Chapter 4

Fractional-Order Sliding Mode Control under Uncertainty

This chapter proposes a sliding surface which renders the system dynamics to start directly from itself without a reaching phase. More specifically, the system dynamics is insensitive to matched disturbances/uncertainties throughout the entire system response. The classical control design methodology based on reduced-order subsystem is still preserved. It is different from integral sliding mode control in which the design is based on the full order of the system to reach the same objective [198]. The simulation results of its application to a fractional inverted pendulum system is demonstrated.

4.1 Introduction

The control under heavy uncertainties is one of the most challenging control tasks. *Sliding Mode Control (SMC)* is one of the most efficient control strategies to deal with uncertainties [198] [200] [176]. Nowadays, it is used in both control and observation of several classes of problems such as that related to power converters, vehicle motion control, etc. Sliding mode control has been widely used for fractional-order systems [7] [28] [55] [221]. Remarkable improvements have been obtained with fractional-order sliding mode based control law design in various applications. In this context, the technique of using the control scheme free from any reaching phase has been used for a class of fractional-order uncertain systems to obtain robustness throughout the state trajectories [55]. It uses the Caputo's definition (1.3). This work has been presented here in detail to illustrate the beauty of using fractional calculus in the context of sliding mode control.

The main objective of this class of controllers is to force the system states to stay in a predefined manifold (sliding surface) and maintain it there in spite of the presence of uncertainties in the system. Therefore, the sliding mode based design consists of two phases (i) *Reaching Phase* in which the system states are driven from the initial state to reach the sliding manifold in finite time and (ii) *Sliding Phase* in which the closed-loop system is induced into sliding motion. However, when the system reaches sliding phase, the consideration of robustness and order reduction come into picture which are the most important aspects of the sliding mode based design. It is worth noting that during the reaching phase, there is no guarantee of robustness [198] [200]. In order to address robustness issue throughout the entire space, *Integral Sliding Mode Control (ISMC)* has been proposed in the SMC literature but its design methodology has been based on full order of the system. However, the system exhibits a reduced-order dynamics after it has reached the sliding surface i.e. the system order gets reduced by one due to the introduction of the sliding variable, s such that $s = 0$ in finite time. As a consequence, the simplicity and flexibility of the design procedure which is provided by reduced-order subsystem in classical SMC is lost in ISMC. The motivation behind this work is to preserve the robustness in the system by eliminating the reaching phase such that the system remains on the sliding manifold from the very initial time.

The main aim of the present work is to address robustness from the very initial time and also maintain the design methodology based on order reduction for uncertain fractional-order systems. In order to achieve this, two different methodologies have been adopted:

- An integer reaching law approach is used proposing a sliding surface which eliminates the reaching phase and also, its stability is proved.

- Secondly, a sliding surface using fractional reaching law approach is proposed followed by the same procedure as in the case of integer reaching law approach.

Using the theory of fractional calculus, sliding mode control law design using two approaches are presented here which are integer reaching law approach and fractional reaching law approach.

4.2 Motivation

Classical sliding mode control scheme does not give guarantee of robustness in the reaching phase and only handles uncertainty in the sliding phase of the design. Integral sliding mode control exists in the literature which ensures robustness throughout the evolution of the system states [199] [200] [202]. However, the beauty of reduced-order design methodology is lost in integral sliding mode control. So, a fractional-order sliding mode control approach has been proposed which is free from any reaching phase and also preserves the reduced-order design methodology.

4.3 Fractional-Order Sliding Mode Control

Consider a controllable commensurate fractional-order linear time-invariant system [53] [54] [60] [201] [190] [81],

$$ {}_{t_0}^{c}D_t^{\alpha}\bar{x}(t) = \bar{A}\bar{x}(t) + \bar{B}(u(t) + d(t)) \tag{4.1} $$

where, $\bar{x}(\cdot) \in \mathbb{R}^n$ are pseudo states , $\bar{A} \in \mathbb{R}^{n\times n}$ is the system matrix, $\bar{B} \in \mathbb{R}^{n\times m}$ is the input matrix, $u(\cdot) \in \mathbb{R}^m$ is the control input, $d(\cdot) \in \mathbb{R}^m$ is the disturbance which is assumed to be bounded. It is important to mention here that for fractional-order systems, the knowledge of the initial state, $x(t_0)$ is not sufficient to determine the future state of the system. So, the physical variables do not strictly represent the actual states of the fractional-order system. Therefore, the terminology of *Pseudo States* is coined to represent these physical variables [62] [117] [118] [64]. The same philosophy is followed throughout this presentation.

There always exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that using the linear transformation $z(t) = T\bar{x}(t)$, (4.1) can be transformed into the regular form,

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\alpha}z_1(t) &= A_{11}z_1(t) + A_{12}z_2(t) \\ {}_{t_0}^{c}D_t^{\alpha}z_2(t) &= A_{21}z_1(t) + A_{22}z_2(t) + B_2(u(t) + d(t)) \end{aligned} \tag{4.2} $$

where, $\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = z(t)$, $z_1(\cdot) \in \mathbb{R}^{n-m}$, $z_2(\cdot) \in \mathbb{R}^m$.

As the pair $(\bar{A}, \bar{B})$ is assumed controllable, the pair (A_{11}, A_{12}) will also be controllable. The above system of equations can be represented as,

$$ {}_{t_0}^{c}D_t^{\alpha}z(t) = Az(t) + B(u(t) + d(t)) \tag{4.3} $$

where, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$.

Assumption 4.1 *For a non-smooth controller, the notions of existence and uniqueness of solutions of the system are usually defined in the Filippov's sense [7] [127] i.e., considering x as the pseudo states of the entire system ${}_{t_0}^{c}D_t^{\alpha} x(t) = f(x(t), d(t))$, $\alpha > 0$, disturbance $d \in \mathbb{R}^m$ and assuming $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ to be locally bounded, then the solutions are defined using the differential inclusion,*

$$ {}_{t_0}^{c}D_t^{\alpha} x(t) \in \bigcap_{\delta>0} \bigcap_{\mu N=0} \mathrm{cl}(\mathrm{co}(\zeta(B_\delta(x) \setminus N))) $$

where, cl *and* co *denote the closure and the convex hull respectively. $B_\delta(x)$ is the unit ball and the sets N are all sets of zero Lebesgue Measure.*

The sliding surface has been designed using fractional reaching law and integer reaching law in [7] [197]. Using integer reaching law, the sliding surface for (4.2) is,

$$ s(z,t) = {}_{t_0}I_t^{1-\alpha}(c_1 z_1(t) + z_2(t)) \tag{4.4} $$

where, $s : \mathbb{R}^n \times (t_0, \infty) \to \mathbb{R}$, $c_1 \in \mathbb{R}^{1\times(n-1)}$ and for,

$$ u(t) = B_2^{-1}(v - c_1\{(A_{11} - A_{12}c_1)z_1(t)\} + A_{12}{}_{t_0}I_t^{1-\alpha} s - A_{21}z_1(t) - A_{22}z_2(t)) $$

where, $v = -k_1 \mathrm{sign}(s)$, it has been proved in [7] that for s to be zero in finite time, $k_1 > |B_2||d|$. In case of fractional reaching law,

$$ s(z,t) = c_1 z_1(t) + z_2(t) \tag{4.5} $$

Using fractional-order derivative of s in (4.5), (4.2) becomes,

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\alpha} z_1(t) &= (A_{11} - A_{12}c_1)z_1(t) + A_{12}s \\ {}_{t_0}^{c}D_t^{\alpha} s &= c_1 {}_{t_0}^{c}D_t^{\alpha} z_1(t) + {}_{t_0}^{c}D_t^{\alpha} z_2(t) \\ &= c_1\{(A_{11} - A_{12}c_1)z_1(t) + A_{12}s\} + A_{21}z_1(t) + A_{22}z_2(t) + B_2 u(t) + B_2 d \end{aligned} $$

Here, the control $u(t)$ is chosen as,

$$ u(t) = B_2^{-1}(v - c_1(A_{11} - A_{12}c_1)z_1(t) + A_{12}s - A_{21}z_1(t) - A_{22}z_2(t)) $$

where, $v = -k_1 \text{sign}(s)$. Using the control, the following closed-loop system results,

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha} z_1(t) &= (A_{11} - A_{12}c_1)z_1(t) + A_{12}s, \\
{}_{t_0}^{c}D_t^{\alpha} s &= -k_1 \text{sign}(s) + B_2 d
\end{aligned}
\tag{4.6}
$$

Here, some stability concepts need to be discussed. The *Lyapunov's Theory* for general nonlinear systems has also been extended for fractional-order systems in the literature [101] [103] [104].

4.4 Extension of Lyapunov's Theory to Fractional-Order Systems

Using Caputo definition, an $n-$dimensional fractional-order system can be defined as,

$$
{}_{t_0}^{c}D_t^{\alpha} x(t) = f(x,t); \quad \forall t \geq t_0 \tag{4.7}
$$

where, $\alpha \in (0,1)$ and $f(x,t)$ is locally bounded in x and piecewise continuous in t for all $t \geq t_0$ and $x \in \mathbb{D}$, where $\mathbb{D} \subset \mathbb{R}^n$ is a domain which contains the origin $x = 0$. There are various efforts for the stability analysis of fractional-order systems in the literature [107]. For stability analysis of system (4.7), a fractional-order extension of the Lyapunov's direct method was proposed in [101] which is based on the following definition:

Definition 4.2 *A continuous function* $\gamma : [0,\mathrm{t}) \to [0,\infty)$ *is a Class-*$\mathcal{K}$ *Function if it is strictly increasing and* $\gamma(0) = 0$.

Theorem 4.3 *Let* $x = 0$ *be an equilibrium point for the non-autonomous fractional-order system i.e.,* $f(x,t) = 0, \forall t \geq t_0$. *If there exists a Lyapunov function* $V(t,x(t)) : [t_0,\infty) \times \mathbb{D} \to \mathbb{R}$ *and a Class-*$\mathcal{K}$ *function* $\gamma_i (i = 1,2,3)$ *such that,* $\gamma_1(||x||) \leq V(t,x(t)) \leq \gamma_2(||x||)$ *and* ${}_{t_0}^{c}D_t^{\alpha} V(t,x(t)) \leq -\gamma_3(||x||)$ *where,* $\alpha \in (0,1)$ *then, the system* (4.7) *is asymptotically stable.*

Theorem 4.4 *[100] Let* $x \in \mathbb{R}^n$ *be a continuously differentiable vector-valued function. Then, for any time instant* $t \geq t_0$ *and* $\forall \alpha \in (0,1)$,

$$
\frac{1}{2} {}_{t_0}^{c}D_t^{\alpha} x^{\top}(t)x(t) \leq x^{\top}(t) {}_{t_0}^{c}D_t^{\alpha} x(t). \tag{4.8}
$$

The above results will be used in the later sections for the Lyapunov stability analysis of fractional-order systems with the proposed control law [110] [111] [112]. Since this result was derived using Caputo derivatives, the same definition will be used throughout this section unless mentioned otherwise. Here, it is important to consider the following theorem:

Theorem 4.5 *The sliding surface s in* (4.5) *becomes $zero$ in finite time if* $k_1 > |B_2||d|$.

Proof The Lyapunov function is selected as,

$$V = \frac{1}{2}s^2$$

Then,

$${}_{t_0}^{c}D_t^{\alpha}V = \frac{1}{2}{}_{t_0}^{c}D_t^{\alpha}s^2$$

Using (4.8),

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha}V &\leq s\,{}_{t_0}^{c}D_t^{\alpha}s \\
&= s(-k_1\text{sign}(s) + B_2 d) \\
&\leq -k_1|s| + |s||B_2 d| \\
&= -|s|(k_1 - |B_2 d|) \\
&= -(2V)^{\frac{1}{2}}(k_1 - |B_2 d|) \\
&\leq -\eta(2V)^{\frac{1}{2}}
\end{aligned}$$

where, $\eta = k_1 - |B_2||d| > 0$.

Using the above inequality, $s = 0$ is obtained in finite time [7] which can be derived as follows: Putting $t_0 = 0$ in (4.6),

$${}_{0}^{c}D_t^{\alpha}s = -k_1\text{sign}(s) + B_2 d$$

Taking fractional-order integral of order α on both sides,

$${}_{0}I_t^{\alpha}\,{}_{0}^{c}D_t^{\alpha}s = k_1\,{}_{0}I_t^{\alpha}\text{sign}(s) + B_2\,{}_{0}I_t^{\alpha}d \tag{4.9}$$

Since,

$${}_{0}I_t^{\alpha}\,{}_{0}^{c}D_t^{\alpha}s = s(t) - {}_{0}^{c}D_t^{\alpha-1}s(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

and,

$${}_{0}I_t^{\alpha}c = c\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

Eq. (4.9) becomes after finite time $t = T$,

$$s(T) - {}_0^cD_t^{\alpha-1}s(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} = -k_1\mathrm{sgn}(s(0))\frac{T^\alpha}{\Gamma(\alpha+1)} + B_2\,{}_0I_t^\alpha d$$

Multiplying with sign$(s(0))$ and using $s(T) = 0$,

$$-{}_0^cD_t^{\alpha-1}s(0)\mathrm{sign}(s(0))\frac{T^{\alpha-1}}{\Gamma(\alpha)} = -k_1\frac{T^\alpha}{\Gamma(\alpha+1)} + B_2\,{}_0I_t^\alpha(\mathrm{sign}(s(0))d)$$

Using the inequality,

$${}_0I_t^\alpha(\mathrm{sign}(s(0))d) \leq {}_{t_0}I_t^\alpha|d| \leq {}_0I_t^\alpha d_0 = d_0\frac{T^\alpha}{\Gamma(\alpha+1)}$$

Eq. (4.4) becomes,

$$-{}_0^cD_t^{\alpha-1}s(0)\mathrm{sign}(s(0))\frac{T^{\alpha-1}}{\Gamma(\alpha)} \leq -(k_1 - B_2d_0)\frac{T^\alpha}{\Gamma(\alpha+1)}$$

which further results into,

$$T \leq \frac{\Gamma(\alpha+1){}_0^cD_t^{\alpha-1}s(0)\mathrm{sign}(s(0))}{\Gamma(\alpha)(k_1 - B_2d_0)} \tag{4.10}$$

which is always finite.

Remark 1.1 It is clear that sliding mode has been obtained after a finite time $t \geq T$ where, T is such that $s(z, T) = 0$. Further, a modified sliding surface is proposed in which sliding starts from $t \geq t_0$ such that the reduced-order design methodology of the classical approach is preserved.

4.5 Main Results

Consider the system in the same form as in Eq. (4.2). The sliding surface designed for integer reaching law is,

$$s = {}_{t_0}I_t^{1-\alpha}\left\{\left(c_1z_1(t) + z_2(t)\right) - \left(c_1z_1(t_0) + z_2(t_0)\right)e^{-\lambda(t-t_0)}\right\}$$

where $\lambda > 0$ and $c_1 \in \mathbb{R}^{1\times n-1}$ are the design parameters. Note that the sliding variable, $s = 0$ at the initial time $t = t_0$. Then, the system (4.2) is transformed as,

$${}_{t_0}^cD_t^\alpha z_1(t) = (A_{11} - A_{12}c_1)z_1(t) + A_{12}\left\{{}_{t_0}^cD_t^{1-\alpha}s + \left(c_1z_1(t_0) + z_2(t_0)\right)e^{-\lambda(t-t_0)}\right\}$$

$$\begin{aligned}
\dot{s} &= {}^{c}_{t_0}D^{\alpha}_{t}\Big\{\Big(c_1 z_1(t) + z_2(t)\Big) - \Big(c_1 z_1(t_0) + z_2(t_0)\Big)e^{-\lambda(t-t_0)}\Big\} \\
&= c_1\Big[(A_{11} - A_{12}c_1)z_1(t) + A_{12}\Big\{{}^{c}_{t_0}D^{1-\alpha}_{t} s + \Big(c_1 z_1(t_0) + z_2(t_0)\Big)e^{-\lambda(t-t_0)}\Big\}\Big] \\
&\quad + A_{21}z_1(t) + A_{22}z_1(t) + B_2 u(t) + B_2 f - (-\lambda)^{\alpha}\Big(c_1 z_1(t_0) + z_2(t_0)\Big)e^{-\lambda(t-t_0)}
\end{aligned}$$

The control input is designed as,

$$\begin{aligned}
u(t) &= B_2^{-1}\Big[v - c_1\Big\{(A_{11} - A_{12}c_1)z_1(t) + A_{12} \\
&\quad \times \Big\{{}^{c}_{t_0}D^{1-\alpha}_{t} s + (c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}\Big\}\Big\}\Big] - B_2^{-1}\Big(A_{21}z_1(t) + A_{22}z_1(t)\Big)
\end{aligned} \tag{4.11}$$

where, $v = -k_1 \mathrm{sign}(s)$. Hence,

$$\dot{s} = -k_1 \mathrm{sign}(s) + B_2 d + \Xi$$

where, $\Xi = B_2^{-1}[(-\lambda)^{\alpha}((c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}]$.

Here, it is important to note that $|\Xi|$ always remains bounded for any initial condition $z(t_0)$. It is proved that once the trajectories start from the sliding surface s at $t = t_0$, they remain on it and then, asymptotically converge to $z_1(t) = z_2(t) = 0$.

Lemma 4.6 *If $k_1 > |B_2 d| + |\Xi|$, then the trajectories are maintained on the sliding surface $s = 0, \ \forall t \geq t_0$.*

Proof Consider the Lyapunov function,

$$V = \frac{1}{2}s^2$$

By taking the time derivative of the Lyapunov function along closed-loop subsystem $\dot{s} = -k_1 \mathrm{sign}(s) + B_2 d + \Xi$,

$$\begin{aligned}
\dot{V} = s\dot{s} &= s(-\,k_1\,\mathrm{sign}(s) + B_2 d + \Xi) \\
&= -\,k_1\,|s| + s\,B_2\,d + s\Xi \\
&\leq -\,k_1\,|s| + |s|\,|B_2\,d| + |s|\,|\Xi| \\
&= -(2V)^{\frac{1}{2}}(k_1 - |B_2\,d| - |\Xi|) \\
&\leq -\eta(2V)^{\frac{1}{2}}
\end{aligned}$$

where, $\eta = k_1 - |B_2 d| - \Xi$.

When $\eta = k_1 - |B_2 d| - \Xi > 0$, Lyapunov stability theory ($V = 0$ and $\dot{V} \leq 0$) $\Rightarrow V = 0, \forall t \geq t_0$ implies $s = 0, \ \forall t \geq t_0$.

This completes the proof.

The expression for the finite time, T can be obtained as:

$$\frac{dV}{dt} \le -\eta\sqrt{2}V^{1/2}$$
$$\int_0^T dt \le -\int_{V_0}^0 \frac{dV}{\eta\sqrt{2}(V)^{1/2}}$$
$$T \le -\int_{V_0}^0 \frac{dV}{\eta\sqrt{2}(V)^{1/2}}$$
$$= \frac{\sqrt{2}V(0)}{\eta}$$

Lemma 4.7 *If the matrix* $(A_{11} - A_{12}c_1)$ *is negative definite, then the closed-loop system is asymptotically stable.*

Proof Take the Lyapunov function,

$$V = \frac{1}{2} z_1^\top(t) z_1(t)$$

Then,

$${}_{t_0}^{c}D_t^{\alpha} V = \frac{1}{2} {}_{t_0}^{c}D_t^{\alpha} z_1^\top(t) z_1(t)$$

Using (4.8),

$$\begin{aligned} {}_{t_0}^{c}D_t^{\alpha} V &\le z_1^\top(t) {}_{t_0}^{c}D_t^{\alpha} z_1(t) \\ &\le z_1^\top(t)(A_{11} - A_{12}c_1) z_1(t) + z_1^\top(t) A_{12} \left\{ {}_{t_0}^{c}D_t^{1-\alpha} s + \left(c_1 z_1(t_0) + z_2(t_0) \right) e^{-\lambda(t-t_0)} \right\} \end{aligned}$$

As $s = 0$ from time $t = t_0$, the term $(c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)} \to 0$ as $t \to \infty$, $z_1(t)$. If the matrix $(A_{11} - A_{12}c_1)$ is negative definite, the system is asymptotically stable.

This completes the proof.

Remark 1.2 It is important to note that if we take

$$v = -\lambda |s|^{\frac{1}{2}} \operatorname{sign}(s) - \alpha \int_{t_0}^{t} \operatorname{sign}(s) d\tau$$

where $\alpha = 1.1\Delta$ and $\lambda = 1.5\sqrt{\Delta}$ such that $B_2|\dot{d}(t)| \le \Delta$, where Δ is some a priori known constant, then the proposed control (4.11) generates continuous signal and it also proves better for the chattering minimization problem, which is commonly encountered in the practical implementation of discontinuous control laws. The controller suggested above is known as

Super-Twisting in the literature. Again, the trajectories once start from the sliding surface, will remain there for the subsequent time (for more detailed explanation, see [41] and the references cited therein).

Now, using fractional reaching law approach, the sliding surface is designed as,

$$s = c_1 z_1(t) + z_2(t) - (c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)} \tag{4.12}$$

Note that $s = 0$ when $t = t_0$. Using (4.12), (4.2) becomes,

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha} z_1(t) &= (A_{11} - A_{12}c_1)z_1(t) + A_{12}\left\{s + (c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}\right\} \\
{}_{t_0}^{c}D_t^{\alpha} s &= {}_{t_0}^{c}D_t^{\alpha} z_1(t) + {}_{t_0}^{c}D_t^{\alpha} z_2(t) - (c_1 z_1(t_0) + z_2(t_0))\, {}_{t_0}^{c}D_t^{\alpha}(e^{-\lambda(t-t_0)}) \\
&= c_1\Big[(A_{11} - A_{12}c_1)z_1(t) + A_{12}\Big\{s + (c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}\Big\}\Big] \\
&\quad + A_{21}z_1(t) + A_{22}z_2(t) + B_2(u(t) + d(t)) \\
&\quad - (-\lambda)^{\alpha}((c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}
\end{aligned} \tag{4.13}$$

The control input is designed as,

$$\begin{aligned}
u(t) = B_2^{-1}\Big[v - c_1\Big\{(A_{11} - A_{12}c_1)z_1(t) + A_{12} \times \Big(s + (c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}\Big)\Big\}\Big] \\
- B_2^{-1}\Big(A_{21}z_1(t) + A_{22}z_2(t)\Big)
\end{aligned} \tag{4.14}$$

where, $v = -k_1 \mathrm{sign}(s)$. From (4.13) and (4.14),

$${}_{t_0}^{c}D_t^{\alpha} s = -k_1 \mathrm{sign}(s) + B_2 d + \Xi$$

where, $\Xi = B_2^{-1}(-\lambda)^{\alpha}((c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}$.

Again, the trajectories remain on the sliding surface $s = 0$ from the very initial time $t = t_0$, provided,

$$k_1 > |B_2||d| + |\Xi|$$

Here, the logic of the associated proof remains the same as previously in the case of Lemma 4.6. The related condition for the asymptotic convergence of $z(t)$ at the equilibrium point also remains the same as in Lemma 4.7. This can be shown as follows:

Consider the Lyapunov function,

$$V = \frac{1}{2} z_1^{\top}(t) z_1(t)$$

Taking the fractional-order derivative,

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha} V &\leq z_1^{\top}(t)\, {}_{t_0}^{c}D_t^{\alpha} z_1(t) \\
&\leq z_1^{\top}(t)(A_{11} - A_{12}c_1)z_1(t) + z_1^{\top}(t)A_{12}\Big\{s + (c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}\Big\}
\end{aligned}$$

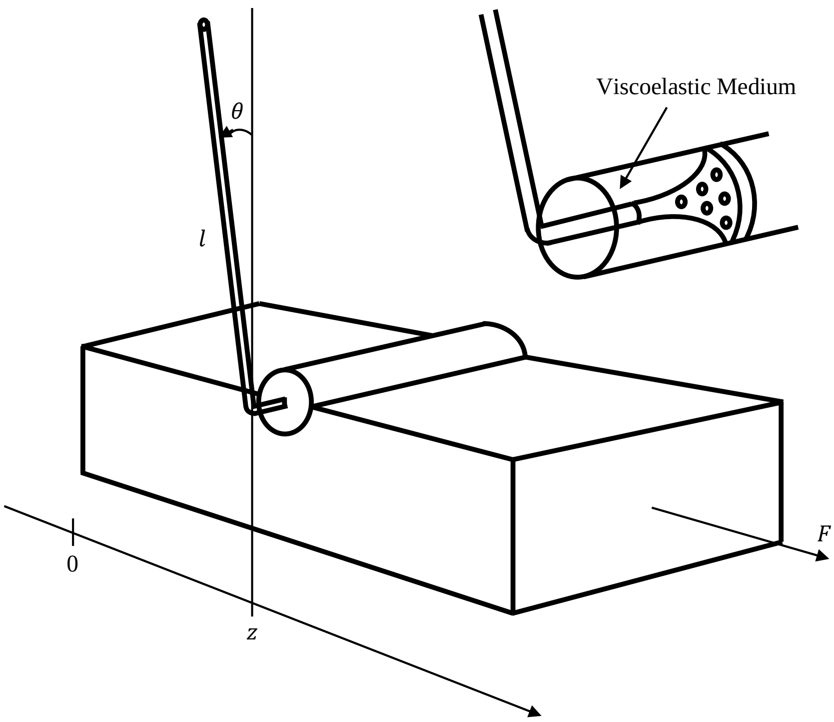

Figure 4.1: Fractional Inverted Pendulum [235]

As $s = 0$ from time $t = t_0$, $(c_1 z_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)} \rightarrow 0$ as $t \rightarrow \infty$. Further, $z_1(t)$ and hence the system is asymptotically stable if $(A_{11} - A_{12}c_1)$ is negative definite.

4.6 Illustrative Example

A commensurate fractional-order uncertain system is considered to illustrate the theoretical results obtained in the paper. The example of a fractional inverted pendulum system is taken as shown in Fig. 4.1. In this system, an inverted pendulum is mounted on the top of a cart such that the pendulum is attached to an extension immersed in a viscoelastic solution [235]. The cart is able to move back and forth. The whole system can be represented by,

$$\ddot{x} = \frac{1}{(m_c + m_p)} \left(\frac{1}{2} m_p l (\ddot{\theta} cos\theta - (\dot{\theta})^2 sin\theta) - f\dot{x} + F \right) \tag{4.15}$$

$$\ddot{\theta} = \frac{1}{(J + \frac{1}{4} m_p l^2)} \left(\frac{1}{2} m_p l (\ddot{x} cos\theta + g sin\theta) + \tau \right) \tag{4.16}$$

$$\frac{d^\alpha \tau}{dt^\alpha} = -\omega_l^\alpha \tau - k\omega_l^\alpha \dot{\theta} - k \left(\frac{\omega_l}{\omega_h} \right)^\alpha \frac{d^{(\alpha+1)}\theta}{dt^{(\alpha+1)}} \tag{4.17}$$

where, x is the position of the cart, θ is the angle of deflection of the pendulum, m_c is the mass of the cart, m_p is the mass of the pendulum, f is the friction coefficient of the cart, τ is the applied torque, k is the damping coefficient of the viscoelastic solution, α is the derivation order of the damper, ω_l and ω_h are the lower and higher frequencies of the bandwidth of the fractional derivative. The state vector is chosen as,

$$X = \begin{bmatrix} x & \frac{d^{0.5}x}{dt^{0.5}} & \frac{dx}{dt} & \frac{d^{1.5}x}{dt^{1.5}} & \theta & \frac{d^{0.5}\theta}{dt^{0.5}} & \frac{d\theta}{dt} & \frac{d^{1.5}\theta}{dt^{1.5}} & \tau \end{bmatrix}^T$$

The above equations can be represented in pseudo state-space form having commensurate order 0.5 which can be further linearized about the equilibrium point of the system resulting into:

$$\frac{d^{0.5}X(t)}{dt^{0.5}} = AX(t) + B(u(t) + d(t)) \tag{4.18}$$

where,

$$A = \alpha \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & 0 & a_{45} & 0 & 0 & 0 & a_{49} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & a_{83} & 0 & a_{85} & 0 & 0 & 0 & a_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{97} & a_{98} & a_{99} \end{bmatrix}$$

$$B = \alpha \begin{bmatrix} 0 & 0 & 0 & 0.116 & 0 & 0 & 0 & 0.338 & 0 \end{bmatrix}^T$$

where, $\alpha = \frac{1}{4J(m_c+m_p)+m_c m_p l^2}$, $a_{43} = -4fJ - fm_p l^2$, $a_{45} = m_p^2 l^2 g$, $a_{49} = 2m_p l$, $a_{83} = -2fm_p l$, $a_{85} = 2m_p gl(m_c + m_p)$, $a_{89} = 4(m_c + m_p)$, $a_{97} = -k\alpha(\omega_l)^{0.5}$, $a_{98} = -k\alpha(\frac{\omega_l}{\omega_h})^{0.5}$, $a_{99} = -\alpha(\omega)^{0.5}$, $b_4 = J + m_p l^2$, $b_8 = 2m_p l$. Here, J is the moment of inertia of the pendulum and l is its length. The values taken are m_p = 0.53 kg, m_c = 3.2 kg, l = 0.36 m, f = 6.2 kg.sec^{-1}, J = 0.065 kg.m^2, k = 0.1 N.m.sec.$^{\alpha}$rad^{-1}, ω_l = 0.1 rad.sec.$^{-1}$, ω_h = 10 rad.sec.$^{-1}$, g = 9.81 m.sec.$^{-2}$,

$d(t) = 0.1 \sin(t)$. The sliding surface is chosen as,

$$s(t) = ([c_1 \ \ c_2 \ \ c_3 \ \ c_3 \ \ c_4 \ \ c_5 \ \ c_6 \ \ c_7 \ \ c_8 \ \ 1]\, X(t)) \\ - ([c_1 \ \ c_2 \ \ c_3 \ \ c_3 \ \ c_4 \ \ c_5 \ \ c_6 \ \ c_7 \ \ c_8 \ \ 1]\, X_0) e^{-\lambda(t-t_0)}$$

where, c_1 to c_8 are the gain values selected such that the reduced-order dynamics is stable. The controller parameter k_1 has to be selected such that $k_1 > |B_2 d| + |\Xi|$. We know that $|B_2 d| = 0.1$ and $|\Xi|$ is also small. Hence, we choose $k_1 = 10$ and $\lambda = 0.4$. The evolution of states, sliding surface and control input with time are shown in Figs. 4.1, 4.2 and 4.3 respectively.

As obtained in the simulation results, stabilization of the state trajectories about the origin is obtained in finite time. At the same time, the states remain on the sliding surface from the starting time $t = 0$ due to the designed sliding surface. The exponential nature of the control action can be seen during the finite time in which the state transients can be observed. After finite time, the discontinuous sliding mode control action maintains the states $X = 0$ for all time.

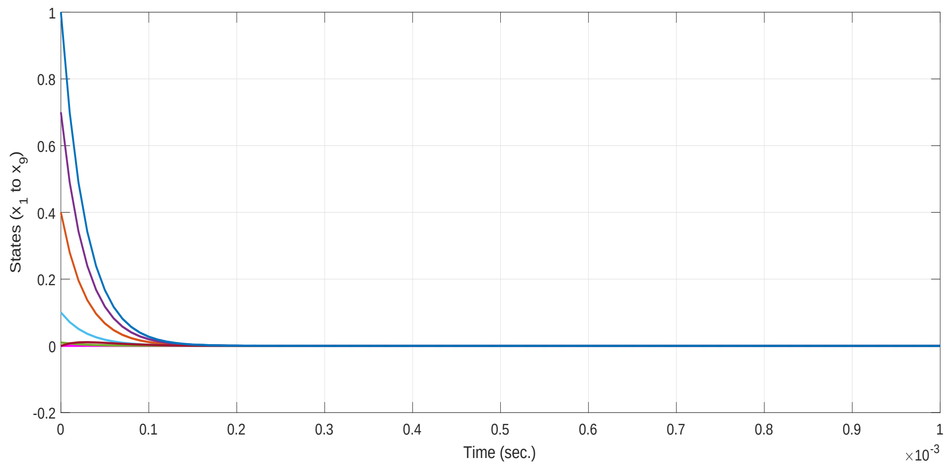

Figure 4.2: Evolution of States (x_1 to x_9) with time

4.7 Summary

The work presented in the chapter proposes a new sliding mode control based controller for uncertain fractional-order systems. Two different control schemes, one based on integer reaching law and the other on fractional reaching law have been used in order to maintain the trajectories on the sliding surface from the very initial time preserving robustness. The simplicity of the technique lies in the control design being based on the reduced order subsystem as in the

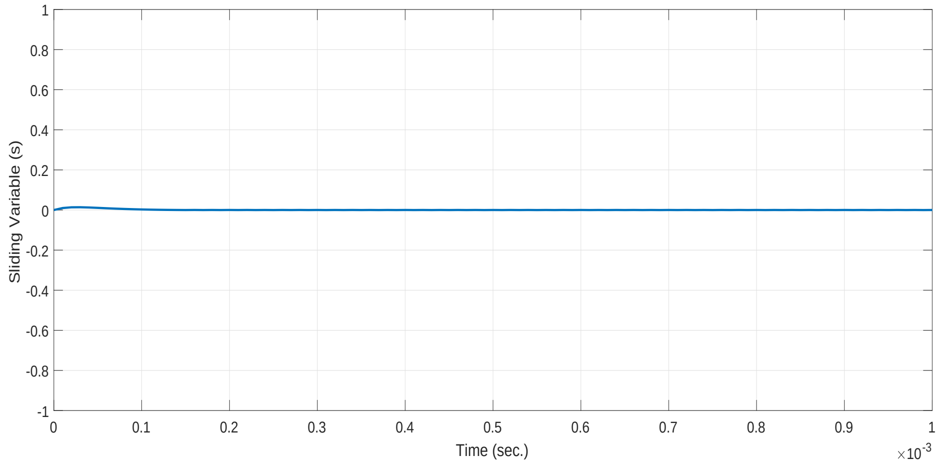

Figure 4.3: Evolution of Sliding Surface (s) with time

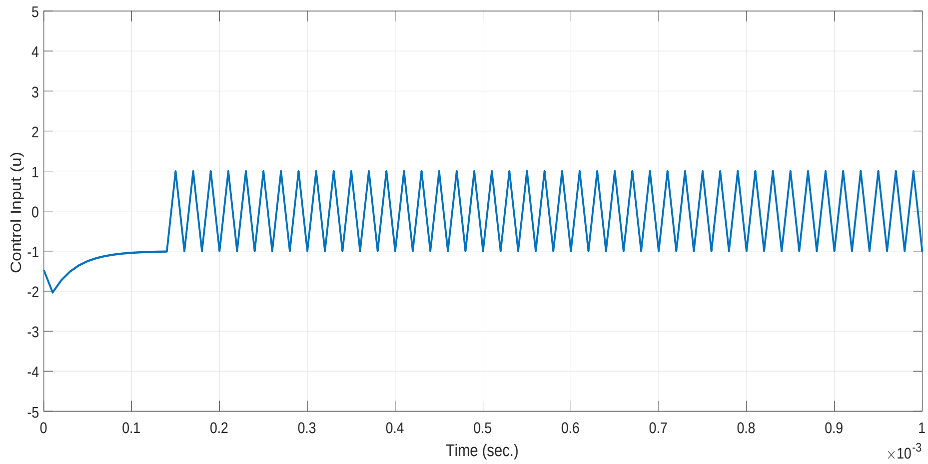

Figure 4.4: Evolution of Control Input (u) with time

case of classical sliding mode control. Robustness is ensured throughout the evolution of the trajectories form the very initial time. The effectiveness of the proposed approach is verified through numerical simulation for the case of a fractional inverted pendulum system.