CHAPTER 7
CONTROL SYSTEM DESIGN USING STABILITY BOUNDARY LOCUS

In this chapter, we have discussed briefly about interval plant stabilization. The significant results about stabilization of interval systems are available in the literature. In [216], it was shown that a constant gain controller stabilizes an interval plant family if and only if it stabilizes a set of eight of the extreme plants. In [217], it was shown that a first order controller stabilizes an interval plant if it stabilizes the set of extreme plants. The best results regarding this subject were given in [218]- [219] where it was proved that a first order controller stabilizes an interval plant if and only if it simultaneously stabilizes the sixteen Kharitonov plants family. In [220], the generalized version of the Hermite-Biehler theorem has been used for the stabilization of interval systems. In this section, instead of using Routh tables, which are used in [218] in order to characterize all the parameters of a first-order controller which stabilize an interval plant, the stability boundary locus is used to find all the values of the parameters of a PI controller for which the given interval plant is Hurwitz stable.

7.1 INTERVAL PLANT STABILIZATION USING PI CONTROLLER

Consider an interval plant

$$G_n(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots\cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots\cdots + a_0} \tag{7.1}$$

where $b_i \in \left[b_i^-, b_i^+\right], i = 0,1,2,\cdots\cdots m$ and $a_j \in \left[a_j^-, a_j^+\right], j = 0,1,2,\cdots\cdots n$. Let the Kharitonov polynomials associated with $N(s)$ and $D(s)$ be respectively:

$$\begin{aligned}
N_1(s) &= b_0^- + b_1^- s + b_2^+ s^2 + b_3^+ s^3 + \cdots\cdots \\
N_2(s) &= b_0^- + b_1^+ s + b_2^+ s^2 + b_3^- s^3 + \cdots\cdots \\
N_3(s) &= b_0^+ + b_1^- s + b_2^- s^2 + b_3^+ s^3 + \cdots\cdots \\
N_4(s) &= b_0^+ + b_1^+ s + b_2^- s^2 + b_3^- s^3 + \cdots\cdots
\end{aligned} \tag{7.2}$$

and

$$
\begin{aligned}
D_1(s) &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + \cdots\cdots \\
D_2(s) &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + \cdots\cdots \\
D_3(s) &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + \cdots\cdots \\
D_4(s) &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + \cdots\cdots
\end{aligned} \tag{7.3}
$$

By taking all combinations of the $N_i(s)$ and $D_j(s)$ for $i, j = 1, 2, 3, 4$ the following sixteen Kharitonov plant family can be obtained

$$G_K(s) = G_{ij}(s) = \frac{N_i(s)}{D_j(s)} \tag{7.4}$$

Consider the single-input single output (SISO) control system of Fig. 7.1 where $G_n(s)$ is the plant to be controlled and $C(s)$ is a PI controller of the form

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s} \tag{7.5}$$

The problem is to compute the parameters of the PI controller of Eq. (7.5) which stabilize the system of Fig. 7.1.

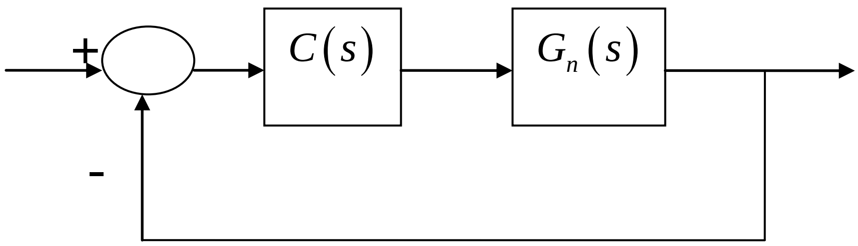

Fig.7.1. A SISO control system

Decomposing the numerator and the denominator polynomials of Eq. (7.1) into their even and odd parts, and substituting $s = j\omega$ gives

$$G_n(j\omega) = \frac{N_e(-\omega^2) + j\omega N_o(-\omega^2)}{D_e(-\omega^2) + j\omega D_o(-\omega^2)} \tag{7.6}$$

The closed loop characteristic polynomial of the system can be written as

$$\begin{aligned}\Delta(s) &= \left[k_i N_e\left(-\omega^2\right) - k_p\omega^2 N_o\left(-\omega^2\right) - \omega^2 D_o\left(-\omega^2\right)\right] \\ &+ j\left[k_p\omega N_e\left(-\omega^2\right) - k_i\omega N_o\left(-\omega^2\right) + \omega D_e\left(-\omega^2\right)\right] = 0\end{aligned} \tag{7.7}$$

Then, equating the real and imaginary parts of $\Delta(s)$ to zero, one obtains

$$k_p\left(-\omega^2 N_o\left(-\omega^2\right)\right) + k_i N_e\left(-\omega^2\right) = \omega^2 D_o\left(-\omega^2\right) \tag{7.8}$$

and

$$k_p\left(N_e\left(-\omega^2\right)\right) + k_i\left(N_0\left(-\omega^2\right)\right) = -D_e\left(-\omega^2\right) \tag{7.9}$$

Let

$$\begin{aligned}Q(\omega) &= -\omega^2 N_o\left(-\omega^2\right) \\ R(\omega) &= N_e\left(-\omega^2\right) \\ S(\omega) &= N_e\left(-\omega^2\right) \\ U(\omega) &= N_o\left(-\omega^2\right) \\ X(\omega) &= \omega^2 D_o\left(-\omega^2\right) \\ Y(\omega) &= -D_e\left(-\omega^2\right)\end{aligned} \tag{7.10}$$

Then, Eq. (7.8) and Eq. (7.9) can be written as

$$\begin{aligned}k_p Q(\omega) + k_i R(\omega) &= X(\omega) \\ k_p S(\omega) + k_i U(\omega) &= Y(\omega)\end{aligned} \tag{7.11}$$

From this equation

$$k_p = \frac{X(\omega)U(\omega) - Y(\omega)R(\omega)}{Q(\omega)U(\omega) - R(\omega)S(\omega)} \tag{7.12}$$

and

$$k_i = \frac{Y(\omega)Q(\omega) - X(\omega)S(\omega)}{Q(\omega)U(\omega) - R(\omega)S(\omega)} \tag{7.13}$$

Solving these two equations simultaneously, the stability boundary locus, $l(k_p, k_i, \omega)$ in (k_p, k_i) -plane can be obtained. The stability boundary locus divides the parameter plane into stable and unstable regions. Choosing a test point within each region, the stable region which contains the values of stabilizing k_p and k_i parameters can be determined.

Define the set $S(C(s)G_n(s))$ which contains all the values of the parameters of the controller $C(s)$ which stabilize $G_n(s)$, then the set of all the stabilizing values of parameters of a PI controller which stabilize the interval plant of Eq. (7.1) can be written as

$$\begin{aligned} &S(C(s)G(s)) = S(C(s)G_K(s)) = \\ &S(C(s)G_{11}(s)) \cap S(C(s)G_{12}(s)) \cap \cdots\cdots \cap S(C(s)G_{44}(s)) \end{aligned} \tag{7.14}$$

where $G_K(s)$ represents the sixteen Kharitonov plant family which is given in Eq. (7.4).

7.2 ILLUSTRATIVE EXAMPLE

Example 7.1: Consider the control system of Fig.1 with an interval transfer function

$$G(s) = \frac{K}{s^4 + a_2 s^3 + a_1 s^2 + a_0 s} \tag{7.15}$$

where $K \in [10, 30], a_2 \in [85, 95], a_1 \in [1900, 2000]$ and $a_0 \in [3450, 3750]$. The objective is to calculate all the parameters of a PI controller which stabilize $G(s)$.

Consider the first Kharitonov plant ($i = 1$ and $j = 1$) which is

$$G_{11}(s) = \frac{10}{s^4 + 95s^3 + 2000s^2 + 3450s} \quad (7.16)$$

Since $\operatorname{Im}\left[G_{11}(j\omega)\right] = 0$ is only satisfied for $\omega = 6.0263 rad/\sec$, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.0263)$. Then, from Eqs. (7.12) and (7.13)

$$k_p = -0.1\omega^4 + 200\omega^2 \quad (7.17)$$

and

$$k_i = -9.5\omega^4 + 345\omega^2 \quad (7.18)$$

All stabilizing values of k_p and k_i are shown in Fig. 7.2.

$$G_{12}(s) = \frac{10}{s^4 + 85s^3 + 1900s^2 + 3750s} \quad (7.19)$$

Since $\operatorname{Im}\left[G_{12}(j\omega)\right] = 0$ is only satisfied for $\omega = 6.6421 rad/\sec$, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.6421)$. Then, from Eqs. (7.12) and (7.13)

$$k_p = -0.1\omega^4 + 190\omega^2 \quad (7.20)$$

and

$$k_i = -8.5\omega^4 + 375\omega^2 \quad (7.21)$$

$$G_{13}(s) = \frac{10}{s^4 + 85s^3 + 2000s^2 + 3750s} \quad (7.22)$$

Since $\operatorname{Im}\left[G_{13}(j\omega)\right] = 0$ is only satisfied for $\omega = 6.6421 rad/\sec$, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.6421)$. Then, from Eqs. (7.12) and (7.13)

$$k_p = -0.1\omega^4 + 120\omega^2 \quad (7.23)$$

and

$$k_i = -8.5\omega^4 + 375\omega^2 \tag{7.24}$$

$$G_{14}(s) = \frac{10}{s^4 + 95s^3 + 1900s^2 + 3450s} \tag{7.25}$$

Since $\mathrm{Im}\left[G_{14}(j\omega)\right] = 0$ is only satisfied for $\omega = 6.0263 rad/\sec$, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.0263)$. Then, from Eqs. (7.12) and (7.13)

$$k_p = -0.1\omega^4 + 190\omega^2 \tag{7.26}$$

and

$$k_i = -9.5\omega^4 + 345\omega^2 \tag{7.27}$$

$$G_{21}(s) = \frac{30}{s^4 + 95s^3 + 2000s^2 + 3450s} \tag{7.28}$$

Since $\mathrm{Im}\left[G_{21}(j\omega)\right] = 0$ is only satisfied for $\omega = 6.0263 rad/\sec$, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.0263)$. Then, from Eqs. (7.12) and (7.13)

$$k_p = -0.333\omega^4 + 66.6667\omega^2 \tag{7.29}$$

and

$$k_i = -3.1667\omega^4 + 115\omega^2 \tag{7.30}$$

$$G_{22}(s) = \frac{30}{s^4 + 85s^3 + 1900s^2 + 3750s} \tag{7.31}$$

Since $\mathrm{Im}\left[G_{22}(j\omega)\right] = 0$ is only satisfied for $\omega = 6.6421 rad/\sec$, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.6421)$. Then, from Eqs. (7.12) and (7.13)

$$k_p = -0.333\omega^4 + 63.333\omega^2 \tag{7.32}$$

and

$$k_i = -2.8334\omega^4 + 125\omega^2 \tag{7.33}$$

$$G_{23}(s) = \frac{30}{s^4 + 85s^3 + 2000s^2 + 3750s} \tag{7.34}$$

Since $\text{Im}\left[G_{23}(j\omega)\right] = 0$ is only satisfied for $\omega = 6.6421 rad/\sec$, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.6421)$. Then, from Eqs. (7.12) and (7.13)

$$k_p = -0.333\omega^4 + 66.6667\omega^2 \tag{7.35}$$

and

$$k_i = -2.8334\omega^4 + 125\omega^2 \tag{7.36}$$

$$G_{24}(s) = \frac{30}{s^4 + 95s^3 + 1900s^2 + 3450s} \tag{7.37}$$

Since $\text{Im}\left[G_{24}(j\omega)\right] = 0$ is only satisfied for $\omega = 6.0262 rad/\sec$, it is necessary to obtain stability boundary locus for $\omega \in (0, 6.0262)$. Then, from Eqs. (7.12) and (7.13)

$$k_p = -0.333\omega^4 + 63.333\omega^2 \tag{7.38}$$

and

$$k_i = -3.1667\omega^4 + 115\omega^2 \tag{7.39}$$

Fig. 7.3 shows the stability regions of the eight Kharitonov plants (the interval plant has eight Kharitonov plants since there are only two Kharitonov polynomials for the numerator) where the intersection of these regions, which can be obtained from the stability region $G_{21}(s)$ and $G_{24}(s)$ as shown in Fig. 7.3 and the stability region which is shown in Fig. 7.4.

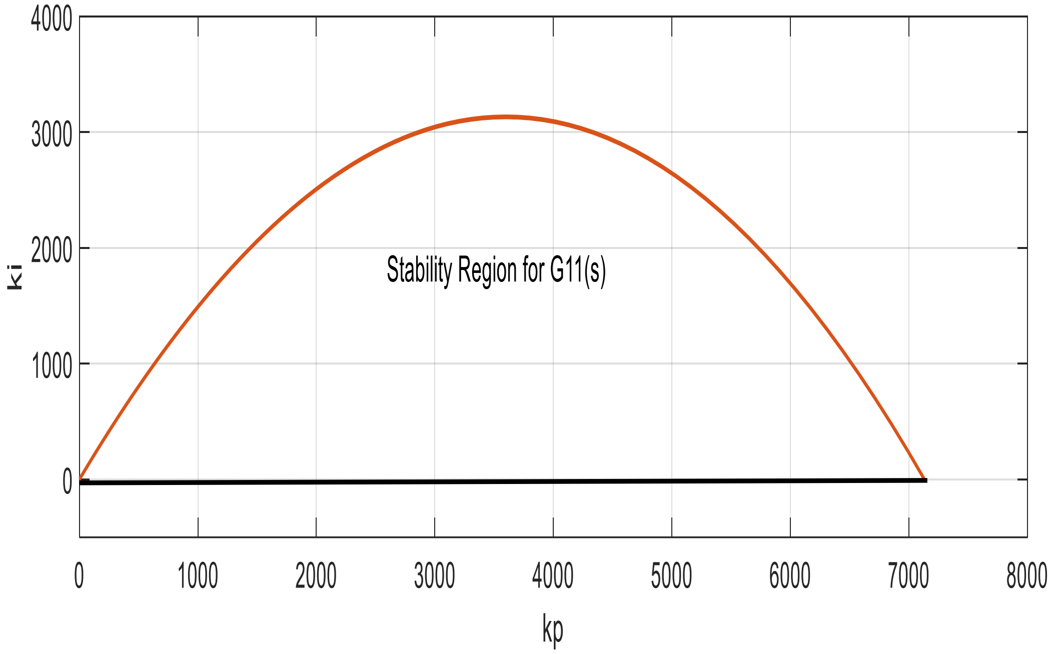

Fig. 7.2. Stability region for $G_{11}(s)$

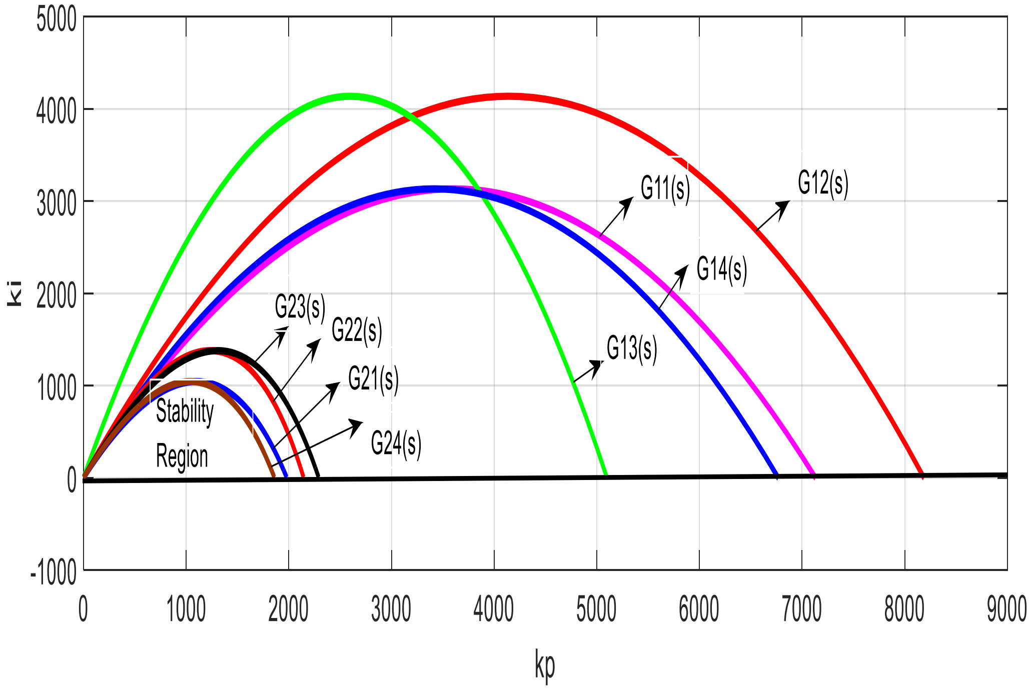

Fig. 7.3. Stability region for eight Kharitonov plants

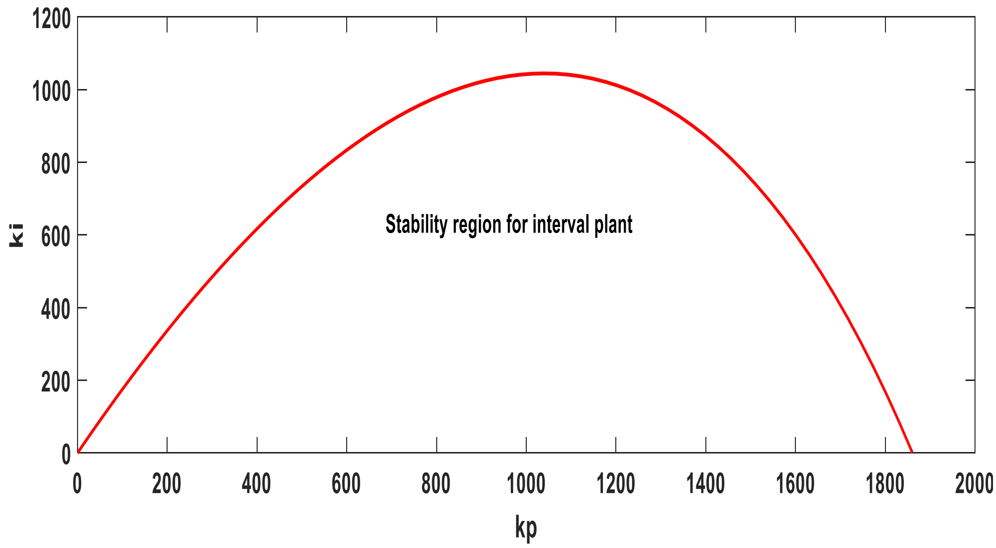

Fig 7.4 Stability region for interval plant $G(s)$

This approach is existing for the computation of the boundaries of the limiting values of PI controllers parameters that guarantee stability. The method is constructed on the stability boundary locus which can be easily obtained by equating the real and the imaginary parts of the characteristic to zero.