CHAPTER 6
PROPOSED TECHNIQUES FOR THE REDUCTION OF DISCRETE TIME INTERVAL SYSTEMS

6.1 INTRODUCTION

For discrete time interval systems also the same arguments as for the continuous interval systems hold as far as the need for reduced order modeling is concerned. Moreover the fast development and usage of small digital computers and the processors in the design and implementation of control systems have increased the importance of reduced order modeling methods for discrete systems. In this chapter some of the techniques developed in previous sections (4.4.1 & 5.3) for continuous interval systems are extended to the discrete time case.

It is shown in [143] that the bilinear transformation can be used to extended Routh approximation, Hurwitz polynomial approximation, stability equation and retaining dominant poles to reduced z transfer functions in the $w-$ domain. The major drawback that due to the nature of bilinear transformation, the initial value of the step response of reduced order models may not be zero even though the initial value of the step response of the original system is zero. This draw back has been removed in the proposed new methods by using linear transformation $z = (w+1)$.

6.2 PRELIMINARIES

Let the transfer function of a higher order discrete time interval systems be

$$G_n(z) = \frac{[b_0^-, b_0^+] + [b_1^-, b_1^+]z + \ldots.. + [b_{n-1}^-, b_{n-1}^+]z^{n-1}}{[a_0^-, a_0^+] + [a_1^-, a_1^+]z + \ldots.. + [a_n^-, a_n^+]z^n} = \frac{N(z)}{D(z)} \tag{6.1}$$

The k^{th} order reduced model of $G_n(z)$ is expressed as

$$R_k(z) = \frac{\left[d_0^-, d_0^+\right] + \left[d_1^-, d_1^+\right]z + \ldots. + \left[d_{k-1}^-, d_{k-1}^+\right]z^{k-1}}{\left[c_0^-, c_0^+\right] + \left[c_1^-, c_1^+\right]z + \ldots. + \left[c_k^-, c_k^+\right]z^k} = \frac{N_k(z)}{D_k(z)} \tag{6.2}$$

6.3 ERROR INDEX

The error index $'J'$ is specified by the following algorithm

$$J = \sum_{p=0}^{M} \left[y\left(t_p\right) - y_r\left(t_p\right) \right]^2 \qquad (6.3)$$

where y and y_r are the outputs of the original interval system $G_n\left(z\right)$ and the reduced order interval system $R_k\left(z\right)$ respectively at sampling instants t_p, M is the number of sampling periods.

6.4 MODIFIED DIFFERENTIATION METHOD FOR DISCRETE TIME INTERVAL SYSTEMS.

Algorithm to obtain reduced order discrete interval models

Case1: Modified differentiation method

Step 1: The first row formed from the original denominator coefficients of $G_n(z)$ (higher order coefficients).

Step 2: The second row is obtained by differentiation of row 1.

Step 3: The third row is obtained by applying modified Routh approximation. This process will give reduced order denominator of order n-1.

Step 4: The fourth row is obtained by differentiation of row 3.

Step 5: The fifth row can be obtained by modified Routh approximation using row 3 and row 4. This will give reduced denominator of order n-2 and so on

Case2: Appling linear transformation

Step 1: Apply linear transformation $\left(z = w+1\right)$ to the higher order system $G_n\left(z\right)$, then, the first row as is obtained as $G_n(w+1)$ and then denominator coefficients can be formed in terms of w, then the procedure to obtain reduced interval models are same as shown in Table 2.

Step 2: The second row is obtained by differentiation of row 1.

Step 3: The third row is obtained by applying modified Routh approximation. This process will give reduced order denominator of order $n-1$.

Step 4: The fourth row is obtained by differentiation of row 3.

Step 5: The fifth row can be obtained by modified Routh approximation using row 3 and row 4. This will give reduced denominator of order $n-2$ and so on.

Step 6: Substitute $(w = z-1)$ in the reduced order model.

For better understanding the above algorithm, Fig 6.1 is useful.

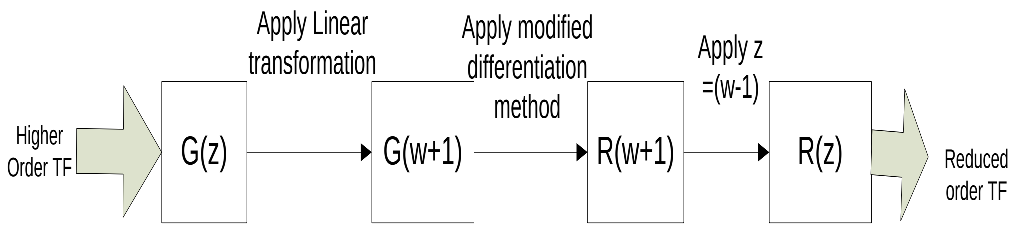

Fig. 6.1. Block diagram of modified differentiation method.

The linear transformation $(z = w+1)$ has the consequence of shifting all the poles and zeros of $G_n(s)$ by a distance -1 unit in the complex plane. Thus for a stable or minimum phase system $G_n(z)$ the poles and/or zeros of the corresponding $G_n(w)$ need not lie within the unit circle.

Case3: Appling bilinear transformation

Step 1: Apply bilinear transformation $z=\left(\frac{1+w}{1-w}\right)$,where $w\neq 1$ to the higher order system $G_n(z)$, then, the first row as is obtained as $G_n\left(\frac{1+w}{1-w}\right)$, where $w\neq 1$ and then denominator coefficients can be formed in terms of w, then the procedure to obtain reduced interval models are same as shown in Table 2.

$$G_n\left(\frac{1+w}{1-w}\right)=\frac{[b_0^-,b_0^+]+[b_1^-,b_1^+]\left(\frac{1+w}{1-w}\right)+.....+[b_{n-1}^-,b_{n-1}^+]\left(\frac{1+w}{1-w}\right)^{n-1}}{[a_0^-,a_0^+]+[a_1^-,a_1^+]\left(\frac{1+w}{1-w}\right)+.....+[a_n^-,a_n^+]\left(\frac{1+w}{1-w}\right)^{n}}$$

Step 2: The second row is obtained by differentiation of row 1.

Step 3: The third row is obtained by applying modified Routh approximation. This process will give reduced order denominator of order $n-1$.

Step 4: The fourth row is obtained by differentiation of row 3.

Step 5: The fifth row can be obtained by modified Routh approximation using row 3 and row 4. This will give reduced denominator of order $n-2$ and so on.

Step 6: Substitute $w=\left(\frac{z-1}{z+1}\right)$ in the reduced order model

6.5 $\alpha-$ TRUNCATION BASED METHOD USING LINEAR TRANSFORMATION

6.5.1 $\alpha-\beta$ TRUNCATION METHOD

Denominator of the reduced order polynomial is reduced by alpha truncation method

Step 1: Reciprocal of higher order interval denominator polynomial

$$\hat{D}(z) = \left[a_0^-, a_0^+\right](z)^n + \left[a_1^-, a_1^+\right](z)^{n-1} + ... + \left[a_n^-, a_n^+\right] \tag{6.4}$$

Step 2: Apply linear transformation $(z = w+1)$

$$\hat{D}(w+1) = \left[a_0^-, a_0^+\right](w+1)^n + \left[a_1^-, a_1^+\right](w+1)^{n-1} + ... + \left[a_n^-, a_n^+\right] \tag{6.5}$$

Step 3: Contraction of $\alpha-$ table

Table 6.1: Construction of $\alpha-$ table

	$a_0^0 = \left[a_0^-, a_0^+\right]$ $a_0^1 = \left[a_1^-, a_1^+\right]$	$a_2^0 = \left[a_2^-, a_2^+\right]$ $a_2^1 = \left[a_3^-, a_3^+\right]$	....	....
$\left[\alpha_1^-, \alpha_1^+\right] = \frac{a_0^0}{a_0^1}$	$a_0^2 = a_2^0 - \left[\alpha_1^-, \alpha_1^+\right] a_2^1$	$a_2^2 = a_4^0 - \left[\alpha_1^-, \alpha_1^+\right] a_4^1$	....	....
$\left[\alpha_2^-, \alpha_2^+\right] = \frac{a_0^1}{a_0^2}$	$a_0^3 = a_2^1 - \left[\alpha_2^-, \alpha_2^+\right] a_2^2$	....		
$\left[\alpha_3^-, \alpha_3^+\right] = \frac{a_0^2}{a_0^3}$	....			
....	....			

From the Table 6.2, we can obtain reduced order polynomial

$$
\left.\begin{aligned}
&\hat{D}_1(w+1) = \alpha_1 w + 1 \\
&\hat{D}_2(w+1) = \alpha_1\alpha_2 w^2 + \alpha_2 w + 1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\hat{D}_k(w+1) = \alpha_1 \hat{D}_{k-1}(w+1) + \hat{D}_{k-2}(w+1) \\
&where \\
&\hat{D}_{-1}(w+1) = 0; \hat{D}_0(w+1) = 0
\end{aligned}\right\} \quad (6.6)
$$

The reduced order polynomial depends on the reciprocal of $\hat{D}_k(w+1)$

Step 4: Reciprocal transformation of $\hat{D}_k(w+1)$

Step 5: Substitute $(w = z - 1)$ in step 4.

Numerator polynomial is reduced by using $\beta -$ truncation.

Step6: Reciprocal of higher order numerator

$$\hat{N}(z) = \left[b_0^-, b_0^+\right](z)^{n-1} + \left[b_1^-, b_1^+\right](z)^{n-2} + \ldots. + \left[b_{n-1}^-, b_{n-1}^+\right] \quad (6.7)$$

Step 7: Apply linear transformation

$$\hat{N}(w+1) = \left[b_0^-, b_0^+\right](w+1)^{n-1} + \left[b_1^-, b_1^+\right](w+1)^{n-2} + \ldots. + \left[b_{n-1}^-, b_{n-1}^+\right] \quad (6.8)$$

Step 8: Contraction of $\beta -$ table

Table 6.2: Construction of $\beta -$ table

	$b_0^1 = \left[b_0^-, b_0^+\right]$ $b_0^2 = \left[b_1^-, b_1^+\right]$	$b_2^1 = \left[b_2^-, b_2^+\right]$ $b_2^2 = \left[b_3^-, b_3^+\right]$	....	....
$\left[\beta_1^-, \beta_1^+\right] = \dfrac{b_0^1}{a_0^1}$	$b_0^3 = b_2^1 - \left[\beta_1^-, \beta_1^+\right] a_2^1$	$b_2^3 = b_4^1 - \left[\beta_1^-, \beta_1^+\right] a_4^1$	....	....

$\left[\beta_2^-, \beta_2^+\right] = \frac{b_0^2}{a_0^2}$	$b_0^4 = b_2^2 - \left[\beta_2^-, \beta_2^+\right] a_2^2$	....		
$\left[\beta_3^-, \beta_3^+\right] = \frac{b_0^3}{a_0^3}$	....			
....	....			

Let $\hat{N}_k\left(w+1\right)$ denote the numerator of the k^{th} Routh convergent respectively

$$\left.\begin{aligned}
&\hat{N}_1\left(w+1\right) = \beta_1 \\
&\hat{N}_2\left(w+1\right) = \alpha_1\beta_1 w + \beta_2 \\
&.................... \\
&\hat{N}_k\left(w+1\right) = \alpha_k w \hat{N}_{k-1}\left(w+1\right) + \hat{N}_{k-2}\left(w+1\right) + \beta_k \\
&where \\
&\hat{N}_{-1}\left(w+1\right) = 0; N_0\left(w+1\right) = 0
\end{aligned}\right\} \qquad (6.9)$$

The reduced order depends up on the order of the system reciprocal of $\hat{N}_k\left(w+1\right)$

Step 9: Reciprocal transformation of step 8.

Step 10: Substitute $\left(w = z - 1\right)$

6.5.2 $\alpha -$ TRUNCATION AND FACTOR DIVISION METHOD

The numerator polynomial is reduced by using factor division method

$$N_k(z)=\frac{N(z)D_k(z)}{D(z)} \tag{6.10}$$

$$N(z)D_k(z)=\left[u_{11}^-,u_{11}^+\right]+\left[u_{12}^-,u_{12}^+\right]z+.....+\left[u_{1,k-1}^-,u_{1,k-1}^+\right]z^{k-1} \tag{6.11}$$

$$\frac{N(z)D_k(z)}{D(z)}=\frac{\left[u_{11}^-,u_{11}^+\right]+\left[u_{12}^-,u_{12}^+\right]z+.....+\left[u_{1,k-1}^-,u_{1,k-1}^+\right]z^{k-1}}{\left[a_0^-,a_0^+\right]+\left[a_1^-,a_1^+\right]z+.....+\left[a_n^-,a_n^+\right]z^{k-1}} \tag{6.12}$$

For mathematical simplification assume

$$\left[a_0^-,a_0^+\right]=\left[c_{11}^-,c_{11}^+\right];\left[a_1^-,a_1^+\right]=\left[c_{12}^-,c_{12}^+\right];........;\left[a_n^-,a_n^+\right]=\left[c_{1,n}^-,c_{1,n}^+\right] \tag{6.13}$$

$$\frac{N(z)D_k(z)}{D(z)}=\frac{\left[u_{11}^-,u_{11}^+\right]+\left[u_{12}^-,u_{12}^+\right]z+.....+\left[u_{1,k-1}^-,u_{1,k-1}^+\right]z^{k-1}}{\left[c_{11}^-,c_{11}^+\right]+\left[c_{12}^-,c_{12}^+\right]z+.....+\left[c_{1,n}^-,c_{1,n}^+\right]z^{k-1}} \tag{6.14}$$

Therefore

$$\left[\alpha_{11}^-,\alpha_{11}^+\right]=\frac{\left[u_{11}^-,u_{11}^+\right]}{\left[c_{11}^-,c_{11}^+\right]}\begin{Bmatrix}\left[u_{11}^-,u_{11}^+\right] & \left[u_{12}^-,u_{12}^+\right] & \left[u_{13}^-,u_{13}^+\right] & ..... \\ \left[c_{11}^-,c_{11}^+\right] & \left[c_{12}^-,c_{12}^+\right] & \left[c_{13}^-,c_{13}^+\right] & .....\end{Bmatrix}$$

$$\left[\alpha_{12}^-,\alpha_{12}^+\right]=\frac{\left[r_{11}^-,r_{11}^+\right]}{\left[c_{11}^-,c_{11}^+\right]}\begin{Bmatrix}\left[r_{11}^-,r_{11}^+\right] & \left[r_{12}^-,r_{12}^+\right] & \left[r_{13}^-,r_{13}^+\right] & ..... \\ \left[c_{11}^-,c_{11}^+\right] & \left[c_{12}^-,c_{12}^+\right] & \left[c_{13}^-,c_{13}^+\right] & .....\end{Bmatrix} \tag{6.15}$$

$$\left[\alpha_{13}^-,\alpha_{13}^+\right]=\frac{\left[s_{11}^-,s_{11}^+\right]}{\left[c_{11}^-,c_{11}^+\right]}\begin{Bmatrix}\left[s_{11}^-,s_{11}^+\right] & \left[s_{12}^-,s_{12}^+\right] & \left[s_{13}^-,s_{13}^+\right] & ..... \\ \left[c_{11}^-,c_{11}^+\right] & \left[c_{12}^-,c_{12}^+\right] & \left[c_{13}^-,c_{13}^+\right] & .....\end{Bmatrix}$$

....

$$\left[\alpha_{1,k-2}^-,\alpha_{1,k-2}^+\right]=\frac{\left[x_{11}^-,x_{11}^+\right]}{\left[c_{11}^-,c_{11}^+\right]}\begin{Bmatrix}\left[x_{11}^-,x_{11}^+\right] & \left[x_{12}^-,x_{12}^+\right] \\ \left[c_{11}^-,c_{11}^+\right] & \left[c_{12}^-,c_{12}^+\right]\end{Bmatrix}$$

$$\left[\alpha_{1,k-1}^-, \alpha_{1,k-1}^+\right] = \frac{\left[y_{11}^-, y_{11}^+\right]}{\left[c_{11}^-, c_{11}^+\right]} \left\{ \frac{\left[y_{11}^-, y_{11}^+\right]}{\left[c_{11}^-, c_{11}^+\right]} \right\}$$

where

$$\left[r_{1,i}^-, r_{1,i}^+\right] = \left[u_{1,i+1}^-, u_{1,i+1}^+\right] - \left[\alpha_{11}^-, \alpha_{11}^+\right]\left[c_{1,i+1}^-, c_{1,i+1}^+\right] \tag{6.16}$$

$$\left[s_{1,i}^-, s_{1,i}^+\right] = \left[r_{1,i+1}^-, r_{1,i+1}^+\right] - \left[\alpha_{12}^-, \alpha_{12}^+\right]\left[c_{1,i+1}^-, c_{1,i+1}^+\right]$$

where, $i = 0,1,2,....,k-2$

....

$$\left[y_{1,i}^-, y_{1,i}^+\right] = \left[x_{1,i+1}^-, x_{1,i+1}^+\right] - \left[\alpha_{1,k-2}^-, \alpha_{1,k-2}^+\right]\left[c_{1,1}^-, c_{1,1}^+\right]$$

The reduced transfer function is

$$R_k(z) = \frac{\left[\alpha_{11}^-, \alpha_{11}^+\right] + \left[\alpha_{12}^-, \alpha_{12}^+\right]z + ... + \left[\alpha_{1,k-1}^-, \alpha_{1,k-1}^+\right]z^{k-1}}{D_k(z)} \tag{6.17}$$

6.6 ILLUSTRATIVE EXAMPLE

Example 1: Consider a third order system describe by the transfer function

$$G_3(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{6.18}$$

Case 1: Modified differentiation method

$$R_2(z) = \frac{[1,2.5]z + [8,10]}{[2.6678,3.5006]z^2 + [3.2335,3.3668]z + [0.8,0.85]} \tag{6.19}$$

Case 2: Modified differentiation method by using linear transformation

$$R_2(z) = \frac{[1,5.5]z + [6.5,15]}{[8.685,9.518]z^2 + [7.878,11.01]z + [1.005,3.954]} \tag{6.20}$$

Case 3: Modified differentiation method by using bilinear transformation

$$R_2(z)=\frac{[-11.9991,16.0001]z^2+[-4.0024,30.6632]+[26.6695,54.6687]}{[34.8844,37.9344]z^2+[35.9658,40.0658]z+[8.4498,11.4998]} \tag{6.21}$$

By using bilinear transformation the obtained reduced order model is an non-minimum phase system. The impulse response of the lower and upper bond of high-order system and reduced order system obtained by using the propose method are shown in Fig. 6.2 and Fig. 6.3. The step response of the lower and upper bond of high-order system and reduced order system obtained by using the propose method are shown in Fig. 6.4 and Fig. 6.5. In Table 6.1, the comparison of error index has been verified and compared with the existing techniques. The important limitation of Kharitonov's theorem is that it cannot be applied directly to discrete time interval polynomials. To overcome this limitation, bilinear transformation is used in the Kharitonov's theorem for studying the stability of the for discrete time interval systems (Mastorakis [204]).

Table 6.3: Comparison of Error index

S. No	Methods	M	"J" Error Index	
			Lower Limit	Upper Limit
1	Proposed method (MDM)	2	107.2958	134.1428
2	Proposed method (MDM by using linear transformation)	2	1.3428	7.4154
3	Proposed method (MDM by using bilinear transformation)	2	12.0829	26.5229

4	Proposed method ($\alpha-\beta$ truncation method)	2	1.30965	0.5976
5	Proposed method ($\alpha-$ truncation and factor division method)	2	8.6074×10^{3}	2.8088×10^{4}
6	Ismail et. al., [178]	2	9.9116	2.4058
7	Singh and Chandra [183]	2	3.4249	0.7721

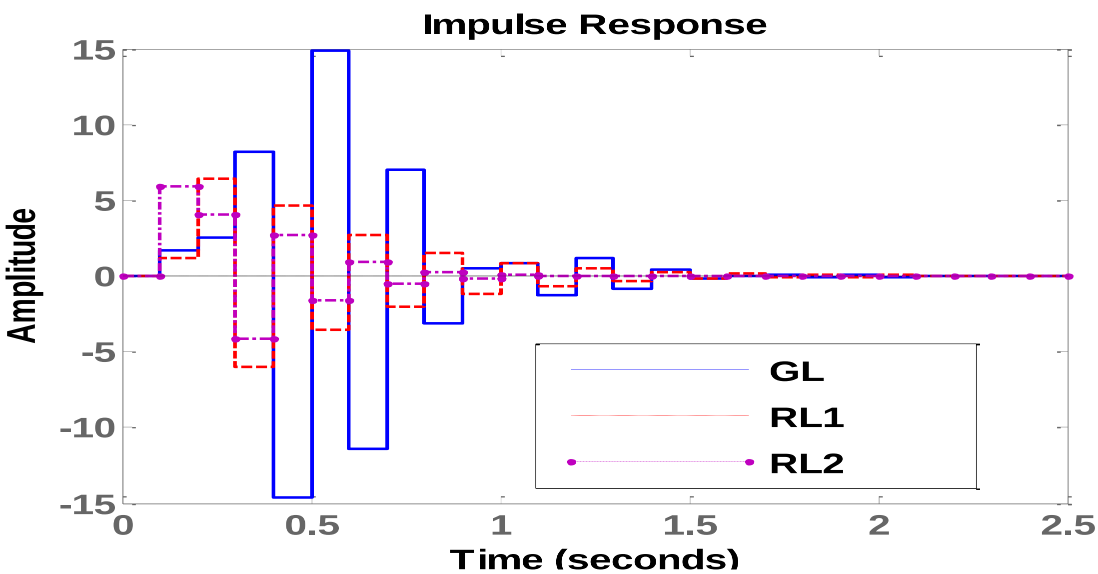

Fig. 6.2. Comparison of impulse response (lower limit)

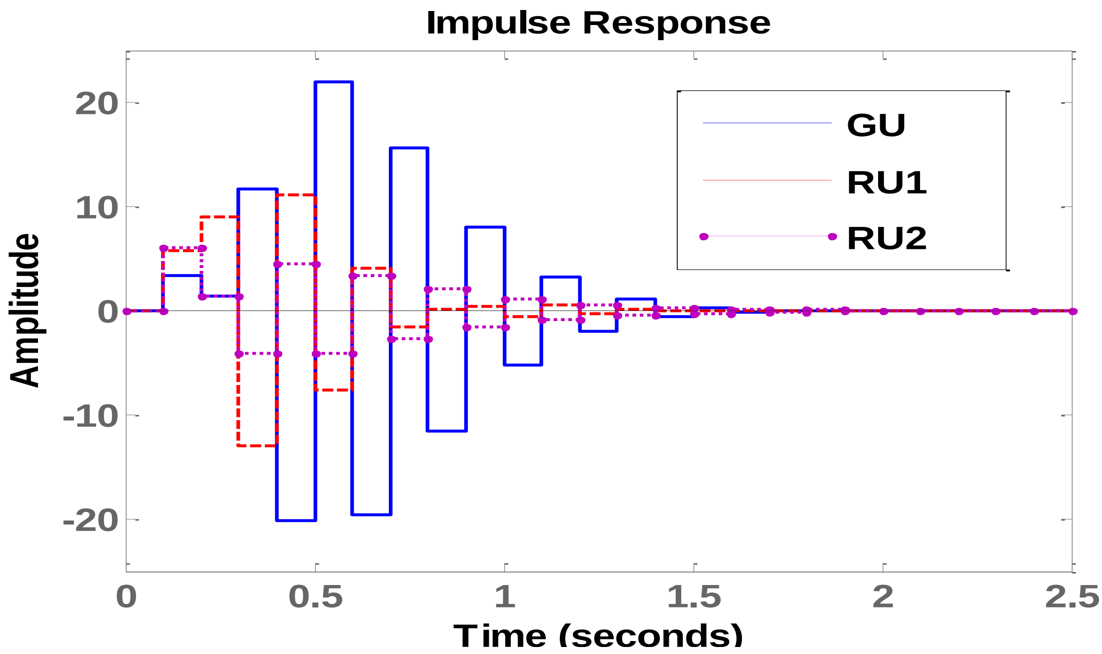

Fig. 6.3. Comparison of impulse response (upper limit)

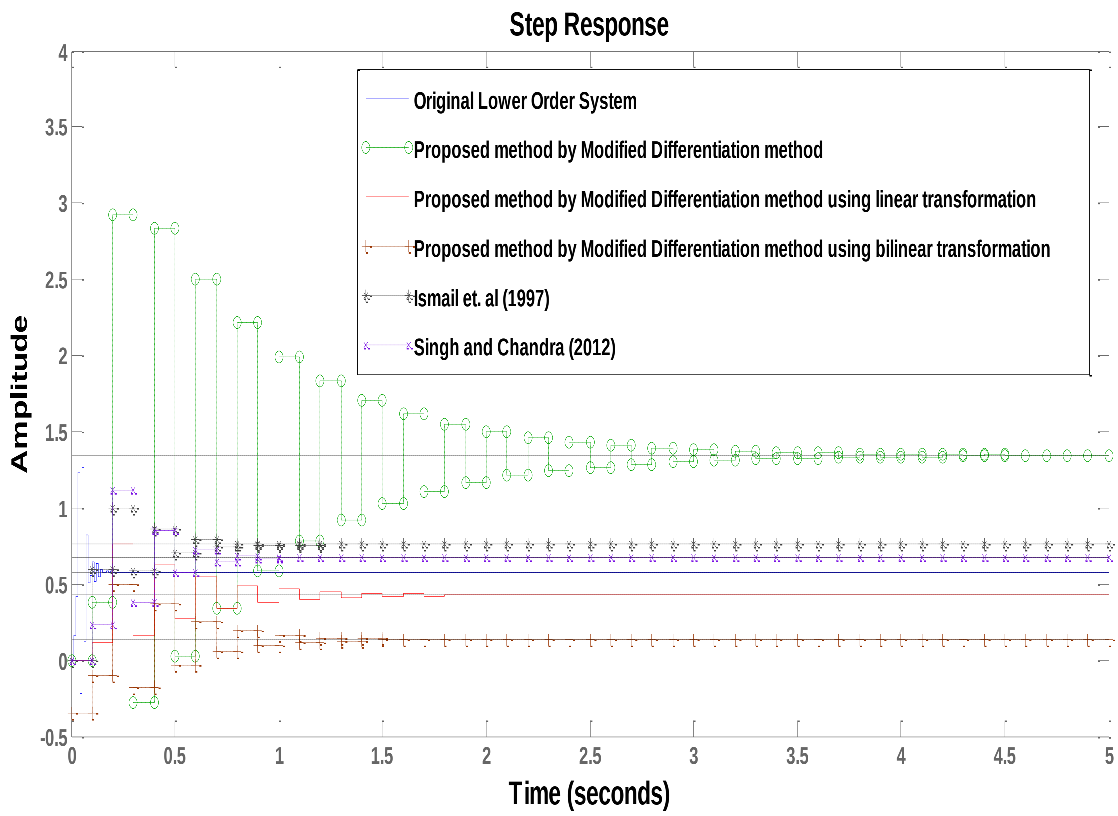

Fig. 6.4. Comparison of step response (lower limit)

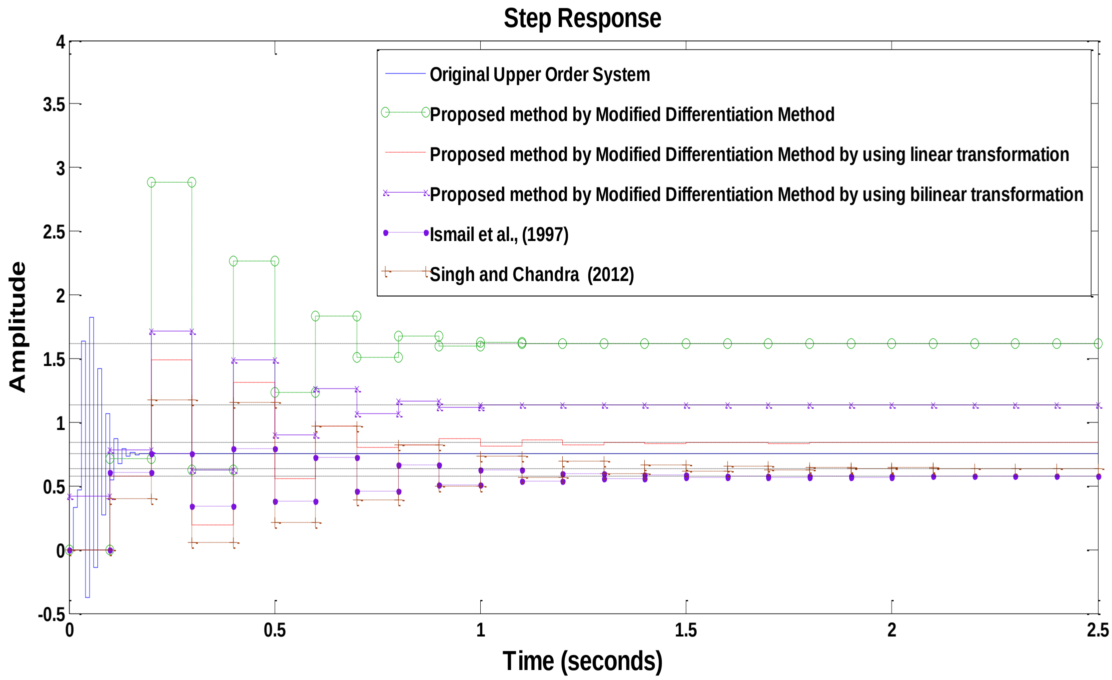

Fig. 6.5. Comparison of step response (upper limit)

Case 4: $\alpha-\beta$ truncation method

$$R_2^1(z)=\frac{[0.2307,0.3352]z+[0.1409,0.4577]}{z^2+[-0.3336,-0.2403]z+[0.0615,0.2522]} \tag{6.22}$$

Case 5: $\alpha-$truncation and Factor division method

$$R_2^2(z)=\frac{[6.2565,11.5084]z+[0.579,3.1525]}{z^2+[-0.3336,-0.2403]z+[0.0615,0.2522]} \tag{6.23}$$

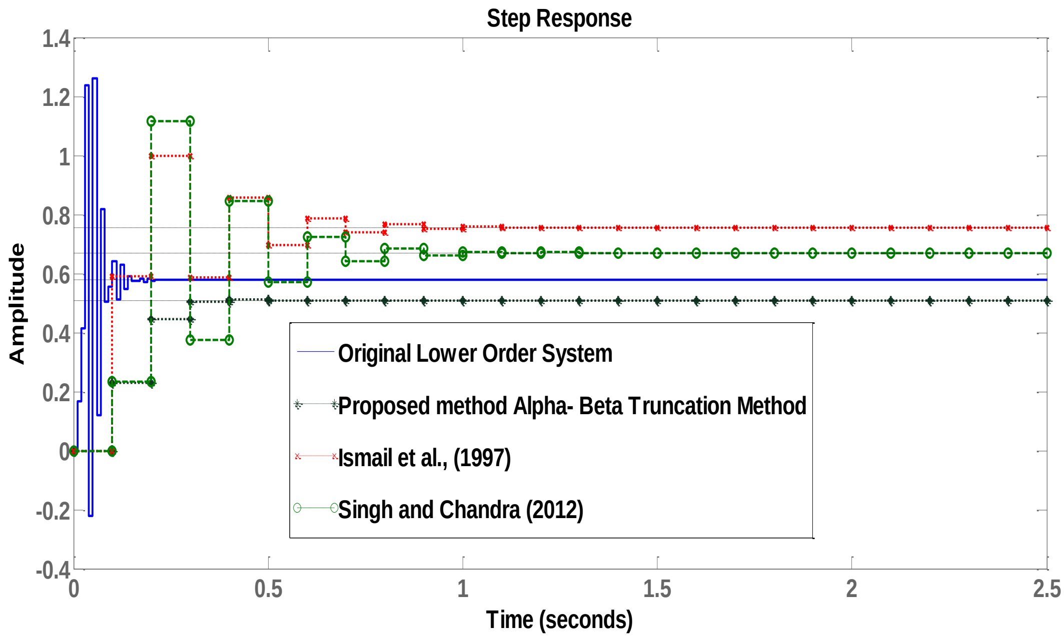

Fig. 6.6. Comparison of step response (lower limit)

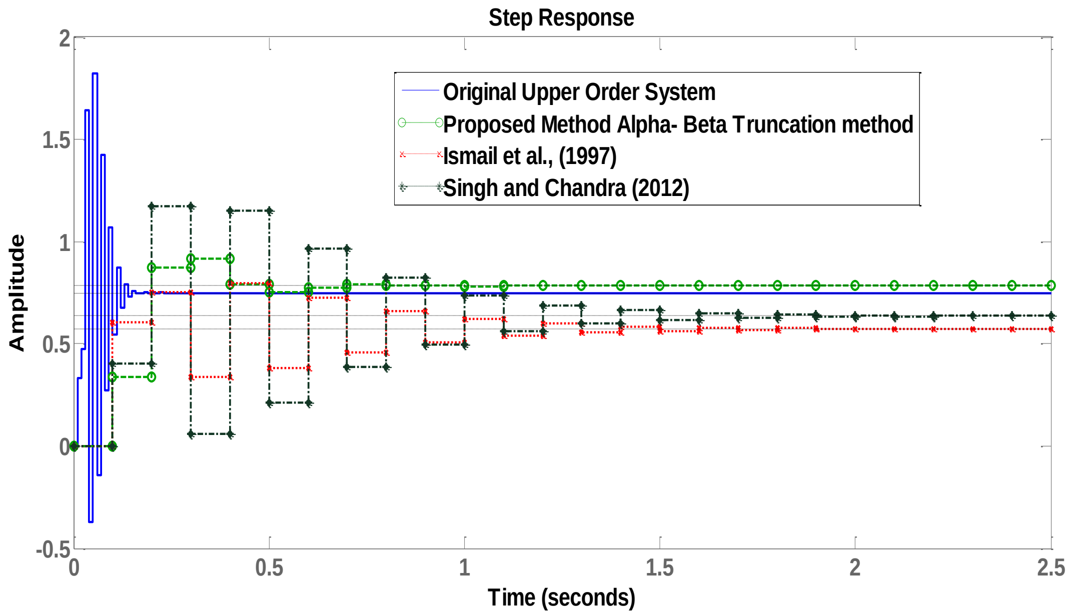

Fig. 6.7. Comparison of step response (upper limit)

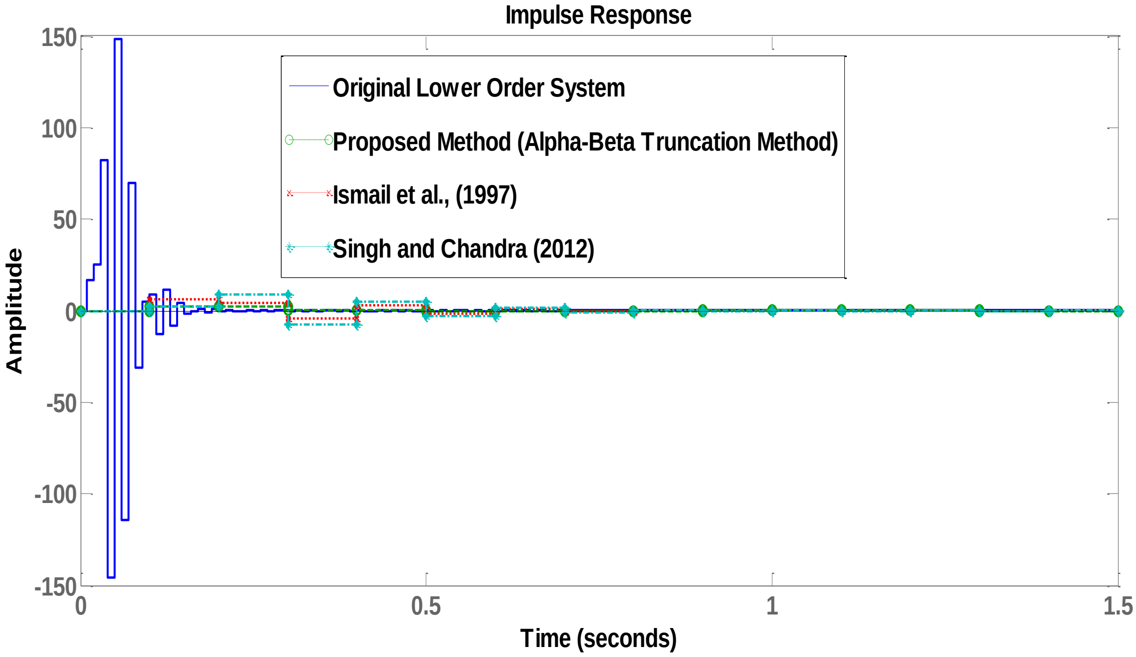

Fig.6. 8. Comparison of impulse response (lower limit)

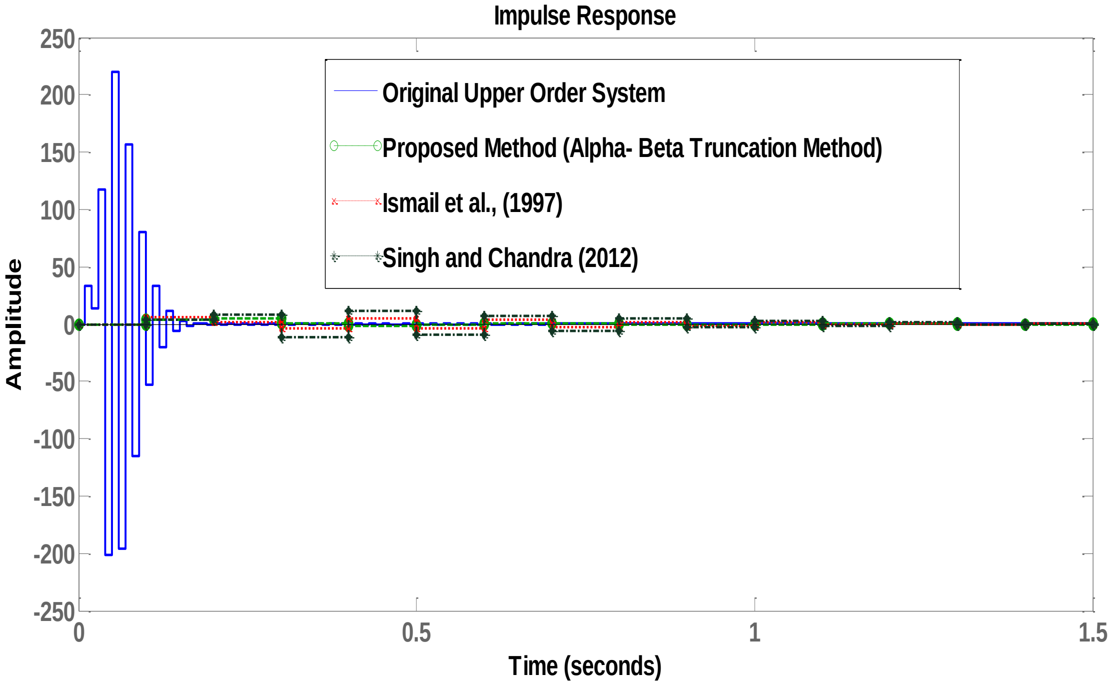

Fig. 6.9. Comparison of impulse response (upper limit)