CHAPTER 3
EXISTING TECHNIQUES ON MODEL ORDER REDUCTION OF INTERVAL SYSTEMS

3.1 INTRODUCTION

The model reduction techniques for both continuous and discrete time interval systems [156]- [184] have received a great deal of attention. Bandyopadhyay et al., [156] has extended Routh-Pade approximation to interval systems for reducing higher-order continuous interval systems. The reduced order denominator polynomial is obtained by direct truncation of the Routh table, and the lower-order numerator polynomial is obtained by matching the coefficients of the power series expansions of the interval systems. Later the concept of γ-δ Routh approximation has been extended to continuous interval systems by Bandyopadhyay et al., [157]. The following is the limitations of above two Routh based approximations claimed by Hwang and Yang [159]: (i) Interval Routh extension formula cannot guarantee the successes in generating a full interval Routh array. (ii) Some interval Routh approximation may not be robustly stable, even if the original interval system is stable. To reduce the computational effort, γ Table formulation [161] has been introduced, instead of γ-δ Table formulation [157]. However, the limitation of this method is that, they obtained reduced interval model may be unstable for the stable original interval model. Later, Dolgin and Zeheb [162] have proven that generalized Routh algorithm to interval systems does not guarantee the stability of the reduced order system. To overcome this problem, Dolgin and Zeheb [162] modified the generalized Routh array and claimed that this method could guarantee the stability of the reduced order system. Later, Yang [165] proved that Dolgin and Zeheb [162] method does not guarantee the stability of the reduced order interval system. To overcome this problem, Dolgin [166] has proposed a modified method of Routh algorithm for obtaining stable reduced order models. It is noted that there is a limitation in this method, that the interval arithmetic subtraction rule has been changed to obtain stable reduced order models. To overcome the limitation of the existing methods [156]-[157], [162], [166] Bandyopadhyay et al., [170] introduced a new method based on stable $\gamma - \delta$ Routh approximation of interval systems using Khartitonov polynomials, which guarantee the stability of the reduced order systems. However, this method does not require any interval arithmetic rules. Another alternative method has also been proposed to overcome the limitation of the $\gamma-\delta$ Routh approximation, which is based on stable Routh- Pade approximation [171]. Ismail [157] and Shingare [167] extended some fixed model reduction techniques to interval systems. In this chapter only few existing techniques has been discussed which are highly motivated for this to develop new algorithms in this thesis.

3.2 PROBLEM FORMULATION

Let the transfer function of a higher order continuous interval systems be

$$G_n(s)=\frac{[b_{1,1}^-,b_{1,1}^+]+[b_{1,2}^-,b_{1,2}^+]s+.....+[b_{1,n}^-,b_{1,n}^+]s^{n-1}}{[a_{1,1}^-,a_{1,1}^+]+[a_{1,2}^-,a_{1,2}^+]s+.....+[a_{1,n+1}^-,a_{1,n+1}^+]s^n}=\frac{\sum_{i=1}^{n}\left[b_{1i}^-,b_{1,i}^+\right]s^{i-1}}{\sum_{i=1}^{n+1}\left[a_{1i}^-,a_{1,i}^+\right]s^{i}}=\frac{N(s)}{D(s)} \tag{3.1}$$

where $[b_{1,i}^-,b_{1,i}^+]$ for i = 1 to n and $[a_{1,i}^-,a_{1,i}^+]$ for i = 1 to n+1 are the uncertain parameters.

The reduced order model $R_k(s)$ is expressed as

$$R_k(s)=\frac{N_k(s)}{D_k(s)}=\frac{[d_{1,1}^-,d_{1,1}^+]+[d_{1,2}^-,d_{1,2}^+]s+.....+[d_{1,k}^-,d_{1,k}^+]s^{k-1}}{[c_{1,1}^-,c_{1,1}^+]+[c_{1,2}^-,c_{1,2}^+]s+.....+[c_{1,k+1}^-,c_{1,k+1}^+]s^k} \tag{3.2}$$

where k = 1 to n-1

The rules of the interval arithmetic have been defined in Chapter 1.

3.3 ROUTH BASED APPROXIMATIONS

3.3.1 Direct Routh Approximation for Interval Systems

This method is used for the model reduction of the order of interval systems. The denominator of the reduced model is obtained by a direct truncation of the Routh table of the interval systems. The numerator of the reduced polynomial is obtained by matching the coefficients of power series expansions of the interval system [156].

The denominator of model order transfer function can be constructed from the Routh stability array of the denominator of the system transfer function $D_k(s)$ as follows. The Routh Table of the denominator of the interval system is given below

Table 3.1: The Routh table for the denominator of the interval system

$\left[a_{1,n+1}^{-}, a_{1,n+1}^{+}\right] = \left[f_{11}^{-}, f_{11}^{+}\right]$	$\left[a_{1,n-1}^{-}, a_{1,n-1}^{+}\right] = \left[f_{1,2}^{-}, f_{1,2}^{+}\right]$	$\left[a_{1,n-3}^{-}, a_{1,n-3}^{+}\right] = \left[f_{1,3}^{-}, f_{1,3}^{+}\right]$	……	……
$\left[a_{1,n}^{-}, a_{1,n}^{+}\right] = \left[f_{2,1}^{-}, f_{2,1}^{+}\right]$	$\left[a_{1,n-2}^{-}, a_{1,n-2}^{+}\right] = \left[f_{2,2}^{-}, f_{2,2}^{+}\right]$	$\left[a_{1,n-4}^{-}, a_{1,n-4}^{+}\right] = \left[f_{2,3}^{-}, f_{2,3}^{+}\right]$	……	……
$\left[f_{3,1}^{-}, f_{3,1}^{+}\right]$	$\left[f_{3,2}^{-}, f_{3,2}^{+}\right]$	$\left[f_{3,3}^{-}, f_{3,3}^{+}\right]$	……	
$\left[f_{4,1}^{-}, f_{4,1}^{+}\right]$	$\left[f_{4,2}^{-}, f_{4,2}^{+}\right]$	………		
……….	……….			
……….	………..			

From the above Table 3.1, the first two rows can be obtained from original systems. The remaining rows can be obtained from Eq. (3.3)

$$\left[f_{i,j}^{-}, f_{i,j}^{+}\right] = \frac{\left(\left[f_{i-2,j+1}^{-}, f_{i-2,j+1}^{+}\right]\cdot\left[f_{i-1,1}^{-}, f_{i-1,1}^{+}\right]\right) - \left(\left[f_{i-2,1}^{-}, f_{i-2,1}^{+}\right]\cdot\left[f_{i-1,j+1}^{-}, f_{i-1,j+1}^{+}\right]\right)}{\left[f_{i-1,1}^{-}, f_{i-1,1}^{+}\right]} \tag{3.3}$$

where $i \geq 3, 1 \leq j \leq \left[\frac{(n-i+3)}{2}\right]$.

3.3.2 $\gamma-\delta$ Approximation for Interval Systems

The method represents the $\gamma-\delta$ Routh approximation [157] for interval systems. The higher order interval systems are evaluated for the intervals $\gamma's$ and $\delta's$ and then reduced order approximant is obtained by retaining the first k, interval $\gamma's$ and $\delta's$.

For reducing the higher order denominator interval polynomial form Eq. (3.1), construct γ Table 3.2.

Table 3.2: γ table

$\left[a_{1,1}^{-},a_{1,1}^{+}\right]=\left[x_{00}^{-},x_{00}^{+}\right]$	$\left[a_{1,3}^{-},a_{1,3}^{+}\right]=\left[x_{0,1}^{-},x_{0,1}^{+}\right]$	$\left[a_{1,5}^{-},a_{1,5}^{+}\right]=\left[x_{0,2}^{-},x_{0,2}^{+}\right]$	…….	……
$\left[a_{1,2}^{-},a_{1,2}^{+}\right]=\left[x_{1,0}^{-},x_{1,0}^{+}\right]$	$\left[a_{1,4}^{-},a_{1,4}^{+}\right]=\left[x_{1,1}^{-},x_{1,1}^{+}\right]$	$\left[a_{1,6}^{-},a_{1,6}^{+}\right]=\left[x_{1,2}^{-},x_{1,2}^{+}\right]$	……..	……
$\left[x_{2,0}^{-},x_{2,0}^{+}\right]$	$\left[x_{2,1}^{-},x_{2,1}^{+}\right]$	$\left[x_{2,2}^{-},x_{2,2}^{+}\right]$	…… …	
…………..	…………..	……..		
……….	……….			
$\left[x_{n-1,0}^{-},x_{n-1,0}^{+}\right]$				
$\left[x_{n,0}^{-},x_{n,0}^{+}\right]$				

The elements for denominator polynomial $\left[x_{i,j}^{-},x_{i,j}^{+}\right]$ can be obtained by applying the algorithm as shown in Eq. (3.4)

$$\left[x_{i,j}^{-},x_{i,j}^{+}\right]=\frac{\left[x_{i-2,j+1}^{-},x_{i-2,j+1}^{+}\right]\left[x_{i-1,0}^{-},x_{i-1,0}^{+}\right]-\left[x_{i-2,0}^{-},x_{i-2,0}^{+}\right]\left[x_{i-1,j+1}^{-},x_{i-1,j+1}^{+}\right]}{\left[x_{i-1,0}^{-},x_{i-1,0}^{+}\right]} \qquad (3.4)$$

where $i=2,3,4,...,n+1$ and $j=0,1,2,3,....,n+1$

The interval parameters can be defined as the ratios of the first column elements of the above Routh like table as given in

$$\left[\gamma_k^-,\gamma_k^+\right]=\frac{\left[x_{k-1,0}^-,x_{k-1,0}^+\right]}{\left[x_{k,0}^-,x_{k,0}^+\right]},k=1,2,3,....,n+1. \quad (3.5)$$

For reducing the higher order numerator interval polynomial form Eq. (3.1), construct $\delta-$ table.

Table 3.3: δ table

$\left[b_{1,1}^-,b_{1,1}^+\right]=\left[y_{10}^-,y_{10}^+\right]$	$\left[b_{1,3}^-,b_{1,3}^+\right]=\left[y_{1,1}^-,y_{1,1}^+\right]$	$\left[b_{1,5}^-,b_{1,5}^+\right]=\left[y_{1,2}^-,y_{1,2}^+\right]$	…….	……
$\left[b_{1,2}^-,b_{1,2}^+\right]=\left[y_{2,0}^-,y_{2,0}^+\right]$	$\left[b_{1,4}^-,b_{1,4}^+\right]=\left[y_{2,1}^-,y_{2,1}^+\right]$	$\left[b_{1,6}^-,b_{1,6}^+\right]=\left[y_{2,2}^-,y_{2,2}^+\right]$	……..	……
$\left[y_{3,0}^-,y_{3,0}^+\right]$	$\left[y_{3,1}^-,y_{3,1}^+\right]$	$\left[y_{3,2}^-,y_{3,2}^+\right]$	……	
…………..	…………..	……..		
……….	……….			
$\left[y_{n-1,0}^-,y_{n-1,0}^+\right]$				
$\left[y_{n,0}^-,y_{n,0}^+\right]$				

The elements for numerator polynomial $\left[y_{i,j}^-,y_{i,j}^+\right]$ can be obtained by applying the algorithm as shown in Eq. (3.6)

$$\left[y_{i,j}^-,y_{i,j}^+\right]=\frac{\left[y_{i-2,j+1}^-,y_{i-2,j+1}^+\right]\left[x_{i-2,0}^-,x_{i-2,0}^+\right]-\left[y_{i-2,0}^-,y_{i-2,0}^+\right]\left[x_{i-2,j+1}^-,x_{i-2,j+1}^+\right]}{\left[x_{i-2,0}^-,x_{i-2,0}^+\right]} \quad (3.6)$$

where $i=2,3,4,...,n+1$ and $j=0,1,2,3,....,n+1$

The interval parameters can be defined as the ratios of the first column elements of the above Routh like table as given in

$$\left[\delta_k^-,\delta_k^+\right]=\frac{\left[y_{k,0}^-,y_{k,0}^+\right]}{\left[x_{k,0}^-,x_{k,0}^+\right]},k=1,2,3,4,....,n+1 \tag{3.7}$$

The reduced order model $R_k\left(s\right)$ can be written as

$$R_k\left(s\right)=\frac{N_k\left(s\right)}{D_k\left(s\right)} \tag{3.8}$$

where

$$D_k\left(s\right)=s^2D_{k-2}\left(s\right)+\left[\gamma_k^-,\gamma_k^+\right]D_{k-1}\left(s\right) \tag{3.9}$$

$$N_k\left(s\right)=\left[\delta_k^-,\delta_k^+\right]s^{k-1}+s^2N_{k-2}\left(s\right)+\left[\gamma_k^-,\gamma_k^+\right]N_{k-1}\left(s\right) \tag{3.10}$$

with

$$D_{-1}\left(s\right)=\frac{1}{s};D_0\left(s\right)=1;N_{-1}\left(s\right)=0;N_0\left(s\right)=0$$

The expression for $D_k\left(s\right)$ and $N_k\left(s\right)$ is the same as for the fixed system with the exception that $\gamma's$ and $\delta's$ are intervals. The first and second order Routh approximates are

$$R_1\left(s\right)=\frac{\left[\delta_1^-,\delta_1^+\right]}{s+\left[\gamma_1^-,\gamma_1^+\right]} \tag{3.11}$$

$$R_2\left(s\right)=\frac{\left[\delta_2^-,\delta_2^+\right]s+\left[\gamma_2^-,\gamma_2^+\right]\left[\delta_1^-,\delta_1^+\right]}{s^2+\left[\gamma_2^-,\gamma_2^+\right]s+\left[\gamma_2^-,\gamma_2^+\right]\left[\gamma_1^-,\gamma_1^+\right]} \tag{3.12}$$

3.3.3 γ Approximation for Interval Systems

This method is computationally simple and it requires only γ table [161] instead of both $\gamma-\delta$ table [157].

The reduced denominator is

For k = 1

$$D_1(s)=[1,1]s+\frac{\left[\gamma_1^-,\gamma_1^+\right]}{\left[a_{11}^-,a_{11}^+\right]}\left[a_{11}^-,a_{11}^+\right] \tag{3.13}$$

For k = 2

$$D_2(s)=[1,1]s^2+\frac{\left[\gamma_1^-,\gamma_1^+\right]\left[\gamma_2^-,\gamma_2^+\right]}{\left[a_{11}^-,a_{11}^+\right]}\left\{\left[a_{11}^-,a_{11}^+\right]+\left[a_{12}^-,a_{12}^+\right]s\right\} \tag{3.14}$$

Therefore the generalised algorithm is

$$D_k(s)=[1,1]s^k+\frac{\left[\gamma_k^-,\gamma_k^+\right]}{\left[a_{11}^-,a_{11}^+\right]}\left\{\left[a_{11}^-,a_{11}^+\right]+\left[a_{12}^-,a_{12}^+\right]s+.....+\left[a_{1k}^-,a_{1k}^+\right]s^{k-1}\right\} \tag{3.15}$$

The reduced numerator polynomial with $k \le n$

For k =1

$$N_1(s)=\frac{\left[\gamma_1^-,\gamma_1^+\right]}{\left[a_{11}^-,a_{11}^+\right]}\left[b_{11}^-,b_{11}^+\right] \tag{3.16}$$

For k = 2

$$N_2(s)=\frac{\left[\gamma_1^-,\gamma_1^+\right]\left[\gamma_2^-,\gamma_2^+\right]}{\left[a_{11}^-,a_{11}^+\right]}\left\{\left[b_{11}^-,b_{11}^+\right]+\left[b_{12}^-,b_{12}^+\right]s\right\} \tag{3.17}$$

The generalised algorithm is

$$N_k(s) = \frac{\left[\gamma_k^-, \gamma_k^+\right]}{\left[a_{11}^-, a_{11}^+\right]} \left\{ \left[b_{11}^-, b_{11}^+\right] + \left[b_{12}^-, b_{12}^+\right] s + ... + \left[b_{1k}^-, b_{1k}^+\right] s^{k-1} \right\} \tag{3.18}$$

The interval coefficients $\left[\gamma_k^-, \gamma_k^+\right]$ where k = 1, 2, 3, ... , n-1 are obtained from the proposed $\gamma = \left[\gamma^-, \gamma^+\right]$ Routh type interval table.

Table 3.4: $\left[\gamma^-, \gamma^+\right]$-table

$\left[\gamma_1^-, \gamma_1^+\right] = \langle \frac{\left[a_{11}^-, a_{11}^+\right]}{\left[a_{12}^-, a_{12}^+\right]}$	$\left[a_{1,1}^-, a_{1,1}^+\right] = \left[P_0^-, P_0^+\right]$	$\left[a_{1,2}^-, a_{1,2}^+\right] = \left[P_1^-, P_1^+\right]$	$\left[a_{1,3}^-, a_{1,3}^+\right] = \left[P_2^-, P_2^+\right]$	……
	$\left[a_{1,2}^-, a_{1,2}^+\right] = \left[P_1^-, P_1^+\right]$	$\left[a_{1,3}^-, a_{1,3}^+\right] = \left[P_2^-, P_2^+\right]$	. . . .	…….
$\left[\gamma_2^-, \gamma_2^+\right] = \langle \frac{\left[Q_1^-, Q_1^+\right]}{\left[Q_2^-, Q_2^+\right]}$	$\left[Q_1^-, Q_1^+\right]$	$\left[Q_2^-, Q_2^+\right]$	$\left[Q_3^-, Q_3^+\right]$	
	$\left[Q_2^-, Q_2^+\right]$	$\left[Q_3^-, Q_3^+\right]$	……..	
$\left[\gamma_3^-, \gamma_3^+\right] = \langle \frac{\left[R_1^-, R_1^+\right]}{\left[R_2^-, R_2^+\right]}$	$\left[R_1^-, R_1^+\right]$	$\left[R_2^-, R_2^+\right]$	$\left[R_3^-, R_3^+\right]$	
	$\left[R_2^-, R_2^+\right]$	$\left[R_3^-, R_3^+\right]$	. . . . .	
…………	. . . .			

for $i = odd$:

$$\left[Q_i^-, Q_i^+\right] = \left[P_i^-, P_i^+\right]; i = 1, 3, 5, ...$$

$$\left[R_1^-, R_1^+\right] = \left[Q_2^-, Q_2^+\right] \tag{3.19}$$

$$\left[R_i^-, R_i^+\right] = \left[P_{i+1}^-, P_{i+1}^+\right]; i = 3, 5, ...$$

for $i = even$

$$\left[Q_i^-, Q_i^+\right] = \left[P_i^-, P_i^+\right] - \frac{\left\{\left[P_{i+1}^-, P_{i+1}^+\right]\left[P_0^-, P_0^+\right]\right\}}{\left[P_1^-, P_1^+\right]}$$

$$\left[R_i^-, R_i^+\right] = \left[Q_{i+1}^-, Q_{i+1}^+\right] - \frac{\left\{\left[Q_1^-, Q_1^+\right]\left[Q_{i+2}^-, Q_{i+2}^+\right]\right\}}{\left[Q_2^-, Q_2^+\right]} \tag{3.20}$$

3.3.4 Modified Routh Approximation for Interval Systems

This method is an alternative method for direct Routh approximation (refer 3.3.1). It is shown that the existing generalization of the direct Routh approximation fails to produce a stable system. The modified Routh approximation pioneered by Dolgin and Zeheb [162].

The method is similar to the 3.3.1, but considers the following changes. Firstly, rewrite the formula (Eq. (3.3)) as follows

$$\left[f_{i,j}^-, f_{i,j}^+\right] = \left[f_{i-2,j+1}^-, f_{i-2,j+1}^+\right] - \frac{\left(\left[f_{i-2,1}^-, f_{i-2,1}^+\right]\cdot\left[f_{i-1,j+1}^-, f_{i-1,j+1}^+\right]\right)}{\left[f_{i-1,1}^-, f_{i-1,1}^+\right]},$$

$$i \geq 3, i \leq j \leq \left[\frac{(n-i+3)}{2}\right] \tag{3.21}$$

Example 3.1: Consider a seventh order polynomial

$$P_7(s) = s^7 + 9s^6 + [31, 34]s^5 + 71s^4 + 111s^3 + 109s^2 + [76, 83]s + 12 \tag{3.22}$$

Case1: Direct Routh Approximation [156]

Based on Table 3.1, construct direct Routh approximation for example 3.1

Table 3.5: Direct Routh Approximation of example 3.1

$[1,1]$	$[31,34]$	$[111,111]$	$[76,83]$
$[9,9]$	$[71,71]$	$[109,109]$	$[12,12]$
$[23.11,26.11]$	$[98.89,98.89]$	$[74.67,81.67]$	
$[28.76,41.71]$	$[68.33,94.07]$	$[12,12]$	
$[9.3,88.51]$	$[43.98,108.81]$		
$[-419.68,759.28]$	$[12,12]$		
$[-196.60,111.34]$			
$[12,12]$			

The first column of the last two rows consists of two negative elements. The reduced order polynomial will not be stable.

Case2: Modified Routh Approximation [162]

Based on Eq. (3.21) construct Table 3.6.

Table 3.6: Modified Routh Approximation of example 3.1

$[1,1]$	$[31,34]$	$[111,111]$	$[76,83]$
$[9,9]$	$[71,71]$	$[109,109]$	$[12,12]$
$[23.11,26.11]$	$[98.89,98.89]$	$[74.67,74.67]$	
$[34.84,34.84]$	$[79.14,81.70]$	$[12,12]$	
$[41.18,42.98]$	$[66.19,73.19]$		
$[18.54,26.90]$	$[12,12]$		
$[43.96,50.96]$			
$[12,12]$			

From Table 3.6, we can obtain 6th order and 5th order polynomials

$$P_6(s) = [9,9]s^6 + [23.11, 26.11]s^5 + [71,71]s^4 + [98.89, 98.89]s^3 + [109,109]s^2 + [74.67, 81.67]s + [12,12] \quad (3.23)$$

$$P_5(s) = [23.11, 26.11]s^5 + [34.84, 34.84]s^4 + [98.89, 98.89]s^3 + [79.14, 81.70]s^2 + [74.67, 81.67]s + [12,12] \quad (3.24)$$

The stability of the above polynomial can be verified by Kharitonov's theorem [192].

Example 3.2: Consider a sixth order polynomial [165] and reduced the system using modified Routh approximation

$$P_6(s) = [2.1, 2.6]s^6 + [76.1, 76.7]s^5 + [119.1, 119.6]s^4 + [111.0, 111.6]s^3 + [71.8, 72.3]s^2 + [31.0, 31.7]s + [9.0, 9.9] \quad (3.25)$$

Table 3.7: Modified Routh Approximation of example 3.2

$[2.1, 2.6]$	$[119.1, 119.6]$	$[71.8, 72.3]$	$[9.0, 9.9]$
$[76.1, 76.7]$	$[111.0, 111.6]$	$[31.0, 31.7]$	
$[115.28, 116.57]$	$[70.72, 71.45]$	$[9.0, 9.9]$	
$[63.46, 65.43]$	$[24.41, 25.83]$		
$[23.28, 28.45]$	$[9.0, 9.9]$		
$[-3.42, 5.76]$			
$[9.0, 9.9]$			

From Table 3.7, we see that the sixth row of first columns have negative element. Therefore the reduced order polynomial will not be stable.

Remark 3.1: From the above two example problems we can conclude that direct Routh approximation and modified approximation cannot give the guarantee the stability of reduced order interval systems. To overcome this problem Dolgin [166] suggested some conditions to improve the modified Routh approximation.

3.3.5 Improved Modified Routh Approximation for Interval Systems

From the previous method it is shown that the generalization of direct Routh approximation and modified Routh approximation for interval fails to produce stable systems. So to improve the modified Routh approximation, Dolgin [166] formulated additional conditions. The failure causes due to the coefficients of the member polynomials of the high-order original stable interval polynomial through the Routh table generated algorithm for fixed coefficients polynomials. Therefore, the stability of the reduced order interval polynomials cannot be guaranteed.

Condition : To Shrink the uncertainty of the elements

$$\left[f_{i,j}^{-}, f_{i,j}^{+} \right] = \left[f_{i-2,j+1}^{-}, f_{i-2,j+1}^{+} \right] - \frac{\alpha_{i-2,1}}{\alpha_{i-1,1}} \left[f_{i-1,j+1}^{-}, f_{i-1,j+1}^{+} \right], \quad i \geq 3, i \leq j \leq \left[\frac{(n-i+3)}{2} \right] \tag{3.26}$$

$$\alpha_{i-2,1} = \left[\alpha_{i-2,1}^{-}, \alpha_{i-2,1}^{+} \right] = \frac{f_{i-2,1}^{-} + f_{i-2,1}^{+}}{2}; \quad \alpha_{i-1,1} = \left[\alpha_{i-1,1}^{-}, \alpha_{i-1,1}^{+} \right] = \frac{f_{i-1,1}^{-} + f_{i-1,1}^{+}}{2} \tag{3.27}$$

$\alpha_{i-2,1}$ and $\alpha_{i-1,1}$ are the midpoints of the coefficients.

This condition is present to ensure the consistency of all elements of each pair of rows, except the first elements of each pair of rows, expect the first element of the first row of each pair, which is treated in the second condition.

The condition is easily satisfied by the following redefinition of interval subtraction in Eq. (3.26) to be

$$\left[e^{-}, e^{+} \right] - \left[f^{-}, f^{+} \right] = \left[e^{-} - f^{-}, e^{+} - f^{+} \right] \tag{3.28}$$

Example 3.3: Consider a seventh order polynomial

$$P_7(s) = [1,2]s^7 + [9,10]s^6 + [31,35]s^5 + [71,72]s^4 + [111,112]s^3 + [109,110]s^2 + [76,84]s + [12,13] \tag{3.29}$$

Using the algorithm 3.3.5, we obtain

Table 3.8: Improved Modified Routh Approximation of example 3.3

$[1,2]$	$[31,35]$	$[111,112]$	$[76,84]$
$[9,10]$	$[71,72]$	$[109,110]$	$[12,13]$
$[19.79,23.63]$	$[93.86,94.56]$	$[77.73,78.33]$	
$[29.93,30.62]$	$[75.16,75.56]$	$[12,13]$	
$[39.97,40.38]$	$[68.95,69.18]$		
$[23.19,23.42]$	$[12,13]$		
$[47.48,47.56]$			
$[12,13]$			

The fifth order reduction of the higher order polynomial is

$$P_5(s) = 21.71s^5 + [29.93,30.62]s^4 + [93.86,94.56]s^3 + [75.16,75.56]s^2 + [77.73,78.33]s + [12,13] \quad (3.30)$$

The stability of the above polynomial can be verified by Kharitonov's theorem [192] has shown in chapter 2.

3.4 LIMITATION USING INTERVAL ROUTH APPROXIMATION

- Unlike the non-interval Routh approximate, the interval Routh approximations don't preserve intervals of the first time moments and markov parameters.
- The drawback of the interval Routh based approximations fails to obtain stable reduced order models

All these five methods given motivation to develop new techniques in this thesis, explained in chapter 4 and chapter 5.