Chapter 1

Introduction

The occurrence of non-differentiability in the modeling and control of various phenomena is well recognized. It plays an important role in areas like functional analysis, control theory, differential equations, optimization, mechanics, etc. Consider a simple example of finding the maxima or minima of a function. The general approach would be to find the derivative of the function and use the idea that it must be zero at the maximum or minimum points of the function. But, what if the function itself is non-differentiable? There are a number of problems in control theory where one or the other associated function needs to be differentiated for the implementation of the control law. Generally, the functions are assumed to be differentiable in the analysis and design related to such problems. There are various phenomena related to the issue of non-differentiability [32]. These issues are very common and are generally ignored in the design aspects. In order to deal with such important issues, there is a need of a mathematical insight into such class of problems at the fundamental level. Another important issue is related with that of uncertainty in the modeling of physical systems. The control must be designed to guarantee robustness to the uncertainties. One such control design is discussed in this text in the last part. This chapter gives a brief overview of *Fractional Calculus* which has been used throughout the work. The issue of non-differentiability has been illustrated by considering tracking problem in nonlinear systems and the design of differentiators in further chapters.

1.1 Literature Review

The issue of non-differentiability has been studied in some remarkable contributions in control theory. One of these work has been discussed in [32]. It talks of the analysis of differential equations in the presence of non-differentiability. This is also referred to as *Non-smooth Analysis*. Certain Banach spaces have smooth norms which can be used in problems arising due to non-differentiability. The notions of proximal subgradients has been discussed in [32]. The theory of generalized gradients is another approach in this direction which also operates in Banach spaces. Value function analysis for constrained optimization has also been studied. Another direction to such type of analysis is based on directional subderivates. In this work, fractional-order operators have been used throughout as a tool to overcome the issues arising due to non-differentiability and uncertainty in control problems.

The first problem involving the non-differentiability issue which has been considered in this text is that of the tracking problem. It is a general problem in control theory which can be reduced to an equivalent stabilization problem as discussed in [214]. The tracking problem has been explored in various contexts in the literature. One of the earlier contributions can be found in [58]. It has also been explored in the field of robotics. A sliding mode control based approach is used in [214]. Passivity based scheme is presented for tracking problem of flexible robot arms [215]. Attitude tracking of spacecrafts has been a major research area in the context of tracking problems. In this direction, some significant contributions are [217] - [219]. Another notable contribution is [223]. In these applications, there are possibilities where a sudden change in the desired reference functions may be required at some point of time. The work presented in this chapter finds an important significance in such problems.

An observer-based controller is designed for output tracking of fractional-order positive switched systems in [3]. The observer includes equivalent-input-disturbance and Smith predictor. It is able to reject the disturbance. In [58], a function space analysis is used to find the optimal input to track the output of the system in a fixed time to minimize the control energy required. Signal transformation approach to track non-differentiable functions has been attempted by some authors [228]. In [223], the technique of signal transformation is used to track triangular function. This approach improves the overall performance of the closed-loop system. Necessary and sufficient conditions for the stability of the control system are stated for tracking triangular reference functions.

In the previous approaches, the desired reference functions have been approximated by piecewise continuous functions. However, it not possible to use this approximation to all class of desired reference functions. This situation calls for a revisit of the tracking problem with some sound mathematical framework. There exist continuous functions for which no first-order derivatives are defined but they possess fractional-order derivatives of all orders less than one [10]. This motivates further to have a look on the beauty of fractional calculus and how it can be utilized in tracking problems where the reference function is not differentiable.

Sliding Mode Control (SMC) is one of the most efficient robust control techniques which performs quite effectively in the presence of disturbances and uncertainties [121] - [126]. In this control scheme, the system states are forced to remain on a predefined manifold called sliding surface and maintained it there. Keeping the states on the sliding surface, they are driven to the origin. Therefore, the sliding mode based control technique consists of two phases. The first one is the *Reaching Phase*, where the system states are driven from their initial values to reach the sliding manifold in finite time. Then, the states undergo the *Sliding Phase* where they are further driven to the origin. On the sliding surface, the order of the system dynamics gets reduced, and robustness is obtained. Recently, the theory has been used for control and observation in several problems [7] [28] [41] [55] [122].

In the context of tracking problems, one of the earlier contribution is [214] which discusses the design of sliding mode control law for robotic manipulators. Some other significant reports are [225] [226] [215] [229]. Using sliding mode techniques for attitude control of rigid body is an important class of tracking problems. Some significant work in this direction are [217] - [219]. Another field of application is described in [220]. However, all of the above mentioned approaches discuss the tracking problems in the integer-order framework. In the above work, it is often required to calculate the derivative of the error function $e(t)$ which itself is the difference between the state $x(t)$ of the system and the desired reference function $x_r(t)$. Ultimately, the information of time-derivative of the reference function $\dot{x}_r(t)$ is needed for the techniques to work. So, the previous approaches restrict the class of reference functions to be tracked.

Another problem involving non-differentiability is that of designing differentiator. There are numerous efforts in this direction which can be found in the literature. One such approach uses sliding mode control strategy for the estimation of the required derivative [137]. It uses the Super-Twisting Algorithm (STA) which is a second-order sliding mode control scheme. Being one of the most popular higher-order sliding mode based technique, it only requires the information of the position variable of the system for its convergence. At the same time, it offers robustness to the uncertainty as an inherent characteristics of sliding mode control. However, the conventional explicit discretization of STA suffers from chattering [150]. Full Euler discretization schemes of the STA are proposed in [148], which achieve only standard first-order accuracy of Sliding Mode Control (SMC). In this discussion, implicit Euler discretization has been used for the integer-order dynamics of the estimator. For the fractional-order dynamics, Fractional-Adams Moulton (FAM) method has been used which is suitable for fractional-order equations with discontinuous right-hand sides [156]. Other approach to suppress chattering is given in [119]. Discretization is the key step to implementation of control laws. A detailed discussion of the significance of discrete-time fractional-order differentiators in physical systems is presented in [142]. There are various methods for discretizing fractional-order operators [147] [174]. They are based on several methods including continued fraction expansion, radial basis function method, Tustin method, Taylor series, Newton series and least squares method [147] [144] [146]. Differentiator design for signals with error is considered in [143].

The existing sliding mode based techniques assume a bound on the second derivative of the signal for determining the first-order derivative. However, not all the practical signals are second-order differentiable which may be the case with commonly used circuits and systems. This imposes a limitation on the class of signals that can be differentiated. Fractional-order operators can be used to overcome this restriction [10] [28] [201] [173] [166] [167] [168] [170] [192]. In this work, the signal is assumed to have only Hölder continuous first-order derivatives. So, this technique addresses a large class of signals to be differentiated. In this paper, Fractional Adams-Moulton (FAM) method has been mainly used as the numerical scheme which is quite effective in dealing with chattering in the case of fractional-order differential equations with discontinuous right hand sides [155] [156]. The Riemann-Liouville definition of fractional-order derivatives has been used throughout the work [56]. In this definition, the signal does not have to be integer-order differentiated for its fractional-order derivative to be defined.

The sliding mode control schemes are robust with respect to the uncertainties and disturbances. The main objective of this class of controllers is to force the system states to stay in a predefined manifold (sliding surface) and maintain it there in spite of the uncertainties in the system [120]. Therefore, the sliding mode based design consists of two phases (i) *Reaching Phase* in which the system states are driven from the initial state to reach the sliding manifold in finite time and (ii) *Sliding Phase* in which the closed-loop system is induced into sliding motion.

However, when the system states reach sliding phase, the consideration of robustness and order reduction come into picture which are the most important aspects of the sliding mode based design. It is worth noting that during the reaching phase, there is no guarantee of robustness [198]. In order to address robustness issue throughout the entire space, *Integral Sliding Mode Control (ISMC)* has been proposed in the SMC literature [198] but its design methodology is based on full order of the system. However, the system exhibits a reduced-order dynamics after it has reached the sliding surface, that is, the system order gets reduced by one due to the introduction of the sliding variable, s such that $s = 0$ in finite time. As a consequence, the simplicity and flexibility of the design procedure which are provided by reduced-order subsystem based design in classical SMC are lost in ISMC. The motivation behind this work is to preserve the robustness in the system by eliminating the reaching phase such that the system remains on the sliding manifold from the very initial time. The robustness is guaranteed only in the sliding phase of design. In this direction, a fractional-order system is considered in the later part of this text [195] [196]. The effectiveness of the approach is shown for such class of systems by taking the example of a fractional inverted pendulum [235].

In control theory, it has been reported in the literature that the introduction of controllers involving integrals and derivatives of arbitrary orders results in higher performance as compared to the classical controllers with integer order operators [50] [8] [22] [172]. It is observed that functions which do not possess the first-order derivative, may possess a fractional-order derivative of order $\alpha < 1$ provided the function is integrable [10]. Therefore, the above limitations of classical control strategies can be overcome by using fractional-order rate of change of the reference function $D^{\alpha} x_r(t)$ in the control law. Various classes of fractional-order controllers can be found in [8].

The tracking problem for a reference function x_r can be considered in two different ways. In one way, a reference generator system can be considered, and it is desired to obtain some reference function x_r. In the other way, a particular reference function is desired where the reference generator system can itself be chosen accordingly. Possible reference generator systems can be networks based on fractional-order elements like supercapacitors and fractional-order inductors or spring-dashpot fractal networks [89] [91] [95]. For both the two situations, the corresponding tracking problems can be formulated and the conditions required to achieve the desired tracking can be derived. This is illustrated by taking an example of switch-controlled RL circuit and also for a general class of nonlinear systems.

1.2 The Non-Differentiability Constraints in Control

Nonlinear systems have the characteristic feature of several interesting phenomena which are inherent to them. These specialities include finite escape time, multiple equilibrium points, chaos, etc. These systems are often desired to behave in a pre-defined fashion in the presence of undesired disturbances and uncertainties. According to the objective, the control problems may be broadly classified into Tracking and Stabilization problems. Several control design techniques have been proposed in the literature to achieve these objectives. However, many of the techniques consider a fundamental assumption on the differentiability of the solutions or the desired trajectories of the system. On the contrary, there is a vast possibility for the functions to be non-differentiable. In order to address the control problems involving non-differentiability issues, there is a demand to re-look into such class of problems with a fundamental viewpoint.

For the integer-order derivatives to be defined, it requires for the function to be integer-order differentiable. However, this seriously restricts the class of functions to which the notion of integer-order derivatives may be applicable. Further, the control techniques relying on the manipulation of the integer-order derivatives of the variables cannot be defined if the variables are not differentiable in the integer-order sense. This limitation comes from the non-differentiability of the associated functions. Such problems can be tackled by allowing the order of differentiation to take fractional values. Arbitrary-order derivatives and integrals constitute the amazing world of Fractional Calculus. Many real-world phenomena are described in a more justified way by their fractional-order models as compared to integer-order models [47] [48] [49]. Fractional-order controllers have been reported to perform better as compared to their integer-order counterparts.

The problem of non-differentiability is encountered in several control problems [227]. Here, non-differentiability is understood in the integer-order sense throughout [34]. Some of the problems has been considered in this work in which the issue is resolved by using fractional-order derivatives. The technique used is based on the work [10]. It has been mentioned in that paper that functions which do not possess the first-order derivatives have well-defined fractional-order derivatives of all orders less than 1. This important result is utilized throughout the work presented here. The use of fractional-order derivatives has been made in the generation and manipulation of control laws instead of the integer-order derivatives [33]. This results into a larger class of functions which can be addressed using the scheme presented. Such functions need only to satisfy the weaker condition of Hölder continuity.

This chapter discusses the importance of fractional-order systems and their fundamental characterization at an introductory level. The physical interpretation of fractional-order derivatives and integrals have been explored. Some physical phenomena have been considered which justify their fractional-order dynamic behaviour. The chapter gives an overall visualization of fractional-order systems which is very important for control design and implementation aspects of these systems.

1.3 The Fractional Calculus

Better results can be obtained by generalizing the order of the operators commonly used in integer-order calculus [6] [9] [19] [20] [27] [36]. Operations with the resulting fractional-order derivatives and integrals constitute *Fractional Calculus* [15] [26]. It is an ancient branch of mathematics which has developed in parallel with the classical calculus [11] - [18]. It has received wide research interests in recent years [51] [52] [115] [116] [232]. Naturally, a number of definitions of the fractional-order operators have been proposed in the literature. However, three of them are widely used. These are *Riemann-Liouville* definition, *Caputo* definition and *Grunwald-Letnikov* definition. There is only one definition of fractional-order integral i.e. Riemann-Liouville definition of fractional-order integral. The fractional-order derivative has different expressions in all the three definitions. All of them are presented in this section. Each one includes the Gamma Function in its definition. The Gamma Function plays an important role in fractional calculus. It is the generalized form of the commonly used exponential function in integer-order calculus [160] [51] [162] [163] [164].

The idea of fractional calculus was discussed for the first time over a letter from Leibnitz to L'Hôpital in 1695. Fractional differential equations have been in use to model physical phenomena in the last couple of decades [43]. The history of fractional-order calculus can be found in [14] [15] and the references cited therein. The state space description is given in [62] [63]. In [2], the authors emphasized on the recent interest of the research community in fractional-order systems. New paths have been paved in the fractional calculus theory in [6]. Due to its wide advantages, in recent years, the study of fractional-order controllers has witnessed considerable interest [7] [24] [29] [35] [160] [39] [113] [164]. The discussions on stability of fractional-order systems can be found in [101] - [106]. Several applications of fractional calculus in various fields have been given in [221] [30] [31] and the references cited therein.

Throughout this work, Riemann-Liouville and Caputo definitions are used [37]. One of the advantages with Riemann-Liouville definition is that it does not restrict the class of the functions to which it can be applied. As it can be observed in the expressions of both the definitions, a mathematical comparison can be made. In the Caputo definition, the function to be operated is assumed to be continuously differentiable. Now, it can be easily concluded that this definition can be applied only to smooth functions. This is a serious mathematical constraint with this definition. On the other hand, the Riemann-Liouville definition of fractional-order derivative operates on a function by first integrating it and then differentiating. So, a simple weaker assumption of integrability applies to the function which is operated upon. However, the initial conditions involved with this definition take values in terms of the fractional-operators while in case of Caputo definition, the initial conditions have the same form as that of the conventional integer-order differential equations. For uniqueness of the solutions of differential equations, one needs to specify the initial conditions. These initial conditions must be such that they may be related or interpreted in terms of some recognized notions. For the fractional-order differential equations involving Caputo operator, the initial conditions are the same as that in the case of integer-order differential equations. This is why Caputo definition is more preferred in applications. On the other hand, the solution of fractional-order differential equation with Riemann-Liouville operator involves initial conditions which are in terms of those operators itself. There is still no concrete interpretation of fractional-order operators existing in the literature though there are few approaches to the same which will be discussed in later section. So, the Riemann-Liouville definition is less preferred over the Caputo definition. However, observing the mathematical constraints with the Caputo definition, Riemann-Liouville definition has been used in this context throughout. A discussion on the initial conditions aspect of fractional-order systems is done in [108]. The interpretation of initial conditions involving R-L fractional-order operators can be found in [97]. An initial-value problem using R-L definition is given in [99].

The beauty of fractional calculus can make the solution of some challenging problems possible for which the classical approach of integer-order calculus fails. Fractional calculus is a generalization of integer-order calculus. It involves generalized expressions of derivatives and integrals of non-integer order. Out of the several definitions of fractional-order derivatives, two are the most commonly used which are *Riemann-Liouville (R-L)* and *Caputo* definitions [1]- [7]

At first, a brief derivation of the expression of fractional-order integral is given here. It comes from the Cauchy's formula for iterated integral which is derived as follows:

$$
\begin{aligned}
D^{-1}f(t) &= \int_0^t f(x)dx \\
D^{-2}f(t) &= \int_0^t \int_0^x f(\tau)d\tau dx \\
\implies D^{-2}f(t) &= \int_0^t \int_\tau^t f(\tau)dxd\tau \\
\implies D^{-2}f(t) &= \int_0^t (t-\tau)f(\tau)d\tau \\
D^{-3}f(t) &= \frac{1}{2}\int_0^t (t-\tau)^2 f(\tau)d\tau
\end{aligned}
$$

Continuing in the same manner, the Cauchy's formula for iterated integral is obtained. In this formula, it is observed that the multiple times integration of the function on the left-hand side has been replaced by a single integral term on the right-hand side. This provides a way to generalize the operation to an arbitrary number of times.

$$D^{-n}f(t) = \int_0^t \ldots\ldots \int_0^t f(\tau)d\tau\ldots\ldots d\tau = \frac{1}{(n-1)!}\int_0^t f(\tau)(t-\tau)^{n-1}d\tau,\ n \in \mathbb{N}$$

The above expression is the Cauchy's formula which is valid for natural numbers. It includes the factorial operator. If it is generalized for arbitrary real values, the expression of Riemann-Liouville definition of fractional-order integral is obtained in which the above factorial operator is replaced with the generalized Gamma function. The Gamma function is a generalized form of the factorial operator. After substituting the Gamma function in place of the factorial function, the expression for fractional-order integral is obtained. The commonly used definitions in fractional calculus are discussed next.

Definition 1.1 *The fractional-order integral of order α of the function $f : (0, \infty) \to \mathbb{R}$ with respect to $t > 0$ and terminal value $t_0 > 0$ is given by*

$$ {}_{t_0}I_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{(1-\alpha)}}d\tau, \tag{1.1}$$

where $\Gamma : (0, \infty) \to \mathbb{R}$ is the Euler's Gamma function:

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1}e^{-x}dx$$

Definition 1.2 *The R-L definition of the fractional-order derivative of order α is given by:*

$$ {}_{t_0}^{RL}D_t^\alpha f(t) := \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{(\alpha-m+1)}}d\tau, \tag{1.2}$$

where $m \in \mathbb{N}$ such that $m \geq \lceil \alpha \rceil$, where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α where $0 < \alpha < 1$.

Definition 1.3 *The Caputo definition of the fractional-order derivative of order α of m times continuously differentiable function $f : (0, \infty) \to \mathbb{R}$ or $f \in C^m((0,\infty), \mathbb{R})$ is given by:*

$$ {}_{t_0}^{c}D_t^{\alpha} f(t) := \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{(\alpha-m+1)}} d\tau. \tag{1.3} $$

A few important properties of fractional-order derivatives and integrals are as follows [21]:

- For $\alpha = n$, where n is an integer, the operation ${}_{t_0}^{c}D_t^{\alpha} f(t)$ gives the same result as the classical differentiation of integer order n.

- For $\alpha = 0$, the operation ${}_{t_0}^{c}D_t^{\alpha} f(t)$ is the identity operation:

$$ {}_{t_0}^{c}D_t^{\alpha} f(t) = f(t). \tag{1.4} $$

- Fractional differentiation is a linear operation:

$$ {}_{t_0}^{c}D_t^{\alpha}(af(t) + bg(t)) = a\,{}_{t_0}^{c}D_t^{\alpha} f(t) + {}_{t_0}^{c}D_t^{\alpha} g(t). \tag{1.5} $$

- The additive index law (semigroup property)

$$ {}_{t_0}^{c}D_t^{\alpha}\,{}_{t_0}^{c}D_t^{\beta} f(t) = {}_{t_0}^{c}D_t^{\beta}\,{}_{t_0}^{c}D_t^{\alpha} f(t) = {}_{t_0}^{c}D_t^{\alpha+\beta} f(t), \tag{1.6} $$

  holds for $f(t) \in C^1[0,T]$ for some $T > 0$ where, $\alpha, \beta \in \mathbb{R}^+$ and $\alpha + \beta \leq 1$ [57].

Before using the fractional-order operators, proper understanding of their operations is very essential. Visualization of a mathematical notion needs a clear interpretation of the operations involved in terms of the previously established concepts or phenomena. In the field of fractional calculus, there are a number of definitions which have been proposed by many researchers. Due to the presence of a large number of varied definitions, it becomes obvious that one should know the physical interpretations of the associated definitions [44] [45] [108] [98]. Having an idea of the interpretations, the definitions can be applied to suitable problems to get the desired results. The search for finding proper interpretations of fractional-order integrals and derivatives have been quite long since their formulations. In this direction, a significant contribution is the work presented in [40]. The fractional-order integral of order α,

$$ {}_{t_0}I_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{(1-\alpha)}} d\tau \tag{1.7} $$

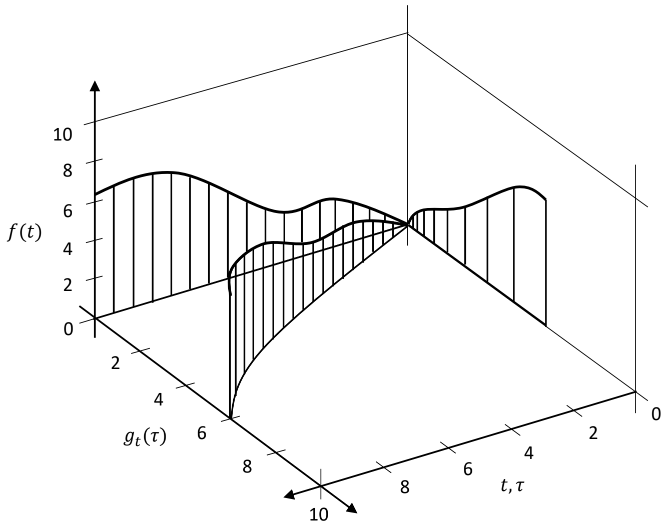

Figure 1.1: Geometrical Interpretation of Fractional-Order Integral [40]

can also be expressed as

$$ {}_{t_0}I_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} f(\tau) dg_t(\tau) \tag{1.8} $$

where,

$$ g_t(\tau) = \frac{1}{\Gamma(\alpha+1)} \{t^{\alpha} - (t-\tau)^{\alpha}\} \tag{1.9} $$

The homogeneous time-scale consists of equal intervals of flowing time. Apart from the concept of classical homogeneous time scale, the fractional-order differentiation and integration can be considered to operate in a non-homogeneous time scale or the *Cosmic* time scale [40] which is composed of non-equal time intervals. The homogeneous time scale can be considered as an ideal notion for the non-homogeneous time scale. The function $g_t(\tau)$ has the scaling property,

$$ g_{t_1}(\tau_1) = g_{kt}(k\tau) = k^{\alpha} g_t(\tau) $$

Consider a three-dimensional space with the axes (τ, g_t, f). The function $g_t(\tau)$ is plotted in the plane (τ, g_t) for $0 \le \tau \le t$. Along the obtained curve, a fence can be formed of varying height $f(\tau)$. So, the top edge of the fence represents a line in three-dimensional space $(\tau, g_t(\tau), f(\tau))$, $0 \le \tau \le t$. For the purpose of further illustrations, consider Fig. 1.1 used here from [40]. The fence can be projected onto the plane (τ, f). The area of this projection corresponds to the integral,

$$ I_1 = \int_{0}^{t} f(\tau) d\tau \tag{1.10} $$

while the same fence can be projected onto the plane (g_t, f), the area of which corresponds to the integral (1.8) or (1.7). This projection on the plane (g_t, f) is the geometric interpretation of the fractional-order integral (1.7) for fixed t. For $g_t(\tau) = \tau$, both the projections overlap each other showing that the classical definition of integral is a special case of the generalized fractional-order integral.

1.4 The Regularization Approach

Consider the system of the following form

$$\dot{x} = f(x, u), \quad x, f \in \mathbb{R}^n, \quad u(x) \in \mathbb{R} \tag{1.11}$$

$$u(x) = \begin{cases} u^+ & \text{if } s(x) > 0 \\ u^- & \text{if } s(x) < 0 \end{cases} \tag{1.12}$$

where the component of vector f, scalar functions u^+, u^- and $s(x)$ are continuous and smooth, and $u^+(x) \neq u^-(x)$.

The regularization approach to find the solution of the discontinuous dynamic system has very important physical interpretation [123]. It is assumed that the sliding mode occurs on the sliding surface, $s(x) = 0$. The equation of motion is derived using the method of regularization. Let the discontinuous control be implemented with some imperfections of unexpected nature, control is assumed to take one of the two extreme values, u^+ or u^-, and the discontinuity points are isolated in time. Since discontinuity points are isolated in time, the solution exists in the conventional sense beyond the sliding surface. Further assume that the state velocity vector $f^+ = f_1 = f(x, u^+)$ and $f^- = f_2 = f(x, u^-)$ to be constant for some point x on the surface $s(x) = 0$ within a short interval $[t, t + \Delta t]$. Let the time interval Δt consists of two sets of intervals Δt_1 and Δt_2 such that $\Delta t = \Delta t_1 + \Delta t_2$, where Δt_1 and Δt_2 is the amount of that time when control of magnitude u^+ and u^- is active.

Mathematically, increment of the state vector in this interval Δt is given by,

$$\Delta x = f_1 \Delta t_1 + f_2 \Delta t_2 \tag{1.13}$$

and the average state velocity is given as,

$$\begin{aligned} \dot{x}_{\text{average}} &= \frac{\Delta x}{\Delta t} = \frac{f_1 \Delta t_1 + f_2 \Delta t_2}{\Delta t_1 + \Delta t_2} = f_1 \frac{\Delta t_1}{\Delta t_1 + \Delta t_2} + f_2 \frac{\Delta t_2}{\Delta t_1 + \Delta t_2} \\ &= \alpha f_1 + (1 - \alpha) f_2 \end{aligned} \tag{1.14}$$

where $\alpha = \frac{\Delta t_1}{\Delta t}$ is relative time for control to take value u^+ and $(1 - \alpha)$ to take value u^- and also $0 \leq \alpha < 1$.

To get the velocity vector $\dot{x}$ along the sliding surface we have to take limit $\Delta t \rightarrow 0$. Hence sliding motion is represented as,

$$\dot{x} = \alpha f_1 + (1 - \alpha) f_2 \tag{1.15}$$

One can also interpret Eqn.(1.15) as the velocity vector in the vicinity of a point on a discontinuous surface which is complemented by a minimal convex set, and the state velocity vector of the sliding motion belongs to this set. Because the state trajectories during sliding mode are in the sliding surface $s = 0$, the parameter α should be selected such that the state velocity vector of the system (1.15) is in the tangential plane to the sliding surface. Mathematically one can write

$$\dot{s} = \nabla[s(x)].\dot{x} = \nabla[s(x)](\alpha f_1 + (1 - \alpha) f_2) = 0 \tag{1.16}$$

where $\nabla[s(x)] = \left[\frac{\partial s}{\partial x_1} \cdots \frac{\partial s}{\partial x_n}\right]$

The solution of above equation is given by

$$\alpha = \frac{\nabla(s).f_2}{\nabla(s).(f_2 - f_1)} \tag{1.17}$$

Substituting the α from Eqn.(1.17) to (1.15), one can get motion in sliding mode as

$$\dot{x} = f_{\text{sliding}} = \frac{\nabla(s).f_2}{\nabla(s).(f_2 - f_1)} f_1 - \frac{\nabla(s).f_1}{\nabla(s).(f_2 - f_1)} f_2 \tag{1.18}$$

1.5 The Equivalent Control Method

Sliding mode occurs in the surface $s(x) = 0$, therefore, the function s and $\dot{s}$ have different signs in the vicinity of the surface and $\dot{s}^+ = \nabla(s).f_1 < 0$, $\dot{s}^- = \nabla(s).f_2 > 0$. Also one can easily check that $\dot{s} = \nabla(s) f_{\text{sliding}} = 0$ for the trajectories of system (1.18) and show that they are confined to the switching surface $s(x) = 0$.

The geometrical interpretation of the equivalent control method is described as follows. In sliding mode control, our main aim is to design a control law so that the state trajectories are confined to a sliding manifold in finite time. From a geometrical point of view, the equivalent control method does the same job. It replace the discontinuous control on the intersection of the switching surface by a continuous one such that, the state velocity vector lies in the tangential manifold. Mathematically, consider the system,

$$\dot{x} = f(x) + B(x)u \;\; x, \; f(x) \in \mathbb{R}^n, \;\; B(x) \in \mathbb{R}, u \in \mathbb{R}^n \tag{1.19}$$

$u(x)$ is defined as,

$$u(x) = \begin{cases} u^+ & \text{if } s(x) > 0 \\ u^- & \text{if } s(x) < 0 \end{cases} \tag{1.20}$$

So,

$$\dot{s} = \frac{\partial s}{\partial x}\dot{x} = G(x)f(x) + G(x)B(x)u_{\text{equivalent}} = 0 \tag{1.21}$$

where $G = \frac{\partial s}{\partial x}$. Assuming the matrix GB is nonsingular for any x, find the equivalent control $u_{\text{equivalent}}$ as the solution of the Eqn.(1.21).

$$u_{\text{equivalent}} = -G(x)B(x)^{-1}G(x)f(x) \tag{1.22}$$

and substituting $u_{\text{equivalent}}$ into (1.19) to yield the sliding mode equation $s = 0$ as,

$$\dot{x} = f(x) - G(x)B(x)^{-1}B(x)G(x)f(x) \tag{1.23}$$

The physical interpretation of the equivalent control method is as follows. For the occurrence of the ideal sliding mode it was assumed that the control changes at high (theoretically infinite) frequency such that the state vector is oriented precisely along the intersection of discontinuity surfaces. In reality however, various imperfections make the state oscillate in some vicinity of the intersection and control components are switched at finite frequency alternatively taking the positive and negative values. These oscillations have high frequency as well as slow components. All most all plants under control act as a low pass filter. Due to this low pass filter characteristic high frequency component is filtered out, and its motion in sliding mode is determined by the slow component. Practically it is reasonable to assume that the equivalent control is close to the slow component of the real control, which can be derived by filtering out the high-frequency components using a low-pass filter.

Mathematically, the output of a low-pass filter,

$$\tau\dot{z} + z = u \tag{1.24}$$

tends to the equivalent control

$$\lim_{\tau \to 0, \frac{\Delta}{\tau} \to 0} z = u_{\text{equivalent}} \tag{1.25}$$

where τ, Δ are the time constant of low pass filter and width of the manifold respectively.

To eliminate the high-frequency component of the control in sliding mode, the frequency should be much higher than $\frac{1}{\tau}$, or $\frac{1}{f} << \tau$, hence, $\Delta << \tau$. Finally, the time constant of the low-pass filter should be made to tend to zero because the filter should not distort the slow component of the control.

1.6 Motivation

Classical control design principles often require the information of the derivative of one function or the other. The function can be the reference function which is to be tracked of the any signal which is to be differentiated. Generally, integer-order derivative is used to solve these problems. However, this operator requires the function to be differentiable for its application. Physical systems possess a wide variety of signals. Not all of the associated functions are necessarily differentiable. In such a scenario, there is a strong need for a more generalized operator which relaxes the condition of differentiability and considers a large class of functions. Fractional-order derivatives can be opted to serve the purpose.

Finite-time tracking of reference functions in nonlinear systems has been one of the major control tasks required to be performed in various applications. Numerous important contributions can be found in the literature in the direction of achieving finite-time stability [183]- [185]. However, the derivative of the reference function is generally required in the implementation of the control law which is a serious limitation of the well-established control techniques in this context. The tracking problem can be approached in a more generalized way by allowing the order of the derivative to take fractional values. This requires the tracking problem to be explored at the fundamental level with a sound mathematical framework.

Sliding mode control is often utilized in achieving the desired tracking objective in finite time. It also has the capability of providing disturbance rejection and robustness to parametric uncertainties. However, the classical sliding mode scheme assumes the reference function to be differentiable with a bound for the convergence of the error dynamics. So, the conventional integer-order framework restricts the allowable class of reference functions that can be tracked.

In this context, fractional-order operators can be utilized so that the reference functions which do not possess the first-order derivative and satisfy the Hölder condition [1] can also be tracked using limited control action. So, a larger class of reference functions satisfying a certain condition can be addressed using the technique proposed in this chapter.

The study of fractional-order derivatives and integrals is done in a branch of mathematics known as *Fractional Calculus* [1] [6] [7]. It has been observed that physical systems can be more accurately represented by their fractional-order models. The nature of some phenomena can inherently be described only by considering them as fractional-order systems [6]. This makes it obvious to explore the tracking problems in a new dimension.

1.7 Organization of the Thesis

The entire thesis is divided into five chapters. Chapter 1. starts with a discussion on the motivation behind the work. Then, the main contribution is briefed. The related work in the literature has been discussed in the next section. Then, a brief outline of fractional calculus is given including the definitions used and the physical interpretation of the fractional-order operators. The properties of these operators have been mentioned which are further used in the entire text.

Then, it comes to the main problem in Chapter 2. The tracking problem is introduced for a general class of nonlinear systems followed by the consideration of a switch-controlled RL circuit as a case study. The required condition on the reference function for the desired trajectory tracking is derived for constant and smooth reference functions. The problem with the integer-order approach in tracking non-differentiable reference functions is highlighted. An approach based on fractional-order operators is proposed resulting into a relaxed condition on the reference function which is derived. This technique uses limited control action. Simulation results have been given for the switch-controlled RL circuit for tracking a sawtooth reference function.

Continuing in the same direction, Chapter 3. highlights the non-differentiability issue in the classical super-twisting algorithm based differentiator. The required condition on the second-order derivative of the signal is discussed. The constraint is relaxed by using fractional-order dynamics in the estimation part resulting into fractional-order differentiator. Further, discretization of the equations has been done with implicit Euler method for chattering suppression. The technique has been tested through simulation on a signal which is not second-order differentiable.

Chapter 4. considers the aspect of uncertainty in the design of control laws. It highlights on the robustness aspects of the design. Classical sliding mode based design lacks robustness in the reaching phase of design and can only guarantee it when the states are on the sliding surface. In order to guarantee robustness in the entire evolution of the states, integral sliding mode control has been proposed in the literature but it lacks the advantage of the reduced-order design technique. In order to guarantee robustness in the entire state-space and also to preserve the reduced-order design methodology, a reaching phase-free approach is proposed in this chapter for uncertain fractional-order systems. Further, its application in the case of a fractional inverted pendulum is shown through simulation.

Finally, Chapter 5. states the overall conclusions of the work with future perspectives.

1.8 Summary

The chapter gives an overview of the non-differentiability and uncertainty aspects of general control problems for nonlinear systems. Some related constraints and various approaches available are discussed. A brief introduction about fractional calculus is given with various definitions which are widely used. Physical interpretation of the definitions has been provided for visualization of the presented approaches.