(KM1) $M(x, y, 0) = 0$.

(KM2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$.

(KM3) $M(x, y, t) = M(y, x, t)$.

(KM4) $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$.

(KM5) The function $M(x, y, .) : (0, \infty) \to [0, 1]$ is left continuous, for all $x, y, z \in X$ and $t, s > 0$.

George and Veeramani in their paper [42] introduced a modification of the above definition to induce a Hausdorff topology on such spaces.

Definition 1.17. GV-fuzzy metric spaces

The 3-tuple (X, M, T) is called a fuzzy metric space in the sense of George and Veeramani if X is a non-empty set, T is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(GV1) $M(x, y, t) > 0$.

(GV2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$.

(GV3) $M(x, y, t) = M(y, x, t)$.

(GV4) $T(M(x, y, s), M(y, z, t)) \leq M(x, z, s + t)$.

(GV5) The function $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous, for all $x, y, z \in X$ and $t, s > 0$.

Fuzzy metric spaces in the sense of Kramosil and Michalek will be referred as KM-fuzzy metric spaces and that of George and Veeramani will be referred as GV-fuzzy metric space. However, in our thesis, we restrict ourself only in GV-fuzzy metric space and we write it simply as fuzzy metric space. The following details of this space are described in the introductory paper of George and Veeramani [42]. Suppose (X, M, T) denote a fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $U(x, \varepsilon, t)$ with center $x \in X$ is defined by $U(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$. A set

$O \subset X$ is open if for each $x \in O$, $\exists\ 0 < \varepsilon < 1$ and $t > 0$ such that $B(x, \varepsilon, t) \subset O$. Suppose that the family of all open subsets of X is denoted by β. Then β induces a Hausdorff topology on X.

Example 1.3. *[42] Let* $X = \mathbb{R}$. *For each* $t \in (0, \infty)$ *and* $x, y \in X$, *let* $M(x, y, t) = \frac{t}{t+|x-y|}$. *Then,* $(\mathbb{R}, M, T_p)$ *is a fuzzy metric space.*

Definition 1.18. Convergent sequence in fuzzy metric space [42, 100]

Let (X, M, T) be a fuzzy metric space. Then A sequence $\{x_n\}$ in (X, M, T) is said to be convergent to $x \in X$, if $M(x_n, x, t) = 1$, $\forall\ t > 0$.

Definition 1.19. Cauchy sequence in fuzzy metric space [42, 100]

A sequence $\{x_n\}$ in (X, M, T) is said to be Cauchy if for each $0 < \epsilon < 1$ and $t > 0$, $\exists\ k \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \epsilon$, for each $m, n \geq k$.

Definition 1.20. Complete fuzzy metric space [42, 100]

A fuzzy metric space (X, M, T) in which every Cauchy sequence is convergent to a point in it, is called complete fuzzy metric space.

Lemma 1.21. *[42]* $M(x, y, .)$ *is non decreasing* $\forall\ x, y \in X$.

Definition 1.22. [42, 100]

Suppose (X, M, T) is a fuzzy metric space, then M is continuous on $X \times X \times (0, \infty)$ if $\lim\limits_{m \to \infty} M(x_m, y_m, t_m) = M(x, y, t)$, where $\{(x_m, y_m, t_m)\}$ be a sequence in $X^2 \times (0, \infty)$ converging to (x, y, t) in $X^2 \times (0, \infty)$.

Lemma 1.23. *[42]*

M is continuous function on $X \times X \times (0, \infty)$.

1.4 Development of fixed point theory in probabilistic metric spaces

The classical Banach contraction principle states that "a contraction mapping on a complete metric space has a unique fixed point", that is,

Theorem 1.24. *If (X, d) be a complete metric space and $f : X \to X$ is a self mapping such that for every $x, y \in X$, f satisfy the following contraction condition:*

$$d(fx, fy) \leq kd(x, y), \quad \textit{where } 0 < k < 1,$$

then f has a fixed point $x \in X$, which is unique.

Banach contraction principle has been extensively generalized in many settings due to its several applications in various disciplines of mathematics. Sehgal and Bharucha-Reid [101] has extended the notion of metric fixed point contraction principle to probabilistic and fuzzy metric spaces (probabilistic k-contraction). Further they obtained the following result:

Theorem 1.25. *Let (X, F, T_m) be a complete Menger PM-space where T_m is the minimum t-norm defined by $T_m(a, b) = \min\{a, b\}$. $f : X \to X$ is a self mapping such that for every $x, y \in X$ and $t > 0$,*

$$F_{fx,fy}(kt) \geq F_{x,y}(t) \quad \textit{where } 0 < k < 1 \tag{1.1}$$

then f has a fixed point $x \in X$, which is unique and $f^n(x_0)$ converges to x for each $x_0 \in X$.

It is interesting to obtain the generalizations of probabilistic k-contraction to establish fixed point theorems in probabilistic metric spaces. A natural generalization of probabilistic k-contraction is the so-called probabilistic φ-contraction . A mapping $f : X \to X$ is called a probabilistic φ-contraction in probabilistic metric space if it satisfies $F_{fx,fy}(\varphi(t)) \geq F_{x,y}(t) \; \forall \; x, y \in X$ and $t > 0$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a control function or a gauge function.

Control functions have been used in many results in fixed point theory. Such use was initiated in metric fixed point theory by khan et al. [57] in 1984. The essential feature of a control function is that it alters the metric distance which makes the triangle inequality directly unavailable in the proofs of the theorems. New methods in the proof are then to be introduced in order to obtain the final results. The idea of control function was extended by Choudhury et al. [18] to fixed point theory in Menger spaces which is twin branch of fuzzy fixed point theory. To establish the fixed point results for φ-contractions on a complete Menger space, many researchers have given various conditions on the control function φ [26, 39, 53] . Most of the results for probabilistic φ-contraction were established with the consideration that φ is nondecreasing and $\sum_{k=0}^{\infty} \varphi^n(t) < \infty$ for $t > 0$. Ciric [26] has pointed out that these conditions were very strong and difficult to apply in general. After that, these conditions have been weakened in the recent papers [26, 39, 53, 108, 111]. Recently, several such control function were generalized in a definition given by Fang [39]. A brief description of the development of control functions in fixed point problem on probabilistic metric spaces are as follows:

Theorem 1.26. *[26, Ciric] Let (X, F, T) be a complete Menger space with a continuous t-norm T of Hadzic-type. If $f : X \to X$ is a probabilistic φ-contraction , that is,*

$$F_{fx,fy}\left(\varphi\left(t\right)\right) \geq F_{x,y}\left(t\right), \text{ for all } x, y \in X \text{ and } t > 0, \tag{1.2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition:

$\varphi(0) = 0, \quad \varphi(t) < t \quad$ *and* $\quad \lim_{n\to\infty} \varphi^n(t) = 0$, *for $t > 0$ then f has a fixed point in X*

which is unique.

To establish the results of the Theorem 1.26, Jaychimski [53] modified the conditions on the control function φ and defined a new type of control function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition: $0 < \varphi(s) < s$ and $\lim_{n\to\infty} \varphi^n(s) = 0$, for $s > 0$. Fang [39] has shown that if control function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that for each $s > 0$ there exists $r \geq s$ with $\lim_{n\to\infty} \varphi^n(r) = 0$, for all $s > 0$ then results of the Theorem 1.26 hold. Wang et al. [111] further weakened the conditions on control function φ by defining a new type of control function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying: for each $s_1,\, s_2 > 0$, there exists $r \geq \max(s_1,\, s_2)$ and $N \in \mathbb{N}$ such that $\varphi^n(r) < \min(s_1,\, s_2)$ for all $s > 0$ and $n \geq N$.

An another notion of contraction different from Sehgal's contraction [101] was introduced by T.L. Hicks [50] which is called as C-contraction on probabilistic metric spaces.

Definition 1.27. Let (X, F, T) be a Menger space. A self mapping $f : X \to X$ is called a C-contraction if there exists $k \in (0, 1)$ such that for every $a, b \in X$, and $r > 0$,

$$F_{a,b}(r) > 1 - r \implies F_{fa,fc}(kr) > 1 - kr \text{ holds.} \tag{1.3}$$

Hicks proved that in a complete Menger space (X, F, T_m) with a strongest t-norm T_m, there exists a unique fixed point of f where $f : X \to X$ is a C-contraction. Radu [47] proved that the above result 1.3 holds in a complete Menger space (X, F, T) if T satisfy $\sup_{t<1} T(t, t) = 1$. A comparative study of the C-contraction and the Sehgal contraction was also made in [47]. By giving an example of C-contraction on a complete Menger space (X, F, T), it was concluded that a probabilistic k-contraction need not be a C-contraction.

As a generalization of the notion of C-contraction 1.3, the concept of (φ, C)-contraction was introduced in [47]:

Definition 1.28. Let (X, F, T) be a Menger space. The mapping $f : X \to X$ is called a (φ, C)-contraction, if for every $a, b \in X$ and for every $r > 0$,

$$F_{a,b}(r) > 1 - r \Rightarrow F_{fa,fb}(\varphi(r)) > 1 - \varphi(r) \tag{1.4}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a control function.

Further, the concept of C-contraction has been generalized and extended in Menger spaces. Consequently, the notion of g-contraction was given in [43]:

Definition 1.29. Let (X, F, T) be a Menger space. The mappings $f, g : X \to X$ be such that g is bijective then f is called a probabilistic g-contraction if for $0 < k < 1$

$$F_{ga,gb}(r) > 1 - r \Rightarrow F_{fa,fb}(kr) > 1 - kr \text{ holds } \forall\ r > 0. \tag{1.5}$$

1.5 Development of fixed point theory in fuzzy metric spaces

In 1988, M. Grabiec [44] introduced the idea of fixed point theory in fuzzy metric space and proved the fuzzy Banach contraction and fuzzy Edelstein contraction theorem. In order to obtain his theorems, Grabiec introduced the following notions.

Recently, many authors have studied the fixed point theory in the fuzzy metric space and number of fixed point theorems have been obtained in fuzzy metric spaces by using the contraction condition of self mappings. In 1983, M. Grabiec [44] extended the known fixed point results of Banach and Edelstein contraction principle in fuzzy metric space in the sense of Kramosil and Michelek.

Theorem 1.30. Fuzzy Banach contraction Theorem *Let (X, M, T) denote a complete fuzzy metric space such that*

(i) $\lim_{t\to\infty} M(x, y, t) = 1,$

(ii) $M(fx, fy, kt) \geq M(x, y, t)$

for all $x, y \in X$ where $0 < k < 1$. Then f has a fixed point, which is unique.

Extensions of this result have been obtained in different settings. Khan et al. gave the notion of altering distance function which alters the distance in metric space.

Definition 1.31. The function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if it is continuous, non-decreasing and $\varphi = 0$ if and only if $t = 0$.

The concept of weak φ-contraction was introduced by Alber and Guerre [1].

Definition 1.32. Let (X, d) be an usual metric space. A self mapping f on X is called weak φ-contraction if $d(fx, fy) \leq d(x, y) - \varphi(d(x, y))$ for each $x, y \in X$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing and continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Dutta et al. [32] gave the notion of $(\varphi - \psi)$-weak contraction mappings:

Definition 1.33. A self mapping f on X is called weak (φ, ψ)-weak contraction if $\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y))$ for each $x, y \in X$ where $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing and continuous function with $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$.

These notions of weak contraction have been extended in fuzzy metric spaces and proved several fixed point results (see, [5, 6, 7, 22, 23]).

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