Chapter 2

Non-Differentiable Function Tracking

An insight into the class of tracking problems in nonlinear systems is provided for which the first-order derivative of the reference function does not exist. Using classical sliding mode control, only a restricted class of reference functions can be tracked. A solution to this problem by using fractional-order operators is proposed. The technique works provided the reference function satisfies the Hölder condition. Notably, its illustration using a switch-controlled *RL* circuit is demonstrated and some of the possible applications have been discussed.

2.1 Introduction

Generally, the design of nonlinear control systems involves two types of problems which are *Stabilization or Regulation Problem* and *Tracking or Servo Problem*. In stabilization problems, the states of the closed-loop system are stabilized around an equilibrium point using a control law. Some of the commonly encountered stabilization problems are altitude control of aircraft, position control of the robotic arm, temperature control of refrigerators. On the other hand, tracking problems involve tracking of a time-varying reference function by the system output. The problems of making an aircraft or a robotic arm follow a specified path fall under the category of tracking problems. Without loss of generality, the tracking problem can be reduced to an equivalent stabilization problem [5]. A generalized approach to the tracking problem is explored in this chapter.

The desired reference function in tracking problems may belong to any class according to the specific application. Conventional sliding mode control works on a restricted class of reference functions. In this chapter, a larger class of reference functions has been considered by using fractional-order derivatives in the design of sliding mode control for tracking problems.

A brief outline of the chapter is as follows. The motivation behind the work is presented in 2.2. The background and formulation of the tracking problem is presented in Section 2.3. Classical approaches are explored and their mathematical limitations are highlighted. A case study of switch-controlled RL circuit is given in Section 2.4. Tracking problems for constant and smooth reference functions are given in 2.5. Section 2.6 gives the main theoretical concepts of the technique proposed in this chapter. The idea is further generalized to nonlinear systems in Section 2.7. Simulation results are used for further illustrations in Section 2.8. Finally, the chapter is concluded with Section 2.9.

2.2 Motivation

The tracking problems in nonlinear systems require the information of the first-order derivative of some functions such as the reference function. So, these functions are often assumed to be differentiable in the design of such systems. However, there lies a strong possibility of some reference functions which do not satisfy the classical differentiability condition. Therefore, a generalized technique is required to utilize the information contained in the non-differentiable functions for the purpose of control design.

2.3 The Tracking Problem

The tracking problem in nonlinear dynamic systems is one of the most important research areas in control theory. The problem can be found in many application areas of science and engineering and can be described by considering the general nonlinear system,

$$\dot{x} = f(x,u) \quad ; x \in D \subset \mathbb{R}^n, \;\; u \in \mathbb{R}^m \tag{2.1}$$

where, $f : D \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and u. The input $u(t)$ is a piecewise continuous, bounded function of $t, \forall t \geq 0$ and domain D contains the origin $x = 0$. Finite-time convergence of the system states x towards the desired states x_r keeping the states x bounded for all time t is required in most of the tracking problems. It is observed that the tracking problem can be reduced to an equivalent stabilization problem of the error dynamics about the origin by defining the error $e = x - x_r$ as the new state variable.

2.4 The Switch-Controlled *RL* Circuit: A Case Study

A switch-controlled *RL* circuit (DC Chopper), supplied by a constant voltage source V is considered as shown in Fig. 2.1.

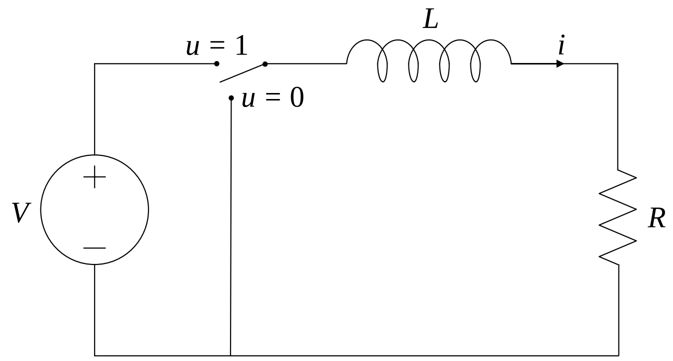

Figure 2.1: Switch-Controlled *RL* Circuit (DC Chopper)

The resulting dynamic behavior of the above circuit can be captured by,

$$L\frac{di(t)}{dt} = u(t)V - i(t)R \tag{2.2}$$

where, R, L, $i(t)$ and $u(t) \in \{0, 1\}$ are resistance, inductance, current through inductor and control input respectively. Here, the control input can take values 0 or 1 corresponding to 'OFF' or 'ON' state of the switch respectively.

Now, suppose that the following classes of reference functions of current through the inductor are desired to be maintained for all time with only limited control action $u(t) \in \{0, 1\}$:

(a) Constant current $i(t) = i_r$, where $i_r \leq i_{\max}$.

(b) Smooth current $i(t) = A\sin(\omega t)$, where $|A| \leq A_{\max}$ and $\omega \leq \omega_{max}$.

(c) Non-smooth current i.e., the desired current i_r which is not differentiable (using Newton's calculus). For example, consider a sawtooth or triangular function as the reference current having a removable discontinuity point:

$$i(t) = \begin{cases} t & \text{if } 0 < t \leq 1, \\ 2 - t, & \text{if } 1 < t < 2. \end{cases} \tag{2.3}$$

Out of the above three problems, both the problems (a) and (b) can be solved directly by using the concept of sliding mode control. More details can be found in [193] and the references cited within.

2.5 Tracking of Constant and Smooth Reference Functions

Using sliding mode control theory, the sliding variable is defined as the error between the current at time t and the desired reference current, i.e.,

$$e = i(t) - i_r(t). \tag{2.4}$$

Further, the rate of change of error can be calculated as in both the above cases (a) and (b), the desired current i_r is assumed to belong to the class of differentiable functions. Therefore, using (2.4),

$$\dot{e} = \dot{i}(t) - \dot{i}_r(t). \tag{2.5}$$

Using (2.2), (2.4) and (2.5),

$$\begin{aligned} \dot{e} &= \frac{R}{L}\left[\frac{V}{R}u(t) - e - \left(i_r(t) + \frac{L}{R}\dot{i}_r(t)\right)\right] \\ &= \frac{R}{L}\left[i_{\max}u(t) - e - \left(i_r(t) + \frac{L}{R}\dot{i}_r(t)\right)\right] \end{aligned} \tag{2.6}$$

where, $\frac{V}{R} = i_{\max}$. Consider that initially, the error is negative i.e., $e < 0$. Therefore, the switch must be turned 'ON' i.e., $u(t) = 1$ as long as e remains negative. Substituting $e < 0$ and $u(t) = 1$ in (2.6),

$$\dot{e} = \frac{R}{L}\left[i_{\max} + |e| - \left(i_r(t) + \frac{L}{R}\dot{i}_r(t)\right)\right] \tag{2.7}$$

Using (2.7), it can be concluded that the time derivative of the error, $\dot{e}$ is guaranteed to remain positive if the desired trajectory satisfies the condition,

$$i_{\max} > i_r(t) + \frac{L}{R}\dot{i}_r(t); \forall t$$

.

If the error is positive i.e., $e > 0$, then $\dot{e}$ should be negative accordingly. Therefore, the switch should go 'OFF' i.e., $u(t) = 0$ as long as e is positive. Substituting $e > 0$ and $u(t) = 0$ in (2.6),

$$\dot{e} = -\frac{R}{L}\left[e + \left(i_r(t) + \frac{L}{R}\dot{i}_r(t)\right)\right] \tag{2.8}$$

Therefore, the time derivative of error, $\dot{e}$ is guaranteed to remain negative provided,

$$\left|i_r(t) + \frac{L}{R}\dot{i}_r(t)\right| > 0; \forall t$$

Therefore, it is clear that tracking of the desired reference function is possible using switch controlled RL circuit provided the following conditions are satisfied:

- the desired trajectory is differentiable
- and $0 < \left| i_r(t) + \frac{L}{R}\dot{i}_r(t) \right| < i_{\max}; \forall t.$

Therefore, a switching strategy can be designed in the following way:

$$u(t) = \begin{cases} 1 & \text{for } e < 0, \\ 0, & \text{for } e > 0, \end{cases} \tag{2.9}$$

or,

$$u = \frac{1}{2}(1 - \mathrm{sgn}(e))$$

where, sgn(.) is defined as,

$$\mathrm{sgn}(e) = \begin{cases} 1 & \text{for } e > 0 \\ -1, & \text{for } e < 0 \end{cases}$$

However, it is not possible to discuss about the existence and uniqueness of the solution of the closed-loop system (2.2) in the classical sense after substitution of the required switching scheme (2.9), due to the associated discontinuity in the right side of the resulting differential equation. Therefore, the solution should be understood using non-smooth theory of differential equations.

There also exist several approaches in the literature which give the existence and uniqueness of the solution for differential equations with discontinuous right hand side. The most straightforward technique among them is *Utkin's Regularization* or *Filippov's Method* [7] which has been followed here for the solution of the differential equation (2.2). Here, it should be ensured that during sliding, the value of the equivalent control u_{eq} should belong to the interval $(0, 1)$ i.e., $0 < u_{\text{eq}} < 1$. The resulting expression of the equivalent control can be obtained by substituting $e = \dot{e} = 0$ into (2.7),

$$u_{\text{eq}} = -\frac{1}{i_{\max}} \left[\left(i_r(t) + \frac{L}{R}\dot{i}_r(t) \right) \right] \tag{2.10}$$

It is clear that the problem (c) in the previous section is not solvable using the above methodology. This is due to the non-differentiable nature of the error signal, which further suppresses the generation and manipulation of the error dynamics based on the ordinary differential equation (2.5). This poses a limitation on the applicability of not only sliding mode-based design, but also that of other classes of controllers like PID (Proportional-Integral-Derivative), adaptive, optimal etc. There may be argument on the possibility to approximate the desired reference function by a piecewise continuously differentiable function and then, applying the same technique with some approximation. However, the resulting analysis will be limited only to some restricted types of functions. For example, the situation may become absurd when the desired reference function is continuous everywhere but not differentiable anywhere like the *Weierstrass Function*:

$$i_r(t) = \sum_{j\geq 1} \lambda^{-\mu j} \sin(\lambda^j t), \qquad 0 < \mu < 1, \lambda > 1 \tag{2.11}$$

The typical cases like that of problem (c) may appear if the derivative of the desired reference function $i_r(t)$:

- does not exist at a finite number of points or
- does not exist almost everywhere

2.6 The Fractional Calculus Approach to Tracking Problem

The results of tracking problems can be improved by using fractional calculus. The allowable set of reference functions in the case of classical sliding mode control can be made larger to include reference functions which are not first-order differentiable. An extensive discussion about various control issues in fractional-order systems can be found in [21]. Here, the tracking problem of a non-differentiable reference function is approached in the following two ways [28]:

- In the first approach, a reference generator system based on fractional-order operators is selected with ON/OFF based controller. For example, networks having resistive-inductive and resistive-capacitive fractional-order elements of order α (where, α is some real number), spring-dashpot fractal elements, etc. can be used with ON/OFF feedback controller.
- The second approach is based on design of control law based on fractional-order operators to track non-differentiable reference functions.

At first, the first approach is presented by using a switch-controlled RL circuit (DC chopper) as an example. The case is discussed in a generalized framework so that the technique can be applied in a general nonlinear system. After that, the second approach is discussed in a generalized sense.

The main idea, on which the solution to this class of problems is based, reflects from the following argument "*Although a class of functions do not possess first-order derivatives at some point (for example, triangular or sawtooth function) or at any point (Weierstrass function) in their respective domains, they do have some level of smoothness which can be measured and analyzed with the help of Fractional Calculus.*"

For the existence of fractional-order derivatives of the given functions, the following remark is important:

Remark 1.3 [65] ${}_{0}^{RL}D_t^{\alpha} f(t)$ exists almost everywhere for the integrable function $f(t)$ even if it possess a finite number of points of discontinuity.

By using simple calculation, it can be verified that the fractional-order derivative of the sawtooth function, $i(t)$ as defined in (2.3) exists almost everywhere and can be expressed as,

$$D^{\alpha} f(t) = \begin{cases} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, & \text{if } 0 < t \leq 1, \\ \frac{t^{1-\alpha} - 2(t-1)^{1-\alpha}}{\Gamma(2-\alpha)}, & \text{if } 1 < t < 2. \end{cases} \tag{2.12}$$

where, $D^{\alpha} f(t)$ is used in place of ${}_{0}^{RL}D_t^{\alpha} f(t)$ for notational convenience. The same notation is followed throughout the chapter.

Since, $\lim_{t\to 1^-} D^{\alpha} f(t) = \lim_{t\to 1^+} D^{\alpha} f(t) = 1/\Gamma(2-\alpha)$; therefore, $D^{\alpha} f(t)$ exists at $t = 1$. However, its first-order derivative $\dot{f}(t)$ does not exist at $t = 1$.

Similar observations can be made in case of the function,

$$f(t) = \begin{cases} t \ln t, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases} \tag{2.13}$$

The above function (2.13) is continuous but has no derivative at the point $t = 0$. However, it has well-defined Riemann-Liouville fractional-order derivatives, ${}_{0}^{RL}D_t^{\alpha} f(t)$ of all orders up to $0 < \alpha < 1$ [1]:

$${}_{0}^{RL}D_t^{\alpha} f(t) = \frac{t^{1-\alpha}}{\Gamma(1-\alpha)} \left[\psi(1) - \psi(2-\alpha) + \ln t\right] \tag{2.14}$$

where, $\psi(x) = \frac{d}{dt} \ln \Gamma(t)$ is the *Psi-Function*. Similarly,

$$f(t) = \prod_{k=0}^{n} (t - t_k) \ln |t - t_k|, t > a = t_0;$$

where, $a = t_0 < t_1 < \cdots < t_n$ is a continuous function. It lacks first-order derivative at a finite number of points. However, it possesses fractional-order derivatives, ${}_{0}^{RL}D_t^{\alpha} f(t)$ of orders $\alpha < 1$.

From the above discussion, it is clear that there are some functions which do not have first-order derivative but possess fractional-order derivatives. It is possible to capture the behaviour of such class of functions using the next theorem.

Lemma 2.1 *[10] Suppose $f(t)$ satisfies the Hölder condition of order β with $0 < \beta \leq 1$, on $[a, b]$, $-\infty \leq a < b \leq \infty$, or $f \in H^{\beta}([a, b])$ if $|f(t+h) - f(t)| \leq L|h|^{\beta}$ with L not depending on h and t; $t, t+h \in [a, b]$. Then, $f(t)$ possesses the fractional-order Riemann-Liouville derivatives of all orders $\alpha < \beta$ and,*

$$ {}_0^{RL}D_t^{\alpha} f(t) = \frac{f(a)}{\Gamma(1-\alpha)(t-a)^{\alpha}} + \psi(t); $$

where, $\psi(t) \in H^{\beta-\alpha}([a, b])$.

Now, using fractional-order operators, the tracking problem for non-differentiable reference functions is formulated. The fractional-order derivatives and integrals can be realized practically by using fractional-order inductors or capacitors with the desired specifications. An important description about the construction and implementation of fractional-order inductors can be found in [90] [91], [95] and the references within. There exists a vast literature devoted to the practical applicability of other such fractional-order elements [92] [93] [94] .

Consider a fractional-order inductor with inductance L. Let the voltage across the element be v. Then, this voltage will be equal to the α-order derivative of the flux $\phi(t)$ through it [89] i.e.,

$$ v(t) = \frac{d^{\alpha}\phi(t)}{dt^{\alpha}}; 0 < \alpha < 1; $$

where, $\frac{d^{\alpha}}{dt^{\alpha}}$ is the Riemann-Liouville (R-L) fractional-order derivative as defined above. Using the relation $\phi = Li(t)$,

$$ v(t) = L\frac{d^{\alpha}i(t)}{dt^{\alpha}}; \tag{2.15} $$

where, $i(t)$ is the current flowing through the fractional-order inductor. Similarly, the following relationship for the fractional-order capacitor (also known as super-capacitor or ultra-capacitor) can be obtained,

$$ i(t) = \frac{d^{\alpha}q(t)}{dt^{\alpha}} = C\frac{d^{\alpha}v(t)}{dt^{\alpha}}; \quad 0 < \alpha < 1; \tag{2.16} $$

where, q and C are the accumulated charge and the capacitance of the fractional-order capacitor respectively.

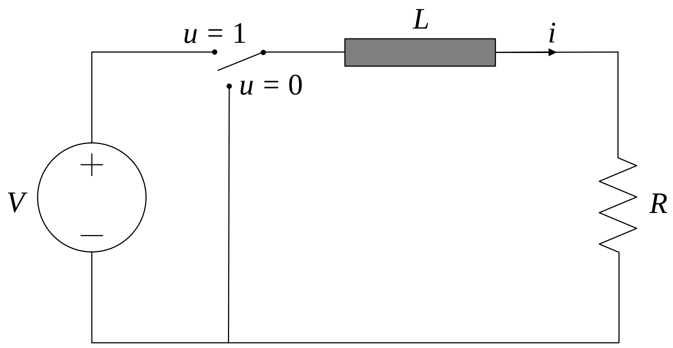

Figure 2.2: Switch-Controlled Fractional RL Circuit

Consider the same DC chopper circuit again as shown in Fig. 2.1. Now, the simple inductor is replaced by a fractional-order inductor as shown in Fig. 2.2.

For the circuit in Fig. 2.2, its dynamic behaviour can be captured by:

$$L\frac{d^{\alpha}i(t)}{dt^{\alpha}} = u(t)V - i(t)R; \quad 0 < \alpha < 1, \tag{2.17}$$

where, R, L, $i(t)$ and $u(t) \in \{0, 1\}$ are the resistance, inductance, current through fractional-order inductor and the control input respectively.

Remark 1.4 For a simple illustration of the proposed technique, a circuit like Fig. 2.2 is considered. However, without loss of generality, the idea can be extended to derive the mathematical equation of electrical circuits composed of any finite number of resistances, fractional-order inductors, fractional-order capacitors and voltage (current) sources. Suppose the state variables are currents flowing in the fractional-order inductors $z_L \in \mathbb{R}^{n_2}$ and voltages across the fractional-order capacitors, $z_C \in \mathbb{R}^{n_1}$. Following (2.4), a generalized representation of the state equations of a fractional-order linear circuit can be obtained as [89] [61]:

$$\begin{bmatrix} \frac{d^{\alpha}z_L}{dt^{\alpha}} \\ \frac{d^{\beta}z_C}{dt^{\beta}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_L \\ z_C \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u; \quad 0 < \alpha, \beta < 1 \tag{2.18}$$

where, the components of $u = u(t) \in \mathbb{R}^m$ are the voltage or current sources and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m}; i, j = 1, 2$.

In order to solve the problem (c) in Section 2.4, one can calculate the fractional-order rate of change of the error (2.4),

$$\frac{d^{\alpha}e}{dt^{\alpha}} = \frac{d^{\alpha}i(t)}{dt^{\alpha}} - \frac{d^{\alpha}i_r(t)}{dt^{\alpha}}. \tag{2.19}$$

Here, a sawtooth function is used as the desired voltage function. It is clear that the first-order or classical derivative of the reference function is not feasible. However, it is possible to calculate and further use the fractional-order derivative of the reference function, $v_r(t)$ for the manipulation of the error (2.4). Using Eqns. (2.17), (2.4) and (2.19),

$$\begin{aligned}\frac{d^\alpha e}{dt^\alpha} &= \frac{R}{L}\left[\frac{V}{R}u(t) - e - \left(i_r(t) + \frac{L}{R}\frac{d^\alpha i_r(t)}{dt^\alpha}\right)\right] \\ &= \frac{R}{L}\left[i_{\max}u(t) - e - \left(i_r(t) + \frac{L}{R}\frac{d^\alpha i_r(t)}{dt^\alpha}\right)\right]; \end{aligned} \tag{2.20}$$

where, $i_{\max} = \frac{V}{R}$. Suppose that the error is negative initially i.e., $e < 0$. Therefore, one should 'ON' the switch i.e., $u(t) = 1$ as long as e remains negative. Substituting $e < 0$ and $u(t) = 1$ in (2.20),

$$\frac{d^\alpha e}{dt^\alpha} = \frac{R}{L}\left[i_{\max} + |e| - \left(i_r(t) + \frac{L}{R}\frac{d^\alpha i_r(t)}{dt^\alpha}\right)\right]. \tag{2.21}$$

Using Eqn. (2.21), it can be concluded that the fractional-order time derivative of the error variable $e(t)$ is guaranteed to remain positive if the desired trajectory satisfies the condition,

$$i_{\max} > \left|i_r(t) + \frac{L}{R}\frac{d^\alpha i_r(t)}{dt^\alpha}\right|; \forall t.$$

Now, it is proved that the first hitting to the error surface (sliding surface) occurs in finite time, T. For simplicity of representation, equation (2.21) is rewritten as,

$$\frac{d^\alpha e}{dt^\alpha} = k|e| + d; \tag{2.22}$$

where, $k = R/L > 0$ and $d = k\left[i_{\max} - \left(i_r(t) + \frac{L}{R}\frac{d^\alpha i_r(t)}{dt^\alpha}\right)\right] > 0$.

Applying fractional-order integral operator to both sides of (2.22),

$$D^{-\alpha}\frac{d^\alpha e}{dt^\alpha} = D^{-\alpha}k|e| + D^{-\alpha}d. \tag{2.23}$$

where, $D^{-\alpha}$ stands for the Riemann-Liouville definition of fractional-order operator, ${}_0^{RL}D_t^{-\alpha}$. The same notation has been used from this point onwards.

Using the relations,

$$D^{-\alpha}\frac{d^\alpha e}{dt^\alpha} = e(t) - D^{\alpha-1}e(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)};$$

and

$$D^{-\alpha}d = d\frac{t^\alpha}{\Gamma(1+\alpha)};$$

Eq. (2.22) can be manipulated so that after finite time $t = T$,

$$e(T) - D^{\alpha-1}e(0)\frac{T^{\alpha-1}}{\Gamma(\alpha)} = k\frac{T^\alpha}{\Gamma(1+\alpha)} + d\frac{T^\alpha}{\Gamma(1+\alpha)}. \tag{2.24}$$

It is straight-forward to conclude that $e(t) = 0$ results if time T is finite. From equation (2.24) and by taking initial condition of e to be negative,

$$D^{\alpha-1}e(0)\frac{T^{\alpha-1}}{\Gamma(\alpha)} = k\frac{T^{\alpha}}{\Gamma(1+\alpha)} + d\frac{T^{\alpha}}{\Gamma(1+\alpha)}$$
$$\Rightarrow T = \frac{\Gamma(1+\alpha)D^{\alpha-1}e(0)}{\Gamma(\alpha)(k+d)}. \tag{2.25}$$

It can be easily checked that the above calculated time, T always comes out to be finite. On the other hand, for positive error i.e. $e > 0$, $\frac{d^{\alpha}e}{dt^{\alpha}}$ should be negative. Therefore, one must 'OFF' the switch i.e. $u(t) = 0$ as long as e remains positive. Substituting $e > 0$ and $u(t) = 0$ in (2.20),

$$\frac{d^{\alpha}e}{dt^{\alpha}} = -\frac{R}{L}\left[e + \left(i_r(t) + \frac{L}{R}\frac{d^{\alpha}i_r(t)}{dt^{\alpha}}\right)\right]. \tag{2.26}$$

So, the fractional-order time derivative of the error is guaranteed to remain negative provided the desired reference function satisfies,

$$\left|i_r(t) + \frac{L}{R}\frac{d^{\alpha}i_r(t)}{dt^{\alpha}}\right| > 0;\ \ \forall t.$$

Therefore, it is clear that the desired tracking of non-differentiable reference function is possible (using fractional-order switch-controlled RL circuit) provided,

$$0 < \left|i_r(t) + \frac{L}{R}\frac{d^{\alpha}i_r(t)}{dt^{\alpha}}\right| < i_{\max};\ \ \forall t.$$

2.7 General Class of Nonlinear Systems Affine in Control

Consider the following general form of an input-affine nonlinear system,

$$\dot{x} = f(x) + g(x)u; \tag{2.27}$$

where, the functions $f(x)$ and $g(x)$ are locally Lipschitz over a domain $D \in \mathbb{R}^n$ with $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$ having bounded region $D \subset \mathbb{R}^n$ with an equilibrium point at $x = 0$. These conditions ensure that a unique solution exists which is defined for all $t \geq t_0$. It is further assumed that the function $g(x)$ is invertible. Now, suppose that it is desired to maintain the output of the system at some reference function x_r, for all $t \geq t_0$. In order to solve the problem (c) discussed in Section 2.4, the fractional-order rate of change of the error is calculated as, $y(t) = x(t) - x_r(t)$;

$$D^{\alpha}y(t) = D^{\alpha}x(t) - D^{\alpha}x_r(t); \tag{2.28}$$

where, $D^{\alpha} = \frac{d^{\alpha}}{dt^{\alpha}}$. By using the property of fractional-order operator,

$$D^{\alpha} y(t) = D^{\alpha-1} Dx(t) - D^{\alpha} x_r(t); \tag{2.29}$$

Using (2.27), Eqn. (2.29) becomes,

$$\begin{aligned} D^{\alpha} y(t) &= D^{\alpha-1} \left(f(x) + g(x)u \right) - D^{\alpha} x_r(t) \\ &= D^{\alpha-1} f(x) + D^{\alpha-1} \left(g(x)u \right) - D^{\alpha} x_r(t). \end{aligned} \tag{2.30}$$

For simplicity of representation, the effect of any type of disturbance is not considered here. It is clear that simple PI^{α} *(Fractional-Order PI) Controller* is enough to stabilize the fractional-order error dynamics at the origin. Moreover, robust control strategies can be used in the presence of disturbances. The same problem formulation can be followed in that case. Now, suppose that the designed control law is $u := g^{-1}(x) \left\{ D^{1-\alpha} v \right\}$. Then, keeping $u(0) = 0$,

$$D^{\alpha} y(t) = D^{\alpha-1} f(x) + v - D^{\alpha} x_r(t). \tag{2.31}$$

Now, defining $v := -D^{\alpha-1} f(x) + D^{\alpha} x_r(t) - \frac{k_i}{k_d} D^{-\alpha} y(t) - \frac{k_p}{k_d} y(t)$; where, $k_p, k_i, k_d > 0$ as the parameters of the $PI^{\lambda} D^{\mu}$ *(Fractional-Order PID) Controller* and substituting v in (2.31), the expression becomes,

$$k_d D^{\alpha} y(t) + k_i D^{-\alpha} y(t) + k_p y(t) = 0. \tag{2.32}$$

Then, the values of the controller parameters $k_p, k_i, k_d > 0$ can be selected based on the work in [66] [67] such that $y(t) \approx 0$, which further implies $x(t) \approx x_r(t)$.

From the above discussion, it can be stated that if the desired reference function is not differentiable, then the classical integer-order derivative becomes unbounded which results in distorted error dynamics at the non-differentiable points. On the other hand, with fractional-order error dynamics (2.28), the terms remain in a bound. This also satisfies the property of *Input-to-State Stability (ISS)* [4]. According to this concept, the states of the fractional-order linear time-invariant system remain in a bound for all time,

$$||x(t)|| \leq ||x_0|| ||E_{\alpha}(A(t-t_0)^{\alpha})|| + || \int_{t_0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha}) d\tau ||$$

where, $E_{\alpha}(z)$ is the one-parameter Mittag-Leffler function defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

and $E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad , \alpha > 0, \quad \beta > 0$$

Further, if the system matrix A satisfies the condition $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$ then, there exists a positive number M such that $|| \int_{t_0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha}) d\tau || \leq M$. In such a case, the states satisfy the condition [59],

$$||x(t)|| \leq ||x_0||||E_{\alpha}(A(t-t_0)^{\alpha})|| + ||B|| \sup_{t_0 \leq \tau \leq t} ||u(\tau)||M$$

Suppose that the information of fractional-order derivative of the reference function is not available. Then, the fractional-order derivative term can be simply removed from the control input. Here, the closed-loop system (2.31) also contains the unknown term. With proper design of the gain values of $PI^{\lambda}D^{\mu}$ controller, it can be shown that the closed-loop system (2.31) is fractional-order input-to-state stable where the input is the fractional-order derivative of the reference function.

Therefore, it is clear that the proposed approach is valid for general class of nonlinear systems. This further enhances the applicability of the technique. The simulation results of the switch-controlled RL circuit has been demonstrated in the next section.

2.8 Illustrative Example

The same switch-controlled RL circuit (DC Chopper) is considered for illustrating the proposed approach to tracking problem [28]. A sawtooth function is taken as the reference current to be tracked. A comparison of the simulation results is done as shown in Fig. 2.3 for both the circuits with fractional-order inductor and simple inductor. The values of the parameters are taken as $\alpha = 0.5$, $V = 1$ V, $R = 10$ and $L = 10$ μH. A sawtooth reference current function having an amplitude of 1 V p-p and a frequency of 100 Hz is used in both the cases. At the corner points where the reference function is not differentiable, the tracking is poor for the circuit with simple inductor as shown in Fig. 2.3. For the circuit with fractional-order inductor, the actual variable is able to track the desired sawtooth reference function at all points accurately. This result properly demonstrates the effectiveness of the technique.

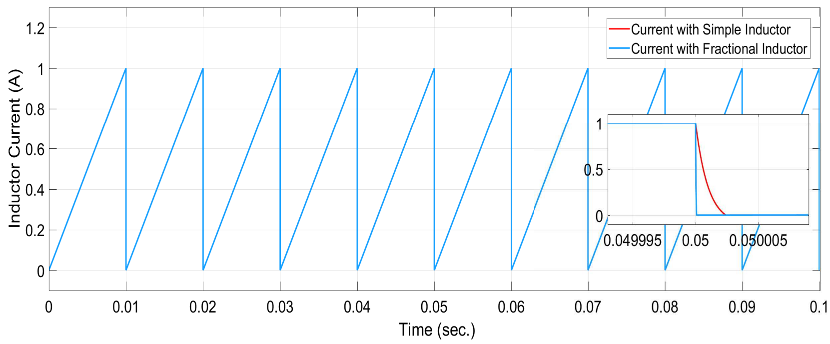

Figure 2.3: Current Trajectories in Switch-Controlled *RL* Circuit with time

2.9 Summary

From the discussion in this chapter, it is clear that the proposed technique using sliding mode control is quite effective in tracking the desired reference function in finite time. At the same time the sliding mode based approach provides robustness to disturbances and parametric uncertainties. It has been demonstrated that an improvement in the fundamental operation by using fractional calculus can result in a large class of reference functions to be tracked. The methodology is applicable irrespective of the non-differentiability of the reference function at some points. This enables accurate tracking in applications where the reference function changes abruptly.

From the theoretical point of view, the tracking problem considered in this chapter can be explored in the direction of implementing various classes of sliding mode controllers. Continuous terminal sliding mode based technique may be utilized to improve the results [186]. Further, arbitrary-time convergence can also be obtained by using the concept presented in [194].

Similar problems are trajectory tracking by robotic manipulators and spacecrafts [215]-[219]. These applications require careful and precise operations of the actuators. There can be sudden obstacles in the path of motion which can dynamically alter the desired reference functions. This requires a fast control response while following the desired trajectory with high accuracy. The methodology presented in this chapter can be explored in this direction to find new results.

Another evolving field is that of nanotechnology. In these applications, atomic force microscopy is frequently used which requires fast, precise and repeatitive motion of the piezo-electric actuator. Fast tracking of triangular signals with high accuracy in scanning probe microscopy is a challenging problem. A triangular function contains all the odd harmonics of the fundamental frequency. When a triangular signal is fed to the piezoelectric actuator, high-frequency excitation results which further reflects as inaccurate triangular signal at the free end of the actuator causing a distorted scanned image [222] [223]. Implementing the proposed technique can greatly improve the performance.

Image edge detection is another commonly encountered problem in which this technique can be used. It consists of mathematical methods which locate the points in a digital image where the image brightness changes sharply. Texture enhancement is also very important image processing aspect in the interpretation of image data, image restoration, pattern recognition, robotics, medical image processing and remote sensing. The fractional derivative operator based control can be used to track sharper edges of the image which can further be used for its reconstruction with higher accuracy. A similar approach is proposed in [231] using Riemann-Liouville (R-L) fractional-order derivative. Further improvements can be obtained by using the proposed technique.

Therefore, it can be concluded that the approach presented in this chapter has the potential to greatly improve the control performance in various fields of diverse applications. Being a generalized approach, it covers a wide range of problems which can be handled quite effectively. The control technique based on fractional calculus consists of interesting dimensions which remains to be explored.