CHAPTER 5
PROPOSED TECHNIQUES FOR THE REDUCTION OF CONTINUOUS TIME INTERVAL SYSTEMS CONSIDERING DEPENDENCY PROPERTY

5.1 INTRODUCTION

In the previous chapters, arithmetic rules are limited to addition, subtraction, multiplication and division but in this chapter dependency property has been considered in the proposed algorithms and attempt to overcome the drawback of the existing techniques.

From the literature it is clear that dependency problem (Brain Hayes, [212]) is a major complicated problem to the application of interval arithmetic. Let consider an example for better understanding the square of an interval function, $\left[e^{-},e^{+}\right]^{2}=\left[e^{-},e^{+}\right]\times\left[e^{-},e^{+}\right]$ seems to work in some cases, such as $\left[2,4\right]^{2}=\left[4,16\right]$. But consider $\left[-3,3\right]^{2}\neq\left[-9,9\right]$ the square of a real number cannot be a negative. The correct answer is $\left[-3,3\right]^{2}=\left[0,9\right]$. The extension of classical model order reduction techniques to interval systems, dependency problem is one of the reasons for obtaining an unstable reduced order model. Let us check the statement that $\left[-1,1\right]-\left[-1,1\right]$ might be zero, or not. There is a dependency issue, well known in interval arithmetic (or reliable computing) circles, in which the lack of ability to determine the "source" of an interval, i.e. whether two intervals are correlated, leads to wider and therefore less useful results.

Consider an example, if the expression means the set $\left\{e-e\,\middle|\,e\in\left[-1,1\right]\right\}$ the answer is 0. If the two intervals are different $\left\{e-f\,\middle|\,e\in\left[-1,1\right]\right\}$ and $f\in\left[-1,1\right]$ then the answer is the interval $\left[-2,2\right]$.

5.2 PROBLEM STATEMENT

Let $G_n(s)$ the transfer function of a higher order stable continuous interval systems given by

$$G_n(s)=\frac{\left[b_0^-,b_0^+\right]+\left[b_1^-,b_1^+\right]s+......+\left[b_{n-1}^-,b_{n-1}^+\right]s^{n-1}}{\left[a_0^-,a_0^+\right]+\left[a_1^-,a_1^+\right]s+......+\left[a_n^-,a_n^+\right]s^n}=\frac{N(s)}{D(s)} \tag{5.1}$$

where $a_j^- < a_j < a_j^+$ for j = *0, 1, 2, 3, ... , n* and $b_i^- < b_i < b_i^+$ for i = *0, 1, 2, 3, ... , n-1*are lower bounds and upper bounds for denominator polynomial and numerator polynomial of original interval transfer function. The order of the original system is *n*. Let $R_k(s)$be the lower order model of the stable original system in the form

$$R_k(s)=\frac{\left[d_0^-,d_0^+\right]+\left[d_1^-,d_1^+\right]s+......+\left[d_{k-1}^-,d_{k-1}^+\right]s^{k-1}}{\left[c_0^-,c_0^+\right]+\left[c_1^-,c_1^+\right]s+......+\left[c_k^-,c_k^+\right]s^k}=\frac{N_k(s)}{D_k(s)} \tag{5.2}$$

where $c_j^- < c_j < c_j^+$ for j = *0, 1, 2, 3, ... , k* and $d_i^- < d_i < d_i^+$ for i = *0, 1, 2, 3, . . . , k-1* are lower bounds and upper bounds for the reduced order denominator polynomial and numerator polynomial interval transfer function. The order of the reduced order interval system is *k*. where *k* is the reduced order model and $k\langle n$.

5.3. MODIFIED DIFFERENTIATION METHOD FOR INTERVAL SYSTEMS.

In this method, we introduced a modified differentiation method for model order reduction of large scale interval systems. This method is an alternative method to the existing differentiation method (see section 4.7.1). The proposed method has been applied for both continuous and discrete time interval systems. The reduction of discrete time interval systems is achieved by bilinear transformation. The proposed method is significant because it is computationally simple and guaranteed stability. The stability of the interval systems is tested with the Kharitonov's theorem. Numerical examples show the effectiveness of the proposed method.

We extended the Lucas differentiation method [116] to interval systems with some modifications. With examples, we proved that many existing methods [156, 157,161, 162, 169,172] fails to preserve the stability of reduced order systems whereas the proposed method guarantee the stability of the reduced order interval systems. The modified differentiation method is computationally simple to apply for both continuous and discrete time interval systems.

Theorem 5.1: If the reduced order interval polynomial $R_k(s)$ satisfies the Kharitonov's theorem then original interval polynomial $D(s)$ is also stable.

Proof: Let the higher order polynomial

$$D(s) = \left[a_n^-, a_n^+\right] s^n + \left[a_{n-1}^-, a_{n-1}^+\right] s^{n-1} + ... + \left[a_1^-, a_1^+\right] s + \left[a_0^-, a_0^+\right] \tag{5.3}$$

be a higher order polynomial. Its differentiated polynomial is

$$\dot{D}(s) = n\left[a_n^-, a_n^+\right] s^{n-1} + (n-1)\left[a_{n-1}^-, a_{n-1}^+\right] s^{n-2} + ... + \left[a_1^-, a_1^+\right] \tag{5.4}$$

To obtain the reduced order polynomial, modified Routh approximation has applied by using the above two Eqs. (5.3) and (5.4)

$$R_k(s) = \left[a_{n-1}^-, a_{n-1}^+\right] s^{n-1} + \left[a_{n-2}^-, a_{n-2}^+\right] s^{n-2} + ... + \left[a_0^-, a_0^+\right] \tag{5.5}$$

The construction of Modified Routh approximation has been given in Table1.

Table.5.1: Modified Routh table

row 1	$\left[a_{11}^-, a_{11}^+\right] = \left[a_n^-, a_n^+\right]$	$\left[a_{12}^-, a_{12}^+\right] = \left[a_{n-1}^-, a_{n-1}^+\right]$	....
row 2	$\left[a_{21}^-, a_{21}^+\right] = (n)\left[a_n^-, a_n^+\right]$	$\left[a_{22}^-, a_{22}^+\right] = (n-1)\left[a_{n-1}^-, a_{n-1}^+\right]$	....
row 3	$\left[a_{n-1}^-, a_{n-1}^+\right] = \left[a_{31}^-, a_{31}^+\right]$	$\left[a_{n-2}^-, a_{n-2}^+\right] = \left[a_{32}^-, a_{32}^+\right]$	....

From Table 5.1, for better calculation, consider $\left[a_{n-1}^{-}, a_{n-1}^{+}\right] = \left[a_{31}^{-}, a_{31}^{+}\right]$, $\left[a_{n-2}^{-}, a_{n-2}^{+}\right] = \left[a_{32}^{-}, a_{32}^{+}\right]$ and so on.

Therefore,

$$\left.\begin{aligned}
\left[a_{31}^{-}, a_{31}^{+}\right] &= \left[a_{12}^{-}, a_{12}^{+}\right] - \frac{\dfrac{\left(a_{11}^{-} + a_{11}^{+}\right)}{2}}{\dfrac{\left(a_{21}^{-} + a_{21}^{+}\right)}{2}}\left[a_{22}^{-}, a_{22}^{+}\right]; \\
\left[a_{32}^{-}, a_{32}^{+}\right] &= \left[a_{13}^{-}, a_{13}^{+}\right] - \frac{\dfrac{\left(a_{11}^{-} + a_{11}^{+}\right)}{2}}{\dfrac{\left(a_{21}^{-} + a_{21}^{+}\right)}{2}}\left[a_{23}^{-}, a_{22}^{+}\right];........
\end{aligned}\right\} \qquad (5.6)$$

The midpoint of the interval $\left[a_{11}^{-}, a_{11}^{+}\right]$ and $\left[a_{21}^{-}, a_{21}^{+}\right]$ is defined by $\alpha_{11} = mid\left(\left[a_{11}\right]\right) = \dfrac{a_{11}^{-} + a_{11}^{+}}{2}; \alpha_{21} = mid\left(\left[a_{21}\right]\right) = \dfrac{a_{21}^{-} + a_{21}^{+}}{2}$.

Note that the first line is the original polynomial $D(s)$ and the second row is the differentiation of the row 1. The third row is the reduced order polynomial obtained by using modified Routh approximation. The third order coefficients are obtained by the following relation

$$\left[a_{ij}^{-}, a_{ij}^{+}\right] = \left[a_{i-2,j+1}^{-}, a_{i-2,j+1}^{+}\right] - \frac{\alpha_{i-2,1}}{\alpha_{i-1,1}}\left[a_{i-1,j+1}^{-}, a_{i-1,j+1}^{+}\right] \qquad (5.7)$$

where *i=3 & j= 1,2,3, ... , n-1*. To overcome the limitations of dependency property midpoints are used in Eqs. (5.6) and (5.7).

The reduced order interval polynomial is stable if it satisfies the Kharitonov's theorem. Based on Kharitonov's theorem the reduced order interval polynomial $R_k(s)$ can be written as four polynomials

$$\left.\begin{aligned}
\Delta^1(s) &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + a_4^- s^4 + a_6^- s^5 + \cdots \\
\Delta^2(s) &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + a_5^- s^4 + a_6^+ s^5 + \cdots \\
\Delta^3(s) &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_5^+ s^4 + a_6^- s^5 + \cdots \\
\Delta^4(s) &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_5^+ s^4 + a_6^+ s^5 + \cdots
\end{aligned}\right\} \qquad (5.8)$$

Let us bring in the hyper rectangle or box Ψ of coefficients of the perturbed polynomials

$$\Psi = \left\{ a \middle| a \in \mathbb{R}^n, a_i^- \le a_i \le a_i^+, i = 1,2,3,\ldots,n-1 \right\} \qquad (5.9)$$

The four Kharitonov's polynomials are constructed from two different even parts $\Delta_{\max}^{even}(s)$ and $\Delta_{\min}^{odd}(s)$ and two different odd parts $\Delta_{\max}^{odd}(s)$ and $\Delta_{\min}^{odd}(s)$ defined below

$$\left.\begin{aligned}
\Delta_{\max}^{even}(s) &= a_0^+ + a_2^- s^2 + a_4^+ s^4 + a_6^- s^6 + \ldots. \\
\Delta_{\min}^{even}(s) &= a_0^- + a_2^+ s^2 + a_4^- s^4 + a_6^+ s^6 + \ldots.
\end{aligned}\right\} \qquad (5.10)$$

and

$$\left.\begin{aligned}
\Delta_{\max}^{odd}(s) &= a_1^+ s + a_3^- s^3 + a_5^+ s^5 + a_7^- s^7 + \ldots. \\
\Delta_{\min}^{odd}(s) &= a_1^- s + a_3^+ s^3 + a_5^- s^5 + a_7^+ s^7 + \ldots.
\end{aligned}\right\} \qquad (5.11)$$

The motivation for obtaining the subscripts "max" and "min" is as follows. Let an arbitrary polynomial $a(s)$ with its coefficients lying in the box Ψ and let $a^{even}(s)$ be its even part. Then

$$\left.\begin{aligned}
\Delta_{\max}^{e}(\omega) &= a_0^+ - a_2^- \omega^2 + a_4^+ \omega^4 - a_6^- \omega^6 + \cdots \\
a^{e}(\omega) &= a_0 - a_2 \omega^2 + a_4 \omega^4 - a_6 \omega^6 + \cdots \\
\Delta_{\min}^{e}(\omega) &= a_0^- - a_2^+ \omega^2 + a_4^- \omega^4 - a_6^+ \omega^6 + \cdots
\end{aligned}\right\} \qquad (5.12)$$

so that

$$\Delta_{\max}^{e}(\omega) - a^{e}(\omega) = \left(a_0^+ - a_0\right) + \left(a_2 - a_2^-\right)\omega^2 + \left(a_4^+ - a_4\right)\omega^4 + \cdots \qquad (5.13)$$

and

$$a^e(\omega)-\Delta_{\min}^e(\omega)=\left(a_0-a_0^-\right)+\left(a_2^+-a_2\right)\omega^2+\left(a_4-a_4^-\right)\omega^4+\cdots \tag{5.14}$$

Therefore

$$\Delta_{\min}^e(\omega)\le a^e(\omega)\le \Delta_{\max}^e(\omega);\omega\in[0,\infty] \tag{5.15}$$

Similarly, if $a^{odd}(s)$ denotes the odd parts of $a(s)$, it can be verified that

$$\Delta_{\min}^o(\omega)\le a^o(\omega)\le \Delta_{\max}^o(\omega);\omega\in[0,\infty] \tag{5.16}$$

To proceed, note that the Kharitonov polynomials in Eq. (5.8) can be rewritten as

$$\left.\begin{aligned}\Delta^1(s)&=\Delta_{\min}^{even}(s)+\Delta_{\min}^{odd}(s)\\ \Delta^2(s)&=\Delta_{\min}^{even}(s)+\Delta_{\max}^{odd}(s)\\ \Delta^3(s)&=\Delta_{\max}^{even}(s)+\Delta_{\min}^{odd}(s)\\ \Delta^1(s)&=\Delta_{\max}^{even}(s)+\Delta_{\max}^{odd}(s)\end{aligned}\right\} \tag{5.17}$$

If all the polynomials with the coefficients in the box Ψ are stable, it is clear that the Kharitonov polynomials in Eq. (5.8) must also be stable since their coefficients lie in Ψ. For the conserve assume that the Kharitonov polynomials are stable, and let $a(s)=a^{even}(s)+a^{odd}(s)$ be an arbitrary polynomial with coefficients in the box Ψ with its even part $a^{even}(s)$ and its odd part $a^{odd}(s)$.

Since $\Delta^1(s)$ and $\Delta^2(s)$ are stable and Eq. (5.16) holds, we conclude from Lemma 1 applied to $\Delta^1(s)$ and $\Delta^2(s)$ in Eq. (5.17) that $\Delta_{\min}^{even}(s)+a^{odd}(s)$ is stable.

Similarly Lemma 1 from Chapter 3 applied to $\Delta^3(s)$ and $\Delta^4(s)$ in Eq. (5.17), we conclude that $\Delta_{\max}^{even}(s)+a^{odd}(s)$ is stable.

Now, since Eq. (5.15) holds, we can apply Lemma 2 from chapter 3 to the two stable polynomials $\Delta_{\max}^{even}(s)+a^{odd}(s)$ and $\Delta_{\min}^{even}(s)+a^{odd}(s)$ and we conclude that $a^{even}(s)+a^{odd}(s)=a(s)$ is stable.

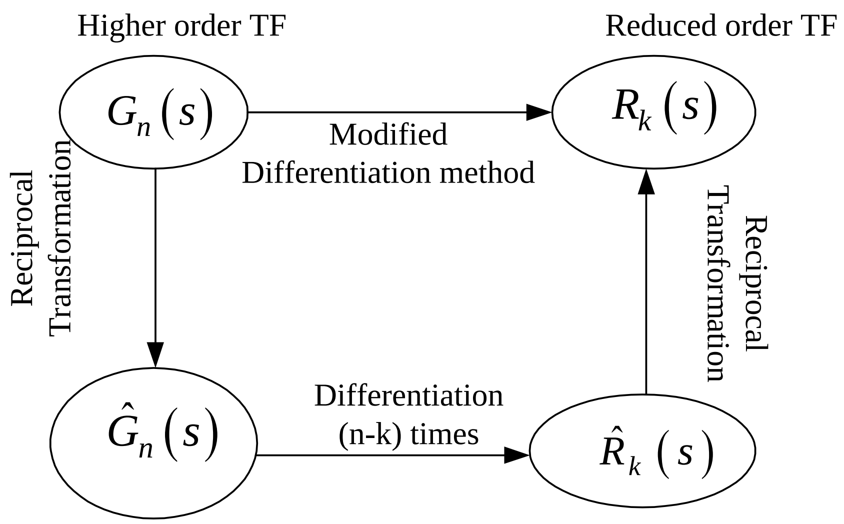

Fig. 5.1: Comparison between proposed method and Differentiation method.

The procedure to obtain reduced order polynomials $D_k(s)$ and $N_k(s)$ in Eq. (5.2) as follows.

Step 1: The first row as shown in Table 5.2 is formed from denominator coefficients of $G_n(s)$ (higher order coefficients).

Step 2: The second row is obtained by differentiation of row 1.

Step 3: The third row can be obtained by applying modified Routh approximation [166]. This process will give reduced order denominator of order n-1.

Step 4: The fourth row can be obtained by differentiation of row 3.

Step 5: The fifth row can be obtained by modified Routh approximation using row 3 and row 4. This will give reduced denominator of order n-2.

The algorithm is illustrated by the following table for obtaining a reduced order denominator.

Table 5.2: Denominator Table for Continues Time Interval Systems

row 1	$\left[a_{11}^{-},a_{11}^{+}\right]=\left[a_{n}^{-},a_{n}^{+}\right]$	$\left[a_{12}^{-},a_{12}^{+}\right]=\left[a_{n-1}^{-},a_{n-1}^{+}\right]$	......
row 2	$\left[a_{21}^{-},a_{21}^{+}\right]=(n)\left[a_{n}^{-},a_{n}^{+}\right]$	$\left[a_{22}^{-},a_{22}^{+}\right]=(n-1)\left[a_{n-1}^{-},a_{n-1}^{+}\right]$	......
row 3	$\left[a_{31}^{-},a_{31}^{+}\right]$	$\left[a_{32}^{-},a_{32}^{+}\right]$	......
row 4	$\left[a_{41}^{-},a_{41}^{+}\right]=(n-1)\left[a_{31}^{-},a_{31}^{+}\right]$	$\left[a_{42}^{-},a_{42}^{+}\right]=(n-2)\left[a_{32}^{-},a_{32}^{+}\right]$	......
row 5	$\left[a_{51}^{-},a_{51}^{+}\right]$	$\left[a_{52}^{-},a_{52}^{+}\right]$	......
row 6	....	....	....

$$\left[a_{ij}^{-},a_{ij}^{+}\right]=\left[a_{i-2,j+1}^{-},a_{i-2,j+1}^{+}\right]-\frac{\alpha_{i-2,1}}{\alpha_{i-1,1}}\left[a_{i-1,j+1}^{-},a_{i-1,j+1}^{+}\right] \tag{5.18}$$

where $i=3,5,7,9,....\&\ j=1,2,3,.......$

$\alpha_{i-2,1}=\dfrac{a_{i-2,1}^{-}+a_{i-2,1}^{+}}{2};\alpha_{i-1,1}=\dfrac{a_{i-1,1}^{-}+a_{i-1,1}^{+}}{2}$ is the middle point of the coefficients.

$$\left[a_{31}^{-},a_{31}^{+}\right]=\left[a_{12}^{-},a_{12}^{+}\right]-\frac{\alpha_{11}}{\alpha_{21}}\left[a_{22}^{-},a_{22}^{+}\right] \tag{5.19}$$

The reduced denominator polynomial $D_k(s)$ for $k=n-1$ coefficients (row 3)

$$\left[c_{k}^{-},c_{k}^{+}\right]=\left[a_{31}^{-},a_{31}^{+}\right];\left[c_{k-1}^{-},c_{k-1}^{+}\right]=\left[a_{32}^{-},a_{32}^{+}\right];..... \tag{5.20}$$

Step 6: The first row as shown in Table 5.3 is formed from numerator coefficients of $G_n(s)$ (higher order coefficients).

Step 7: The second row is obtained by differentiation of row 1 in Table 5.3.

Step 8: The third row can be obtained by applying modified Routh approximation [166]. This process will give reduced order numerator of order n-2.

Step 9: The fourth row can be obtained by differentiation of row 3 in Table 5.3.

Step 10: The fifth row can be obtained by modified Routh approximation using row 3 and row 4 in Table 5.3. This will give reduced denominator of order n-3.

The algorithm is illustrated by the following Table 5.3 for obtaining a reduced order numerator.

Table 5.3: Numerator Table for Continues Time Interval Systems

row 1	$\left[b_{11}^{-}, b_{11}^{+}\right]=\left[b_{n-1}^{-}, b_{n-1}^{+}\right]$	$\left[a_{12}^{-}, a_{12}^{+}\right]=\left[b_{n-2}^{-}, b_{n-2}^{+}\right]$	$\cdots$
row 2	$\left[b_{21}^{-}, b_{21}^{+}\right]=(n-1)\left[a_{n-1}^{-}, a_{n-1}^{+}\right]$	$\left[b_{22}^{-}, b_{22}^{+}\right]=(n-2)\left[a_{n-2}^{-}, a_{n-2}^{+}\right]$	$\cdots$
row 3	$\left[b_{31}^{-}, b_{31}^{+}\right]$	$\left[b_{32}^{-}, b_{32}^{+}\right]$	$\cdots$
row 4	$\left[b_{41}^{-}, b_{41}^{+}\right]=(n-2)\left[b_{31}^{-}, b_{31}^{+}\right]$	$\left[b_{42}^{-}, b_{42}^{+}\right]=(n-3)\left[b_{32}^{-}, b_{32}^{+}\right]$	$\cdots$
row 5	$\left[b_{51}^{-}, b_{51}^{+}\right]$	$\left[b_{52}^{-}, b_{52}^{+}\right]$	$\cdots$
row 6	$\cdots\cdots\cdots\cdots$	$\cdots\cdots\cdots\cdots$	

$$\left[b_{ij}^{-}, b_{ij}^{+}\right]=\left[b_{i-2,j+1}^{-}, b_{i-2,j+1}^{+}\right]-\frac{\beta_{i-2,1}}{\beta_{i-1,1}}\left[b_{i-1,j+1}^{-}, b_{i-1,j+1}^{+}\right] \tag{5.21}$$

where $i=3,5,7,9,....\ \&\ j=1,2,3,.......$

$\beta_{i-2,1}=\dfrac{b_{i-2,1}^{-}+b_{i-2,1}^{+}}{2}; \beta_{i-1,1}=\dfrac{b_{i-1,1}^{-}+b_{i-1,1}^{+}}{2}$ is the middle point of the coefficients.

$$\left[b_{31}^{-}, b_{31}^{+}\right] = \left[b_{12}^{-}, b_{12}^{+}\right] - \frac{\beta_{11}}{\beta_{21}}\left[b_{22}^{-}, b_{22}^{+}\right] \tag{5.22}$$

The reduced numerator polynomial $N_k(s)$ coefficients (row 3)

$$\left[d_{k-1}^{-}, d_{k-1}^{+}\right] = \left[b_{31}^{-}, b_{31}^{+}\right]; \left[d_{k-2}^{-}, d_{k-2}^{+}\right] = \left[b_{32}^{-}, b_{32}^{+}\right]; \ldots.. \tag{5.23}$$

5.4 MODIFIED SCHWARZ APPROXIMATION METHOD FOR INTERVAL SYSTEMS.

In this section, we propose a modified algorithm based on Routh approximation. This algorithm can be considered as modification of many existing techniques such as Schwarz approximation, Direct Routh Approximation Method (DRAM), and $\alpha - \beta$ approximation. The existing model reduction methods (Routh approximation, Modified Routh approximation or Dolgin- Zeheb method, $\gamma - \delta$ approximation, γ approximation, DRAM and Schwarz approximation) when extended to interval systems cannot guarantees a stable reduced model but the proposed method guaranteed the stability of the reduced model if the original system is stable and also preserves the characteristics of the original system.

Algorithm for reduction of interval systems proceeds as follows.

The denominator table associated with $\hat{G}_n(s)$ has the following structure

$$\begin{array}{lllll}
 & \left[a_{1,1}^{-}, a_{1,1}^{+}\right] = \left[a_{0}^{-}, a_{0}^{+}\right] & \left[a_{1,2}^{-}, a_{1,2}^{+}\right] = \left[a_{1}^{-}, a_{1}^{+}\right] & \ldots\ldots & \left[a_{1,n+1}^{-}, a_{1,n+1}^{+}\right] = \left[a_{n}^{-}, a_{n}^{+}\right] \\
\alpha_{11} & & & & \\
 & \left[a_{1,2}^{-}, a_{1,2}^{+}\right] & 0 & \left[a_{1,4}^{-}, a_{1,4}^{+}\right] & \ldots\ldots
\end{array}$$

$$\left[a_{2,1}^-, a_{2,1}^+\right] \quad \left[a_{2,2}^-, a_{2,2}^+\right] \quad \ldots\ldots \quad \left[a_{2,n}^-, a_{2,n}^+\right]$$

$$\alpha_{21}$$

$$\left[a_{2,2}^-, a_{2,2}^+\right] \quad 0 \quad \left[a_{2,4}^-, a_{2,4}^+\right] \quad \ldots\ldots$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$\left[a_{n,1}^-, a_{n,1}^+\right] \quad \left[a_{n,2}^-, a_{n,2}^+\right]$$

$$\alpha_{n1}$$

$$\left[a_{n,2}^-, a_{n,2}^+\right]$$

where

$$\begin{aligned}\left[a_{ij}^-, a_{ij}^+\right] &= \left[a_{i-1,j+1}^-, a_{i-1,j+1}^+\right]; j = odd \\ \left[a_{ij}^-, a_{ij}^+\right] &= \left[a_{i-1,j+1}^-, a_{i-1,j+1}^+\right] - \alpha_{i-1,1}\left[a_{i-1,j+2}^-, a_{i-1,j+2}^+\right]; j = even\end{aligned} \tag{5.24}$$

for $i = 2, 3, 4, ..., n$

$$\alpha_{i,1} = \frac{\left(\dfrac{\left[a_{i,1}^- + a_{i,1}^+\right]}{2}\right)}{\left(\dfrac{\left[a_{i,2}^- + a_{i,2}^+\right]}{2}\right)} = \frac{a_{i,1}}{a_{i,2}}; i = 1, 2, 3, 4, ..., n \tag{5.25}$$

where $\alpha_{i,1}$ is the midpoint of the interval .

To calculate $\alpha_{i,1}$ midpoints of the interval was used suggested by Dolgin [166] to ensure and preserve each row of the point by point properties.

The numerator table for $\hat{G}_n(s)$ is formed with the even numbered rows the same as the denominator table and has the following form

$$\left[b_{1,1}^-, b_{1,1}^+\right] = \left[b_0^-, b_0^+\right] \qquad \left[b_{1,2}^-, b_{1,2}^+\right] = \left[b_1^-, b_1^+\right] \qquad \ldots\ldots$$

$$\left[b_{1,n}^-, b_{1,n}^+\right] = \left[b_{n-1}^-, b_{n-1}^+\right]$$

$$\beta_{11}$$

$$\left[a_{1,2}^-, a_{1,2}^+\right] \qquad 0 \qquad \left[a_{1,4}^-, a_{1,4}^+\right] \qquad 0 \qquad \ldots\ldots$$

$$\left[b_{2,1}^-, b_{2,1}^+\right] \qquad \left[b_{2,2}^-, b_{2,2}^+\right] \qquad \ldots\ldots \qquad [b_{2,n-1}^-, b_{2,n-1}^+]$$

$$\beta_{21}$$

$$\left[a_{2,2}^-, a_{2,2}^+\right] \qquad 0 \qquad \left[a_{2,4}^-, a_{2,4}^+\right] \qquad \ldots\ldots$$

$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$

$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$

$$\left[b_{n,1}^-, b_{n,1}^+\right] \qquad \left[b_{n,2}^-, b_{n,2}^+\right]$$

$$\beta_{n1}$$

$$\left[a_{n,2}^-, a_{n,2}^+\right]$$

where

$$\begin{aligned} \left[b_{ij}^-, b_{ij}^+\right] &= \left[b_{i-1,j+1}^-, b_{i-1,j+1}^+\right];\ j = odd \\ \left[b_{ij}^-, b_{ij}^+\right] &= \left[b_{i-1,j+1}^-, b_{i-1,j+1}^+\right] - \beta_{i-1,1}\left[a_{i-1,j+2}^-, a_{i-1,j+2}^+\right];\ j = even \end{aligned} \tag{5.26}$$

for $i = 2, 3, 4, \ldots., n$

$$\beta_{i,1} = \frac{\left(\frac{\left[b_{i,1}^{-} + b_{i,1}^{+}\right]}{2}\right)}{\left(\frac{\left[a_{i,2}^{-} + a_{i,2}^{+}\right]}{2}\right)} = \frac{b_{i,1}}{a_{i,2}}; i = 1, 2, 3, 4, ..., n \quad (5.27)$$

where $\beta_{i,1}$ is the midpoint of the interval .

The numerator and denominator polynomial of the reduced order model are obtained from the above array by deleting an even number of top rows i.e third row will give polynomial of degree $(n-1)$ and so on. The block diagram of the modified Schwarz approximation is shown in Fig.5.2.

We noticed that dependency problem is a major complicated problem to the application of interval arithmetic. For example squaring an interval. The evident definition $\left[y^{-}, y^{+}\right]^2 = \left[y^{-}, y^{+}\right] \times \left[y^{-}, y^{+}\right]$ seems to work in some cases, such as $[3,4]^2 = [9,16]$. But what about $[-4,4]^2 = [-16,16]$ the square of a real number cannot be a negative. The correct answer is $[-4,4]^2 = [0,16]$. We introduced midpoint in the algorithm to overcome the dependency problem.

Proposed system framework: The procedure to obtain reduced order polynomials via the proposed method named as modified Schwarz approximation (MSA) is as follows.

Step 1: Reciprocal transformation of the higher order interval system $G_n(s)$.

Step 2: The denominator table is obtained from the reciprocal transformation of original denominator interval system as defined in the proposed method. In each even numbered row, the denominator coefficients are separated by the inclusion of zero, at every even numbered column.

Step 3: The numerator table is obtained from the reciprocal transformation of original numerator interval system together with the denominator table. The even row elements are the same as those in the denominator table.

Step 4: The reduced order of the system is obtained from the reciprocal transformation of the coefficients, corresponding to odd numbered rows in the numerator and denominator tables.

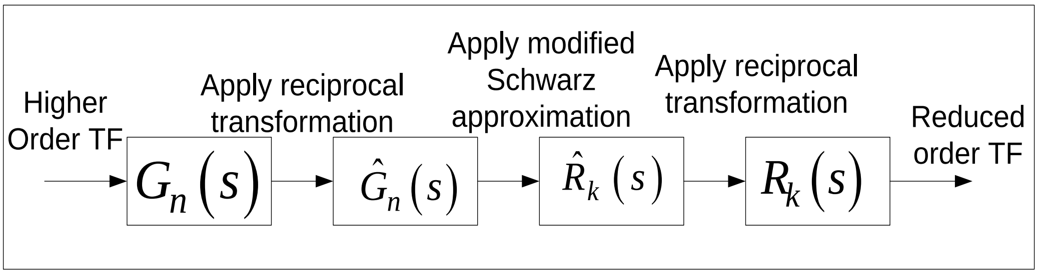

Fig. 5.2. Block diagram of modified Schwarz approximation for continuous time interval systems

5.5 COMPARISON OF METHODS

The proposed method leads to obtain a stable reduced order interval system (ROIS), if the original high-order interval system is stable. The significant features of the proposed method can be summarized as follows

(i) The stability preservation of non-interval Routh based approximations cannot be claimed for the interval Routh based approximation. The chances of Routh based approximation [156, 157,161, 162, 169, 172] failure increase with the order of the approximation. The proposed methods guarantees the stability of the reduced order system.

(ii) The modified Schwarz approximation (MSA) is easily calculated alternative to the Routh based approximation for producing stable reduced order interval system.

(iii) The values of the integral square error (ISE) and integral absolute error (IAE) are compared for the proposed methods and other well know existing order reduction techniques as given in Table 5.10 and showed better accuracy with the original system.

5.6 ILLUSTRATIVE EXAMPLES

Example 5.1: Consider the higher order system reported in Shih feng Yang [165]

$$P(s) = [2.1, 2.6]s^6 + [76.1, 76.7]s^5 + [119.1, 119.6]s^4 + [111.0, 111.6]s^3 + [71.8, 72.3]s^2 + [31.0, 31.7]s + [9.0, 9.9] \quad (5.28)$$

Case1: Modified differentiation method

Following the algorithm, Table 5.4 is constructed

Table 5.4: Modified Differentiation Method of example 5.1

row 1	Higher order	[2.1, 2.6]	[76.1, 76.7]	[119.1,119.6]	. . . . .	[9,9.9]
row 2	Differentiation	[12.6, 15.6]	[380.5, 383.5]	[476.4,478.4]	. . . . .	
row 3	5th order Modified Routh	[12.17,13.27]	[39.35,40.18]	[55.18, 56.08]	. . . . .	
row 4	Differentiation	[60.85, 66.35]	[157.40,160.73]	[165.56,168.26]	. . . . .	
row 5	4th order Modified Routh	[7.20, 8.70]	[21.53, 22.97]	[28.41, 29.11]	. . . . .	
row 6	Differentiation	[28.81, 34.81]	[64.60, 68.92]	[56.83, 58.23]	. . . . .	

row 7	3rd order Modified Routh	[4.30, 6.82]	[13.85, 14.90]	[15.06, 16.28]	. . . . .	
row 8	Differentiation	[12.91, 20.47]	[27.71, 29.81]	[15.06, 16.28]		
row 9	2nd order Modified Routh	[3.92, 5.67]	[9.63, 11.26]	[9,9.9]		

The fifth order reduced order polynomial is formed from the third row of Table 5.4

$$P_5(s) = [12.1706, 13.2707]s^5 + [39.3507, 40.1841]s^4 + [55.1888, 56.0889]s^3 + [47.6952, 48.1952]s^2 + [25.7156, 26.5223]s + [9, 9.9] \tag{5.29}$$

The second order reduced order polynomial is formed from the ninth row of Table 5.4

$$p_2(s) = [3.9205, 5.6705]s^2 + [9.6356, 11.2632]s + [9, 9.9] \tag{5.30}$$

The reduced order polynomial stability can be verified by Kharitonov theorem [192]. For above problem some of the existing methods [156, 157, 161, 162, 169, 172] fails to give the stable reduced order polynomial. It is reported by Yang [165] that Dolgin and Zeheb method [162] will result in an unstable reduced order model. Without applying a modified Routh approximation, i.e extending directly differentiation method proposed by Lucas [116] to interval systems. The reduced order system cannot give any stability guarantee.

Case 2: Modified Schwarz approximation

The reciprocal transformation on $P(s)$ yields

$$\hat{P}(s) = [9, 9.9]s^6 + [31, 31.7]s^5 + [71.8, 72.3]s^4 + [111.0, 111.6]s^3$$
$$+ [119.1, 119.6]s^2 + [76.1, 76.7]s + [2.1, 2.6] \quad (5.31)$$

Following the algorithm in section 5.4, Table 5.5 is constructed

Table 5.5: Modified Schwarz Approximation of example 5.1

[9,9.9]	[31,31.7]	[71.8,72.3]	[111,111.6]	…..
[31,31.7]	0	[111,111.6]	0	…. .
[31,31.7]	*[38.1638, 38.8446]*	[111,111.6]	*[95.9826, 96.6635]*	……
[38.1638,38.8446]	0	[95.9826, 96.6635]	0	…….
[38.1638,38.8446]	*[40.3966, 41.451]*	[95.9826, 96.6635]	*[73.9831, 74.9902]*	……
[40.3966,41.451]	0	[73.9831,74.9902]	0	
[40.3966,41.451]	*[25.4243, 27.0528]*	[73.9831, 74.9902]	[2.1,2.6]	
[25.4243,27.0528]	0	[2.1,2.6]		
[25.4243,27.0528]	*[69.9279, 71.7148]*	[2.1,2.6]		
[69.9279,71.7148]	0			
[69.9279,71.7148]	[2.1,2.6]			

[2.1,2.6]				

The italic elements in the table are required to be calculated.

From Table 5.5, we obtain the following polynomials (coefficients of the odd row in reverse order)

The fifth order reduced order polynomial is formed from the third row of Table 5.5

$$P_5(s) = [2.1, 2.6]s^5 + [76.1, 76.7]s^4 + [95.9826, 96.6635]s^3 + [111, 111.6]s^2 + [38.1638, 38.8446]s + [31, 31.7] \tag{5.32}$$

The fourth order reduced order polynomial is formed from the third row of Table 5.5

$$P_4(s) = [2.1, 2.6]s^4 + [73.9831, 74.9902]s^3 + [95.9826, 96.6635]s^2 + [40.3996, 41.451]s + [38.1638, 38.8446] \tag{5.33}$$

The third order reduced order polynomial is formed from the third row of Table 5.5

$$P_3(s) = [2.1, 2.6]s^3 + [73.9831, 74.9902]s^2 + [25.4243, 27.0528]s + [40.3996, 41.451] \tag{5.34}$$

The second order reduced order polynomial is formed from the ninth row of Table 5.5

$$P_2(s) = [2.1, 2.6]s^2 + [69.9279, 71.7148]s + [25.4243, 27.0528] \tag{5.35}$$

The reduced order polynomial stability can be verified by Kharitonov theorem [192]. For above problem some of the existing methods [156, 157, 161, 162, 169, 172] fails to give the stable reduced order polynomial.

Example 5.2: We consider a numerical example reported by Hwang and Yang [159], where the $\gamma - \delta$ method of interval system proposed by Bandyopadhyay et al., [157] fails to produce a fifth order reduced model. The proposed method successfully provides a stable reduced order system. The sixth order polynomial is as follows

$$P(s) = [9, 9.5]s^6 + [31, 31.5]s^5 + [71, 71.5]s^4 + [111.0, 111.5]s^3 + [119, 119.5]s^2 + [76, 76.5]s + [2, 2.5] \tag{5.36}$$

Case1: Modified differentiation method

The fifth order reduced order polynomial is

$$P_5(s) = [4.7448, 5.6615]s^5 + [23.3238, 24.1572]s^4 + [55.2389, 55.9889]s^3 + [79.1587, 79.8254]s^2 + [63.2475, 63.8008]s + [2, 2.5] \quad (5.37)$$

The second order reduced order polynomial is

$$p_2(s) = [6.7646, 9.1468]s^2 + [24.8576, 25.9643]s + [2, 2.5] \quad (5.38)$$

The reduced order polynomial stability can be verified by Kharitonov theorem [192].

Case2 : modified Schwarz approximation

The reciprocal transformation on $P(s)$ yields

$$\hat{P}(s) = [2, 2.5]s^6 + [76, 76.5]s^5 + [119, 119.5]s^4 + [111.0, 111.5]s^3 + [71, 71.5]s^2 + [31, 31.5]s + [9, 9.5] \quad (5.39)$$

Following the algorithm in section 5.4, Table 5.6 is constructed

Table 5.6: Modified Schwarz Approximation of example 5.2

[2,2.5]	[76,76.5]	[119,119.5]	[111,111.5]	…..
[76,76.5]	0	[111,111.5]	0	……
[76,76.5]	*[115.710, 116.225]*	[111,111.5]	*[70.0708, 70.5855]*	……
[115.710,116.225]	0	[70.0708, 70.5855]	0	…….
[115.710,116.225]	*[87.7986, 88.4677]*	[70.0708, 70.5855]	*[27.8774, 28.5416]*	……

[87.7986,88.4677]	0	[27.8774, 28.5416]	0	
[87.7986,88.4677]	*[32.5156,33.9044]*	[27.8774,28.5416]	[9,9.5]	
[32.5156,33.9044]	0	[9,9.5]		
[32.5156,33.9044]	*[2.6663,4.6575]*	[9,9.5]		
[2.6663,4.6575,]	0			
[2.6663,4.6575]	[9,9.5]			
[9,9.5]				

The italic elements in the table are required to be calculated.

From Table 5.6, we obtain the following polynomials (coefficients of the odd row in reverse order)

The fifth order reduced order polynomial is

$$\begin{aligned} P_5(s) = [9,9.5]s^5 + [31,31.5]s^4 + [70.0708,70.5855]s^3 + [111,111.5]s^2 \\ + [115.7108,116.2255]s + [76,76.5] \end{aligned} \quad (5.40)$$

The fourth order reduced order polynomial is formed from the third row of Table 5.6

$$\begin{aligned} P_4(s) = [9,9.5]s^4 + [27.8774,28.5416]s^3 + [70.0708,70.5855]s^2 \\ + [87.7986,88.4677]s + [115.7108,116.2255] \end{aligned} \quad (5.41)$$

The third order reduced order polynomial is formed from the third row of Table 5.6

$$P_3(s) = [9,9.5]s^3 + [27.8774,28.5417]s^2 + [32.5156,33.9044]s + [87.7986,88.4677] \quad (5.42)$$

The second order reduced order polynomial is formed from the ninth row of Table 5.6

$$P_2(s) = [9,9.5]s^2 + [2.6663,4.6575]s + [32.5156,33.9044] \quad (5.43)$$

The reduced order polynomial stability can be verified by using Kharitonov theorem [192].

Example 5.3: We consider an interval system given by [157,161]

$$G(s) = \frac{[2,3]s^2 + [17.5,18.5]s + [15,16]}{[2,3]s^3 + [17,18]s^2 + [35,36]s + [20.5,21.5]} \tag{5.45}$$

Case1: Modified differentiation method

By using proposed method the second order model is obtained as

$$R_2(s) = \frac{[8.25,9.75]s + [15,16]}{[5.12,6.78]s^2 + [23.12,24.45]s + [20.5,21.5]} \tag{5.46}$$

The step response of the high-order system and reduced order models by proposed method is shown in Fig. 5.3 and Fig. 5.4. A comparison has been made with the existing methods

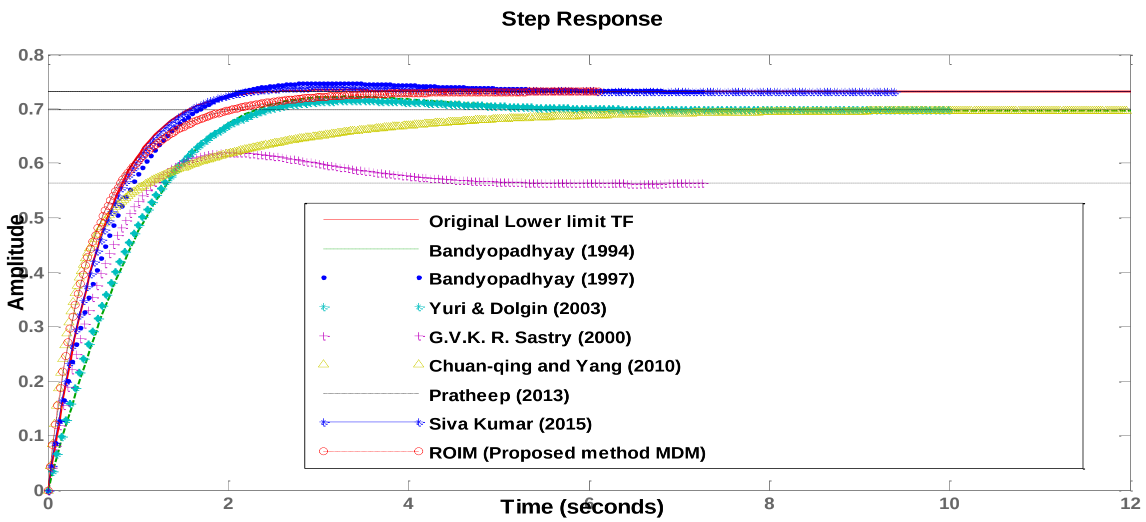

Fig.5.3. Step response for Modified Differentiation method (lower limit TF)

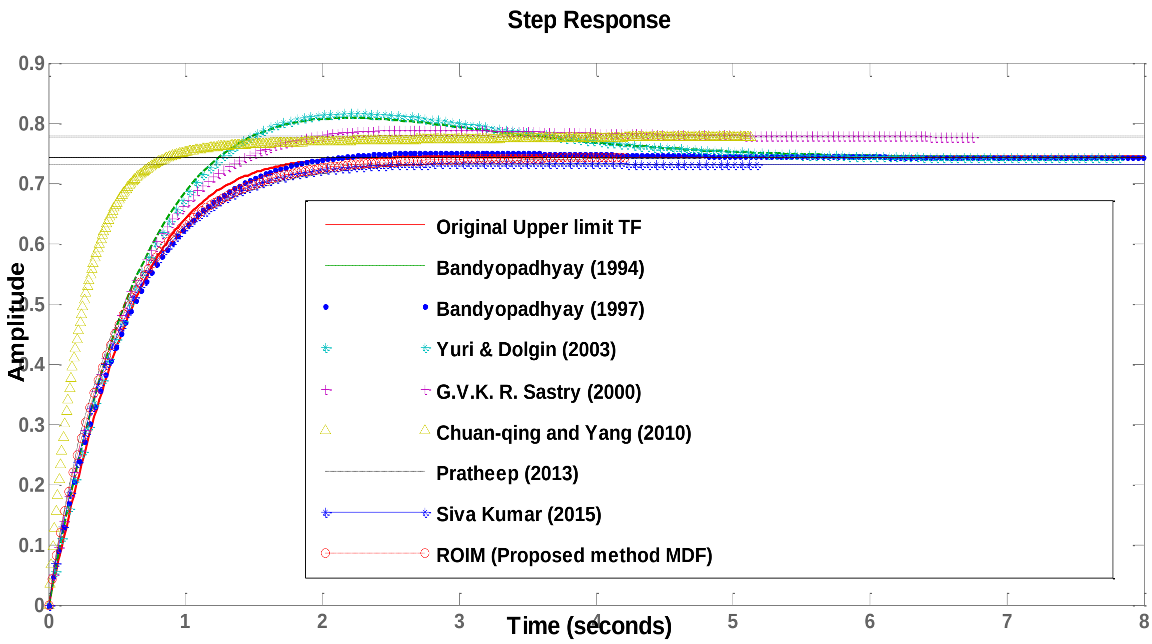

Fig.5.4. Step response for Modified Differentiation method (upper limit TF)

The impulse response of the high-order system and reduced order models by proposed method is shown in Fig. 5.5 and Fig. 5.6. A comparison has been made with the existing methods

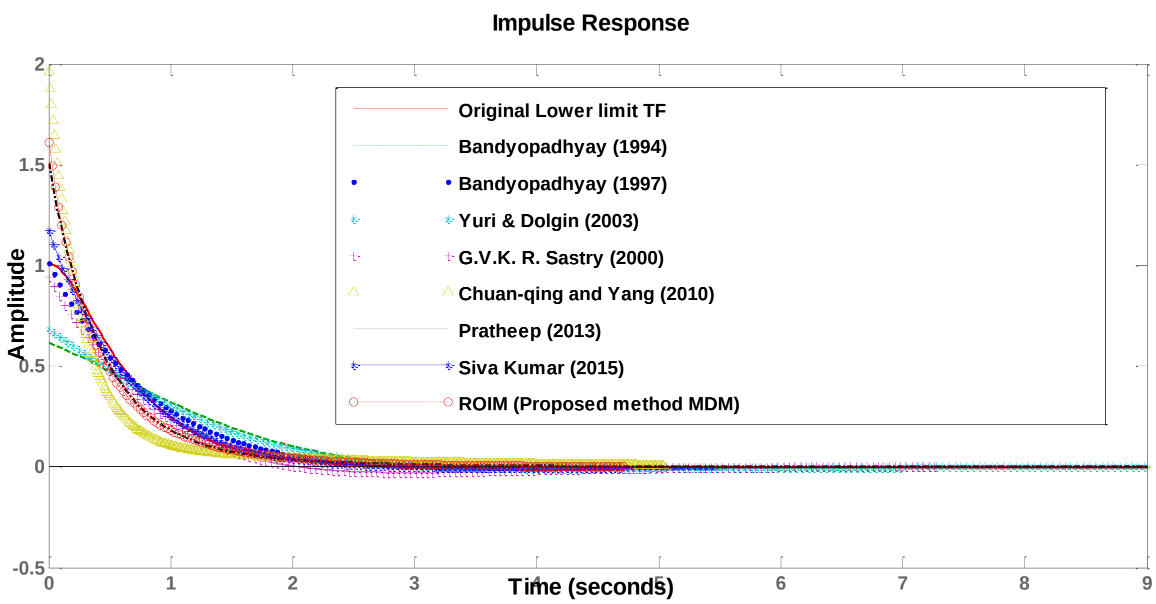

Fig.5.5. Impulse response for Modified Differentiation method (lower limit TF)

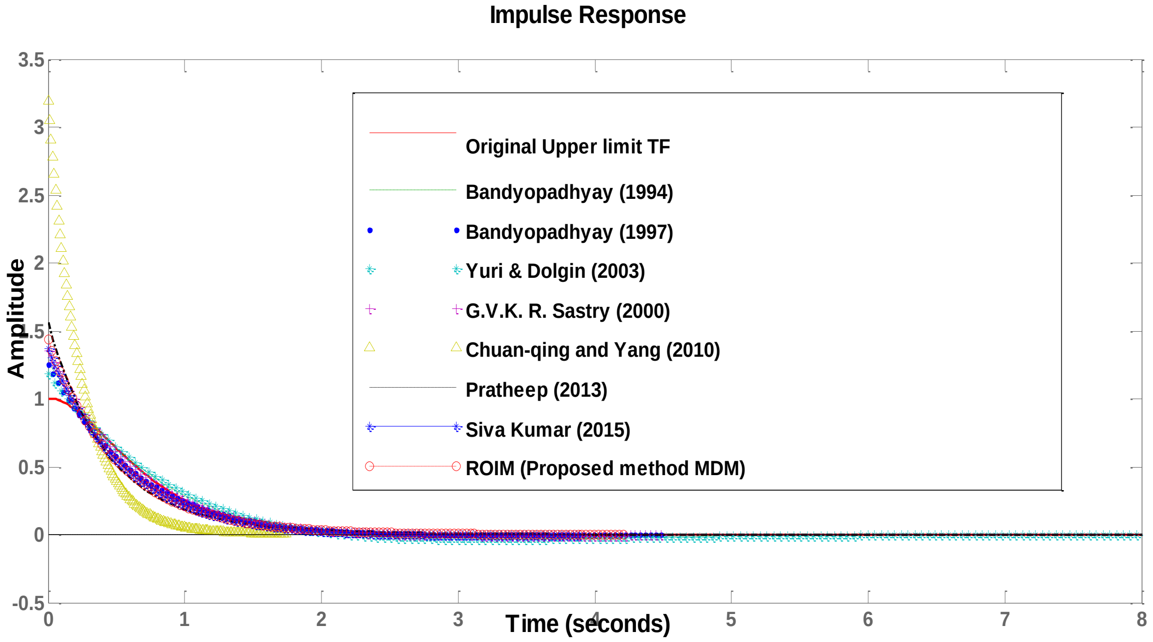

Fig.5.6. Impulse response for Modified Differentiation method (upper limit TF)

The bode plot of the high-order system and reduced order models by proposed method is shown in Fig. 5.7 and Fig. 5.8. A comparison has been made with the existing methods

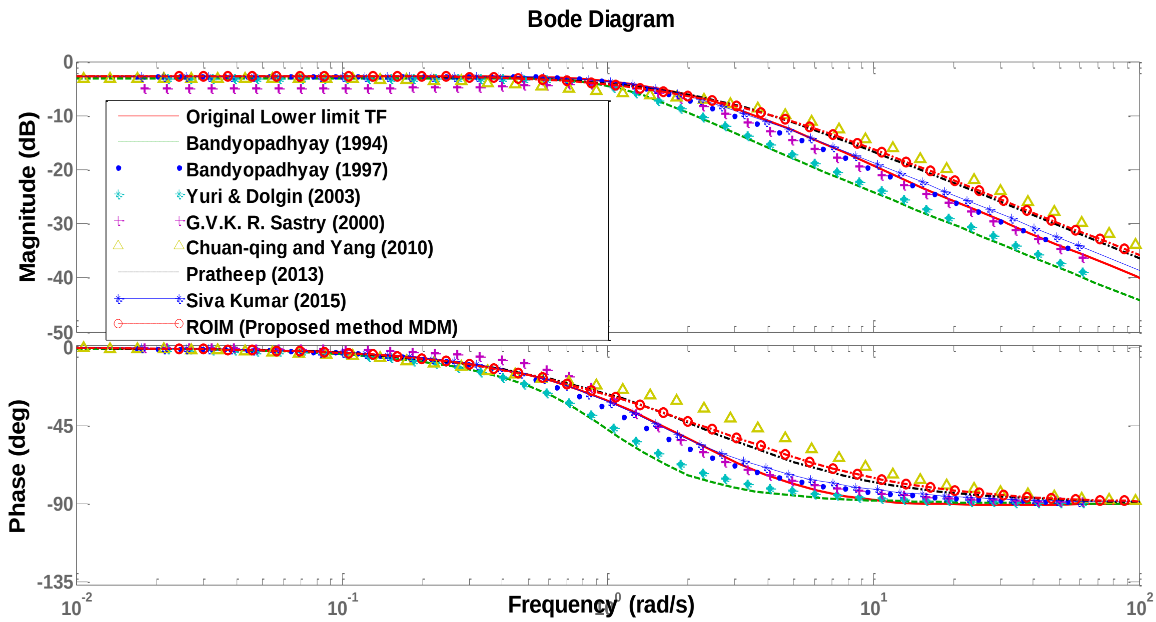

Fig.5.7. Bode plot for Modified Differentiation method (lower limit TF)

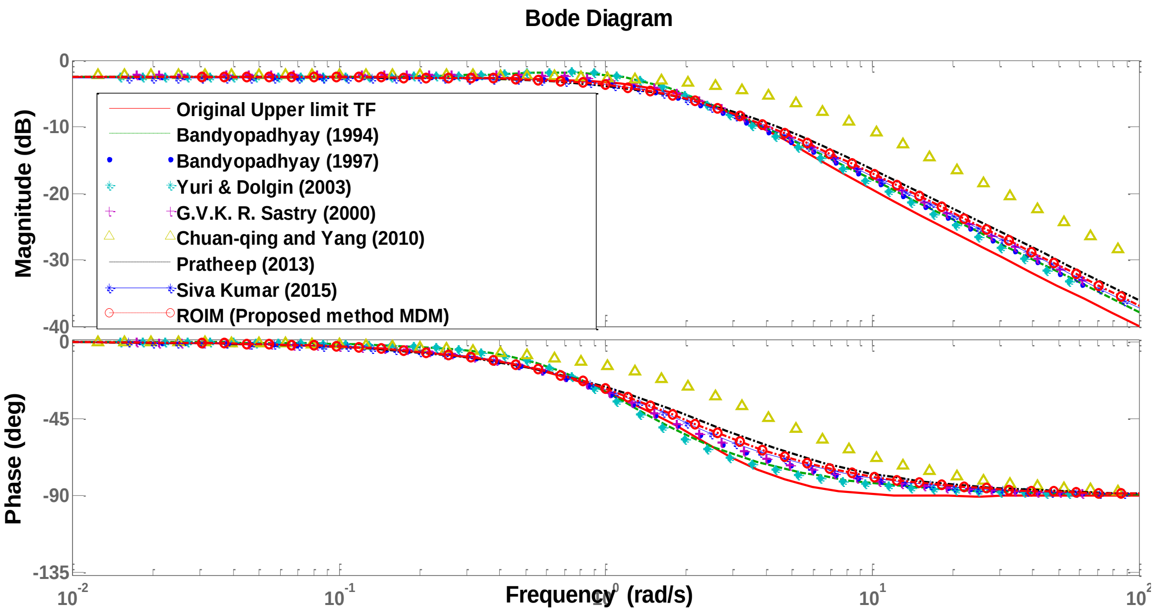

Fig.5.8. Bode plot for Modified Differentiation method (upper limit TF)

The nyquist plot of the high-order system and reduced order models by proposed method is shown in Fig. 5.9 and Fig. 5.10. A comparison has been made with the existing methods

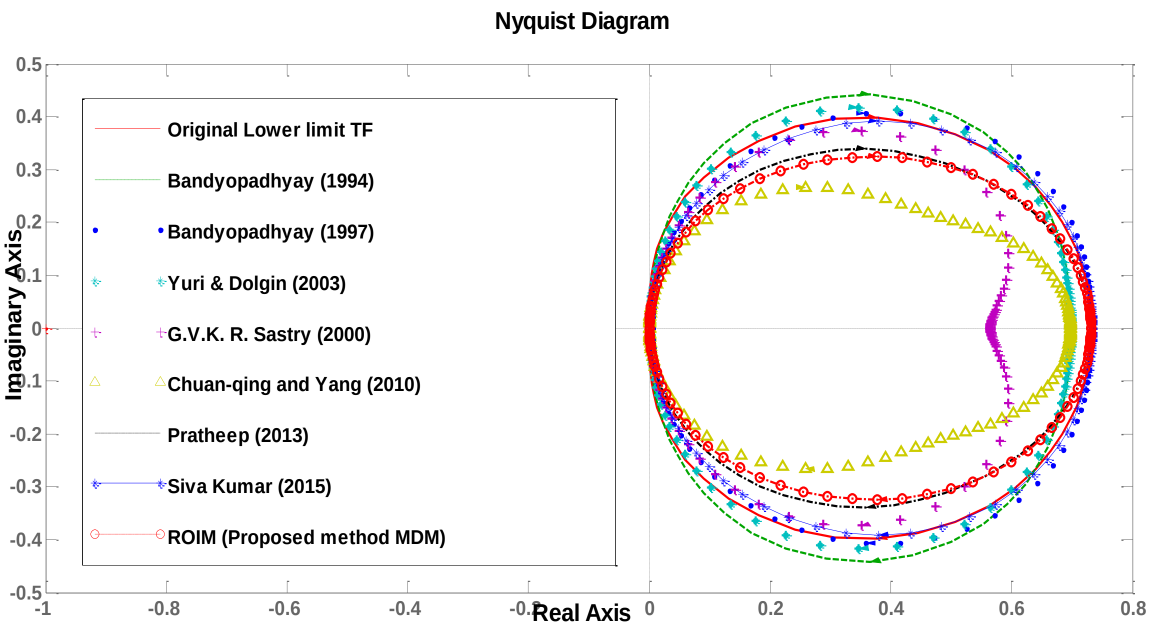

Fig.5.9. Nyquist plot for Modified Differentiation method (lower limit TF)

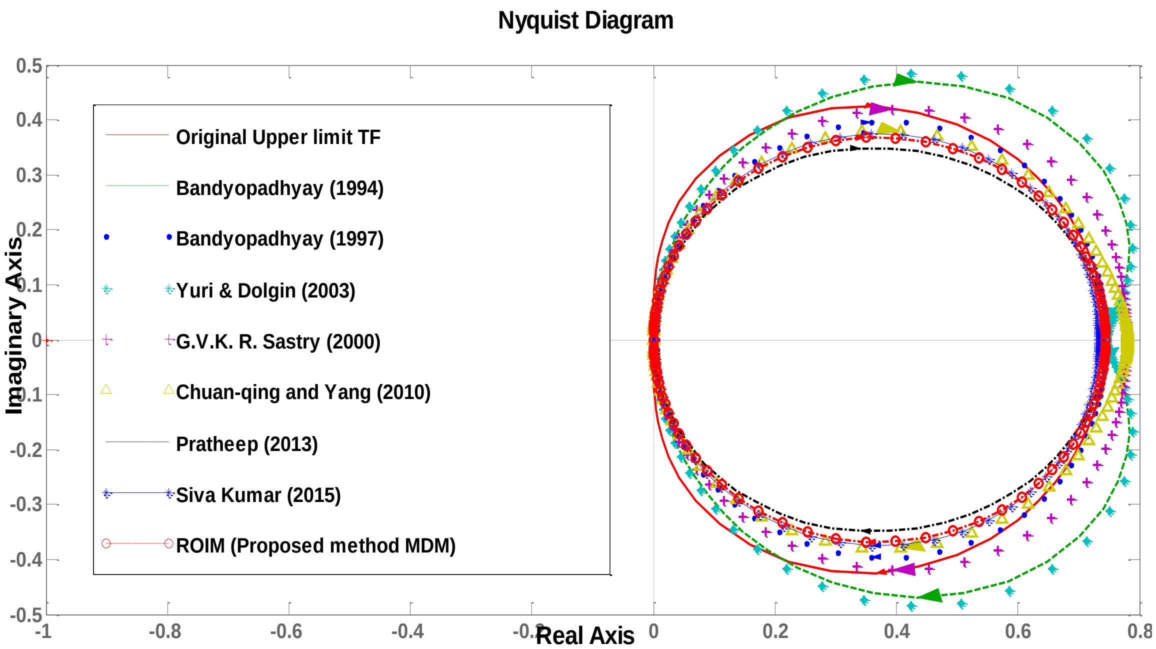

Fig.5.10. Nyquist plot for Modified Differentiation method (upper limit TF)

Table 5.7: Comparison of methods

Methods	Overshoot (%)	Rise time (Sec)	Settling time (Sec)	Steady state
Original system	0.413	1.15	1.86	0.732
Proposed method	0	1.36	2.7	0.732
Bandyopadhyay et al., [157]	2.13	1.36	3.47	0.732
Sastry et al., [161]	9.93	0.847	4.1	0.564

Case 2: Modified Schwarz approximation

The second order reduced model is obtained directly from third row of Table 5.7 and Table 5.8.

Table 5.8: Construction of denominator array of example 5.3

[20.5,21.5]	[35,36]	[17,18]	[2,3]
[35,36]	0	[2,3]	
[35,36]	*[15.2255,16.817]*	[2,3]	
[15.2255,16.817]	0		
[15.2255,16.817]	[2,3]		
[2,3]			

Table 5.9: Construction of numerator array of example 5.3

[15,16]	[17.5,18.5]	[2,3]
[35,36]	0	[2,3]
[17.5,18.5]	*[0.6902,2.1268]*	
[15.2255,16.817]	0	
[0.6902,2.1268]		
[2,3]		

The italic elements in the table are required to be calculated

$$\hat{R}_2(s) = \frac{[0.6902, 2.1268]s + [17.5, 18.5]}{[2,3]s^2 + [15.2255, 16.817]s + [35, 36]} \tag{5.47}$$

The above model has large steady state error in the step response. This can be removed by comparing dc gain of the higher order system and reduced order model which results in a gain factor $K=[1.32, 1.6056]$. Finally the reduced order model is

$$R_2(s)=\frac{K\left([0.6902, 2.1268]s+[17.5, 18.5]\right)}{[2,3]s^2+[15.2255, 16.817]s+[35,36]}$$
$$=\frac{[0.9111, 3.4148]s+[23.1, 29.7036]}{[2,3]s^2+[15.2255, 16.817]s+[35,36]} \quad (5.48)$$

The model obtained by Bandyopadhyay et al., [157] is as follows

$$R_2(s)=\frac{[1.0091, 1.2552]s+[0.8409, 1.1168]}{s^2+[2.0181, 2.4430]s+[1.1492, 1.5007]} \quad (5.49)$$

The model obtained by Sastry et al., [161] is given by

$$R_2(s)=\frac{[0.94, 1.35]s+[0.8409, 1.168]}{s^2+[2.0181, 2.4430]s+[1.1492, 1.5007]} \quad (5.50)$$

A comparison of the step response of the model obtained by proposed method and existing methods is shown in Fig. 5.11and 5.12. The comparison of the proposed method with existing methods for a reduced model is given Table 5.9. The comparison of error for lower limit and upper limit of the proposed method and existing methods is given in Table 5.10.

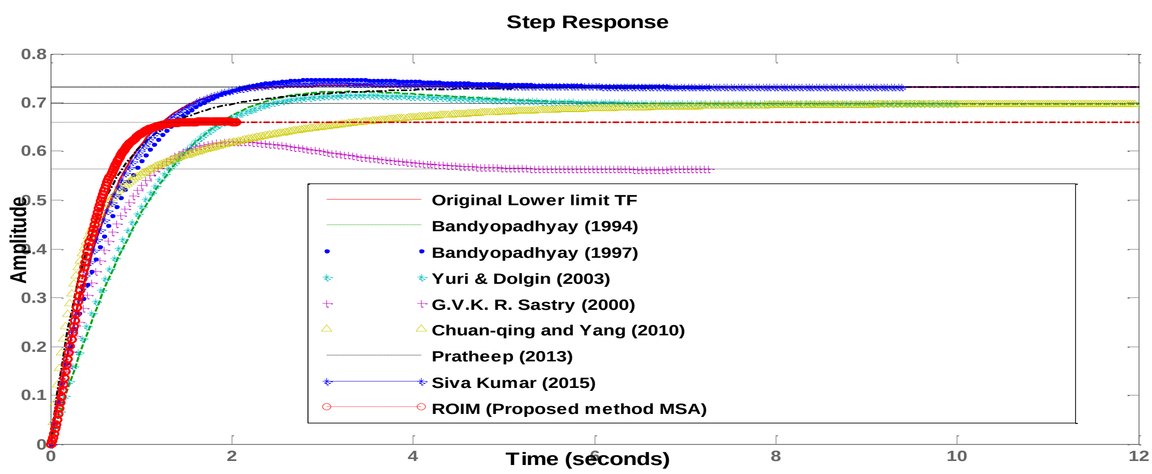

Fig.5.11. Step response for Modified Schwarz Approximation (lower limit TF)

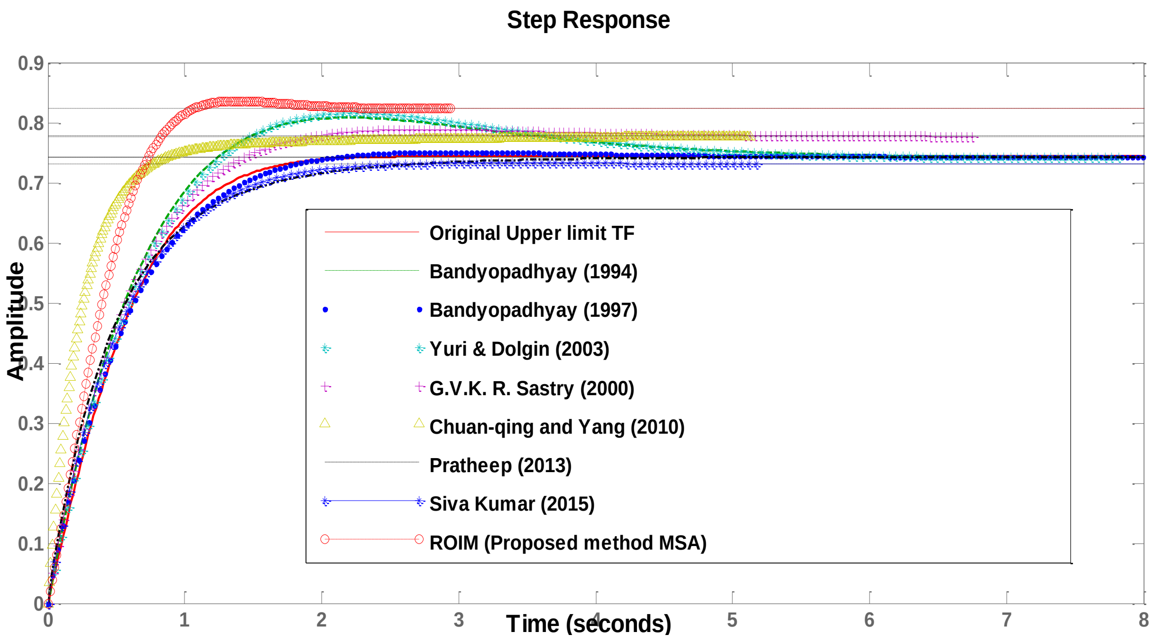

Fig.5.12. Step response for Modified Schwarz Approximation (upper limit TF)

The impulse response of the high-order system and reduced order models by proposed method is shown in Fig. 5.13 and Fig. 5.14. A comparison has been made with the existing methods

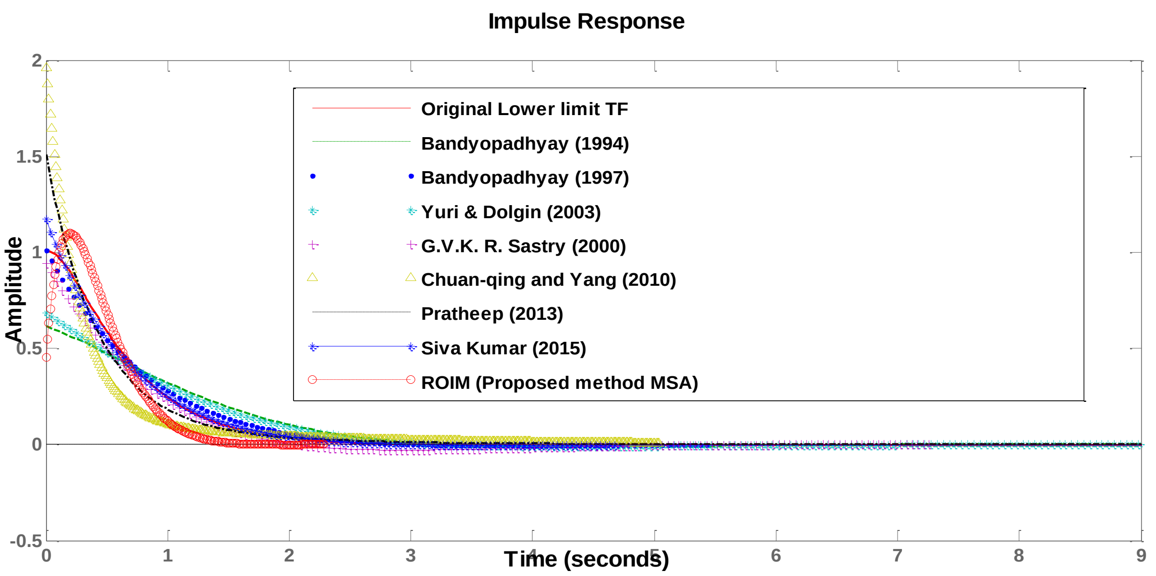

Fig.5.13. Impulse response for Modified Schwarz Approximation (lower limit TF)

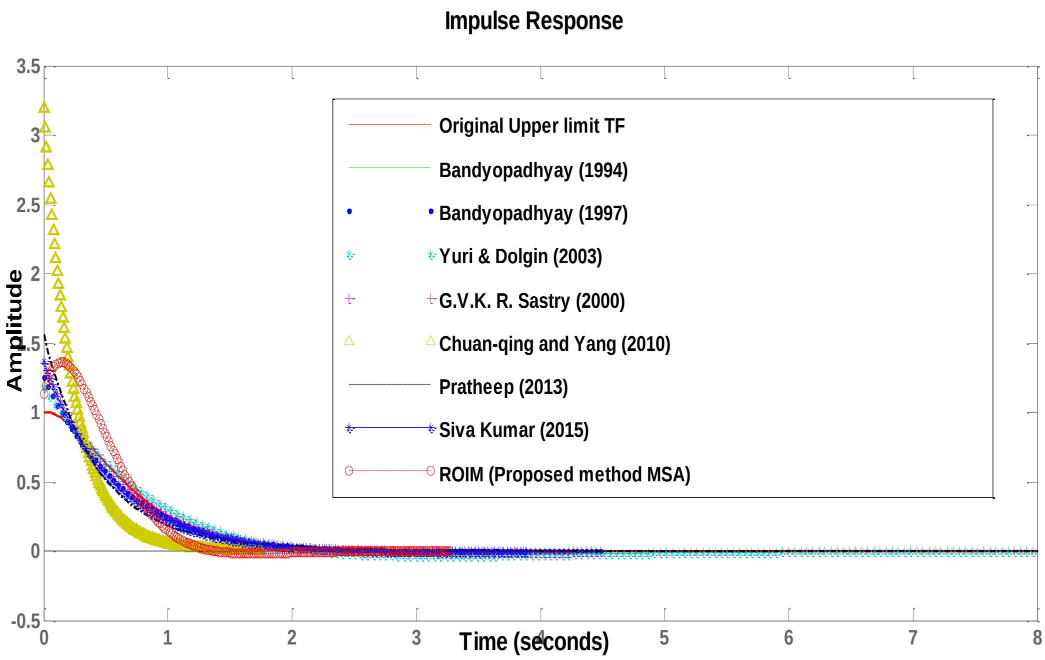

Fig.5.14. Impulse response for Modified Schwarz Approximation (upper limit TF)

The bode plot of the high-order system and reduced order models by proposed method is shown in Fig. 5.15 and Fig. 5.16. A comparison has been made with the existing methods

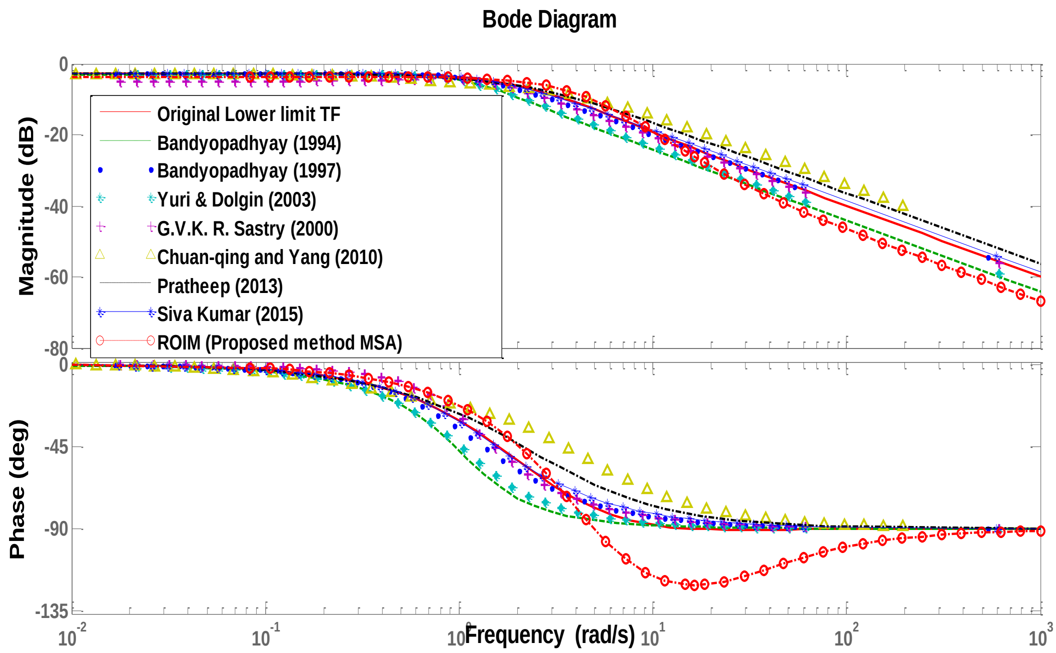

Fig.5.15. Bode plot for Modified Schwarz Approximation (lower limit TF)

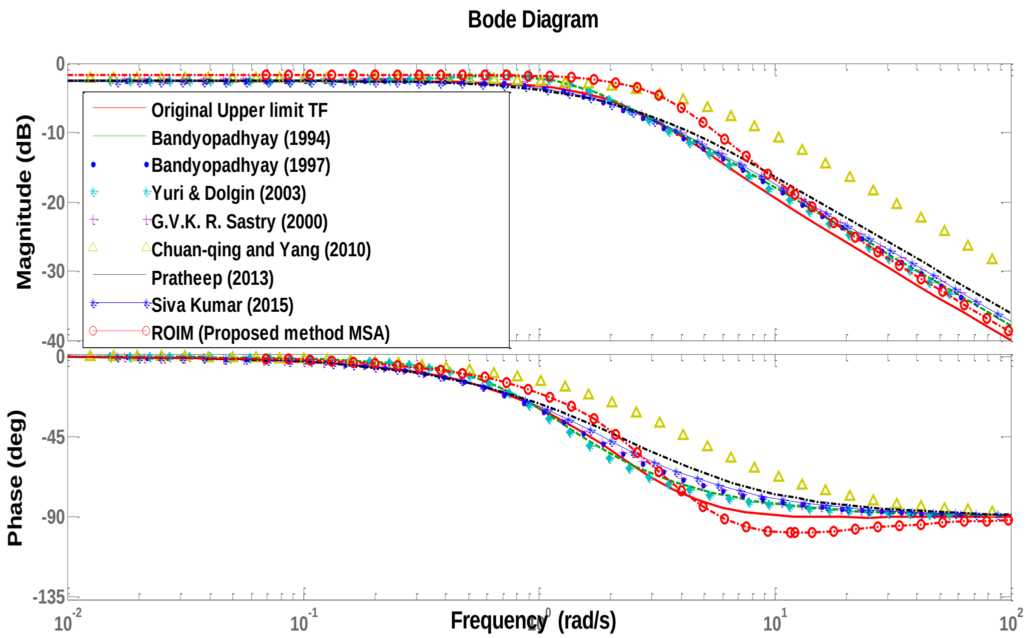

Fig.5.16. Bode plot for Modified Schwarz Approximation (upper limit TF)

The nyquist plot of the high-order system and reduced order models by proposed method is shown in Fig. 5.17 and Fig. 5.18. A comparison has been made with the existing methods

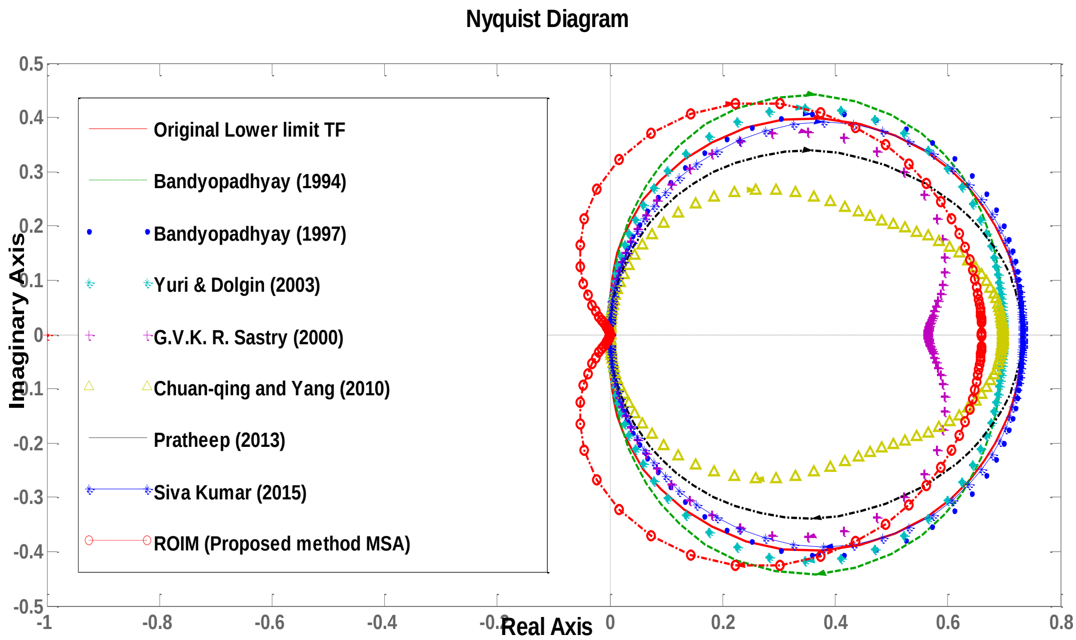

Fig.5.17. Nyquist plot for Modified Schwarz Approximation (lower limit TF)

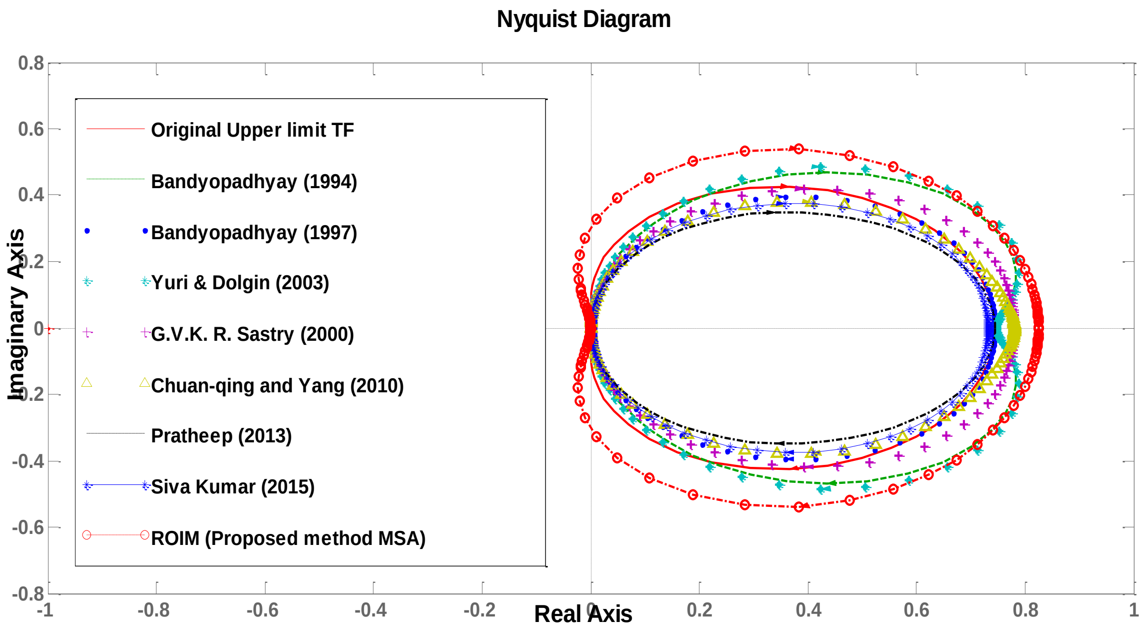

Fig.5.18. Nyquist plot for Modified Schwarz Approximation (upper limit TF)

Table 5.10: Comparison of methods

Methods	Overshoot (%)		Rise time (Sec)		Settling time (Sec)		Steady state	
	Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit
Original system	0.413	0.146	1.15	1.05	1.86	1.68	0.732	0.774
Proposed method	0.103	1.5	0.693	0.666	1.11	0.954	0.66	0.825
Bandyopadhyay [157]	0.0873	1.07	2.04	1.14	3.33	1.76	1.09	0.774
Sastry [161]	9.92	1.49	0.847	1.09	4.1	1.64	0.564	0.778

Table 5.11: Comparison of reduced order models of example 5.3

S.No	Methods	ISE		IAE	
		Lower limit	Upper limit	Lower limit	Upper limit
1	Bandypodayay et al., [156]	0.00878538	8.87354E-05	0.264278635	0.009542405
2	Bandypodayay et al., [157]	2.36319E-05	4.39626E-06	0.005001512	0.002108643
3	Dolgin and Zeheb [162]	0.008876522	8.01481E-05	0.265977631	0.009118821
4	Sastry et al., [161]	0.225673362	0.00949689	1.343641109	0.275550326
5	Chuan-qing and Yang [171]	0.010817047	0.010362791	0.290640962	0.287856756
6	Pratheep et al., [176]	1.19446E-05	9.93222E-07	0.003594415	0.001092456
7	Siva Kumar et. al [177]	2.93246E-06	0.001205443	0.002187121	0.098172255
8	Proposed Method (MDM)	3.43581E-06	6.63119E-08	0.001861249	0.00025824
9	Proposed Method (MSAM)	0.041274021	0.052353858	0.574617967	0.647171326