CHAPTER 2
STABILITY OF INTERVAL SYSTEMS

2.1 INTRODUCTION

Before 1980s, studying the stability analysis of uncertain systems was mostly not considered because lack of general theories for the purpose of analysing and designing of control systems with uncertain parameters. After introducing Kharitononv's theorem [192] most of the researchers involved for analysis of uncertain parametric approaches. Kharithonov's theorem was introduced by Kharitonov, which was published in 1979. However, this theorem remains unknown for many years due to published in Russian Literature and also it difficult to understand the theorem. Later, Barmish [194] simplified the proof and introduced to the western literature.

A polynomial where each coefficient varies in a given interval is called as interval polynomial. The Kharitonov theorem is an extension of the Routh stability criterion to interval polynomials. The Kharitonov theorem states that an interval polynomial family, which has an infinite number of members, is Hurwitz stable if and only if a finite small subset of four polynomials known as the Kharitonov polynomials of the family are Hurwitz stable.

The most significant results following this theorem have been the edge theorem [203] and the generalised Kharitonov theorem [198]. The edge theorem considers a family of polynomials with affine linear uncertainty structure which means that coefficients are not independent as in the case of interval; polynomials. It provided that the whole family is stable if and only if all the exposed edges of the polytopic family are stable. Furthermore, the edge theorem is not restricted to Hurwitz stability and it can be applied to the general problem of robust $\mathcal{D}$ stability. A polynomial is said to be robust $\mathcal{D}$ stable if all its lie in the region $\mathcal{D}$, which is a region in the complex plane. Similar to the edge theorem, the generalized Kharitonov theorem studies the stability problem of polynomials with affine linear uncertainty. However, the advantage of the generalised Kharitonov theorem over the edge theorem is that the number of edges which are required to be studied for stability is dependent on a number of interval polynomials not on the number of uncertain parameters. Later, many researchers developed proofs to simply the Kharitonov's theorem. The complete concept and developments of the generalised Kharitonov theorem has been discussed in the book written by Bhattacharyya et al., [203].

Lemma1: Let

$$\left.\begin{aligned} Q_1(s) &= Q^{even}(s) + Q_1^{odd}(s) \\ Q_2(s) &= Q^{even}(s) + Q_2^{odd}(s) \end{aligned}\right\} \tag{2.1}$$

denote same degree of the two stable polynomials with the same even part $Q^{even}(s)$ and differing odd parts $Q_1^{odd}(s)$ and $Q_2^{odd}(s)$ satisfying

$$Q_1^o(\omega) \le Q_2^0(\omega), \text{ for all } \omega \in [0,\infty] \tag{2.2}$$

Then, $Q(s) = Q^{even}(s) + Q^{odd}(s)$ is stable for every polynomial $Q(s)$ with odd part $Q^{odd}(s)$ satisfying

$$Q_1^o(\omega) \le Q^o(\omega) \le Q_2^o(\omega), \text{ for all } \omega \in [0,\infty] \tag{2.3}$$

Proof: Since $Q_1(s)$ and $Q_2(s)$ are stable, $Q_1^o(s)$ and $Q_2^o(s)$ both satisfy the interlacing property with $Q^e(s)$. In particular, $Q_1^o(\omega)$ and $Q_2^o(\omega)$ are not only one of the same degree, but the sign of their highest coefficients is also the same since it is in fact the same as that of the highest coefficients of $Q^e(\omega)$. Given this is easy to see that $Q^o(\omega)$ cannot satisfy Eq. (2.3) unless it also has the same degree and the same sign for the highest coefficient. Then, the condition in Eq. (2.3) forces the roots of $Q^o(\omega)$ to interlace with those of $Q^e(\omega)$. Therefore, according to the Hermite-Biehler theorem, $Q^{even}(s) + Q^{odd}(s)$ is stable. Here we considered $Q^0 = Q^{odd}; Q^e = Q^{even}$.

Lemma2: Let

$$\left.\begin{aligned} Q_1(s) &= Q_1^{even}(s) + Q^{odd}(s) \\ Q_2(s) &= Q_2^{even}(s) + Q^{odd}(s) \end{aligned}\right\} \tag{2.4}$$

denote same degree of the two stable polynomials with the same odd part $Q^{odd}(s)$ and differing even parts $Q_1^{even}(s)$ and $Q_2^{even}(s)$ satisfying

$$Q_1^e(\omega) \le Q_2^e(\omega), \text{ for all } \omega \in [0, \infty] \tag{2.5}$$

Then, $Q(s) = Q^{even}(s) + Q^{odd}(s)$ is stable for every polynomial $Q(s)$ with even part $Q^{even}(s)$ satisfying

$$Q_1^e(\omega) \le Q^e(\omega) \le Q_2^e(\omega), \text{ for all } \omega \in [0, \infty] \tag{2.6}$$

Proof: Since $Q_1(s)$ and $Q_2(s)$ are stable, $Q_1^e(s)$ and $Q_2^e(s)$ both satisfy the interlacing property with $Q^o(s)$. In particular, $Q_1^e(\omega)$ and $Q_2^e(\omega)$ are not only one of the same degree, but the sign of their highest coefficients is also the same since it is in fact the same as that of the highest coefficients of $Q^o(\omega)$. Given this is easy to see that $Q^e(\omega)$ cannot satisfy Eq. (2.2) unless it also has the same degree and the same sign for the highest coefficient. Then, the condition in Eq. (2.2) forces the roots of $Q^e(\omega)$ to interlace with those of $Q^o(\omega)$. Therefore, according to the Hermite-Biehler theorem, $Q^{even}(s) + Q^{odd}(s)$ is stable.

2.2 KHARITONOV'S THEOREM FOR REAL POLYNOMIALS

Consider the family $\mathbb{F}(s)$ of real polynomials of the degree n of the form $\Delta(s) = x_0 + x_1 s + x_2 s^2 + \cdots + x_n s^n$, where the coefficients lie within given ranges $x_0 \in [x_0^-, x_0^+], x_1 \in [x_1^-, x_1^+], \cdots, x_n \in [x_n^-, x_n^+]$

The Kharitonov's theorem provides a surprisingly simple necessary and sufficient condition for the Hurwitz stability of the entire family.

Theorem 2.1 (Kharitonov's Theorem):

Every polynomial in the family $\mathbb{F}(s)$ is Hurwitz if and only if the four polynomials are Hurwitz.

$$\left.\begin{aligned}
\Delta^1(s) &= x_0^- + x_1^- s + x_2^+ s^2 + x_3^+ s^3 + x_4^- s^4 + x_5^- s^5 + \cdots \\
\Delta^2(s) &= x_0^- + x_1^+ s + x_2^+ s^2 + x_3^- s^3 + x_4^- s^4 + x_5^+ s^5 + \cdots \\
\Delta^3(s) &= x_0^+ + x_1^- s + x_2^- s^2 + x_3^+ s^3 + x_4^+ s^4 + x_5^- s^5 + \cdots \\
\Delta^4(s) &= x_0^+ + x_1^+ s + x_2^- s^2 + x_3^- s^3 + x_4^+ s^4 + x_5^+ s^5 + \cdots
\end{aligned}\right\} \tag{2.7}$$

Proof: [192], [203] The proof given allows for the interpretation of Kharitonov's theorem as a generalization of the interlacing property of Hurwitz polynomials.

Let us introduce the hyper rectangle or box Ψ of coefficients of the perturbed polynomials

$$\Psi = \left\{ x \middle| x \in \mathbb{R}^{n+1}, x_i^- \le x_i^- \le x_i^+, i = 0,1,\cdots,n \right\} \tag{2.8}$$

The four Kharitonov's polynomials are built from two different even parts $\Delta_{\max}^{even}(s)$ and $\Delta_{\min}^{odd}(s)$ and two different odd parts $\Delta_{\max}^{odd}(s)$ and $\Delta_{\min}^{odd}(s)$ defined below

$$\left.\begin{aligned} \Delta_{\max}^{even}(s) &= x_0^+ + x_2^- s^2 + x_4^+ s^4 + x_6^- s^6 + \ldots. \\ \Delta_{\min}^{even}(s) &= x_0^- + x_2^+ s^2 + x_4^- s^4 + x_6^+ s^6 + \ldots. \end{aligned}\right\} \tag{2.9}$$

and

$$\left.\begin{aligned} \Delta_{\max}^{odd}(s) &= x_1^+ s + x_3^- s^3 + x_5^+ s^5 + x_7^- s^7 + \ldots. \\ \Delta_{\min}^{odd}(s) &= x_1^- s + x_3^+ s^3 + x_5^- s^5 + x_7^+ s^7 + \ldots. \end{aligned}\right\} \tag{2.10}$$

The motivation of the subscripts "max" and "min" is as follows. Let $x(s)$ be an arbitrary polynomial with its coefficients lying in the box Ψ and let $x^{even}(s)$ be its even part. Then

$$\left.\begin{aligned} \Delta_{\max}^{e}(\omega) &= x_0^+ - x_2^- \omega^2 + x_4^+ \omega^4 - x_6^- \omega^6 + \cdots \\ x^e(\omega) &= x_0 - x_2 \omega^2 + x_4 \omega^4 - x_6 \omega^6 + \cdots \\ \Delta_{\min}^{e}(\omega) &= x_0^- - x_2^+ \omega^2 + x_4^- \omega^4 - x_6^+ \omega^6 + \cdots \end{aligned}\right\} \tag{2.11}$$

so that

$$\Delta_{\max}^{e}(\omega) - x^e(\omega) = \left(x_0^+ - x_0\right) + \left(x_2 - x_2^-\right)\omega^2 + \left(x_4^+ - x_4\right)\omega^4 + \cdots$$

and

$$x^e(\omega) - \Delta_{\min}^{e}(\omega) = \left(x_0 - x_0^-\right) + \left(x_2^+ - x_2\right)\omega^2 + \left(x_4 - x_4^-\right)\omega^4 + \cdots$$

Therefore

$$\Delta_{\min}^{e}(\omega) \le x^e(\omega) \le \Delta_{\max}^{e}(\omega); \omega \in [0, \infty] \tag{2.12}$$

Similarly, if $x^{odd}\left(s\right)$ denotes the odd parts of $x\left(s\right)$, it can be verified that

$$\Delta_{\min}^{o}\left(\omega\right)\leq x^{o}\left(\omega\right)\leq\Delta_{\max}^{o}\left(\omega\right);\omega\in\left[0,\infty\right] \tag{2.13}$$

To proceed, note that the Kharitonov polynomials in Eq. (7) can be rewritten as

$$\left.\begin{aligned}
\Delta^{1}\left(s\right)&=\Delta_{\min}^{even}\left(s\right)+\Delta_{\min}^{odd}\left(s\right)\\
\Delta^{2}\left(s\right)&=\Delta_{\min}^{even}\left(s\right)+\Delta_{\max}^{odd}\left(s\right)\\
\Delta^{3}\left(s\right)&=\Delta_{\max}^{even}\left(s\right)+\Delta_{\min}^{odd}\left(s\right)\\
\Delta^{1}\left(s\right)&=\Delta_{\max}^{even}\left(s\right)+\Delta_{\max}^{odd}\left(s\right)
\end{aligned}\right\} \tag{2.14}$$

If all the polynomials with the coefficients in the box Ψ are stable, it is clear that the Kharitonov polynomials in Eq. (2.1) must also be stable since their coefficients lie in Ψ. For the conserve assume that the Kharitonov polynomials are stable, and let $x\left(s\right)=x^{even}\left(s\right)+x^{odd}\left(s\right)$ be an arbitrary polynomial with coefficients in the box Ψ with its even part $x^{even}\left(s\right)$ and its odd part $x^{odd}\left(s\right)$.

Since $\Delta^{1}\left(s\right)$ and $\Delta^{2}\left(s\right)$ are stable and Eq.(2.13) holds, we conclude from Lemma 1 applied to $\Delta^{1}\left(s\right)$ and $\Delta^{2}\left(s\right)$ in Eq. (2.14) that $\Delta_{\min}^{even}\left(s\right)+x^{odd}\left(s\right)$ is stable.

Similarly Lemma 1 applied to $\Delta^{3}\left(s\right)$ and $\Delta^{4}\left(s\right)$ in Eq. (2.14), we conclude that $\Delta_{\max}^{even}\left(s\right)+x^{odd}\left(s\right)$ is stable.

Now, since Eq. (2.12) holds, we can apply Lemma 2 to the two stable polynomials $\Delta_{\max}^{even}\left(s\right)+x^{odd}\left(s\right)$ and $\Delta_{\min}^{even}\left(s\right)+x^{odd}\left(s\right)$ and we conclude that

$x^{even}\left(s\right)+x^{odd}(s)=x\left(s\right)$ is stable.

Example 2.1: Consider the higher order system reported in [165] and determine the stability of the interval polynomial

$$\begin{aligned}P_{1}(s)=&\left[2.1,2.6\right]s^{6}+\left[76.1,76.7\right]s^{5}+\left[119.1,119.6\right]s^{4}\\&+\left[111.0,111.6\right]s^{3}+\left[71.8,72.3\right]s^{2}+\left[31.0,31.7\right]s+\left[9.0,9.9\right]\end{aligned} \tag{2.15}$$

The interval polynomial has been divided into four fixed polynomials.

$$\begin{aligned}
\Delta^{1}\left(s\right)&=2.6s^{6}+76.1s^{5}+119.1s^{4}+111.6s^{3}+72.3s^{2}+31s+9\\
\Delta^{2}\left(s\right)&=2.1s^{6}+76.7s^{5}+119.6s^{4}+111s^{3}+71.8s^{2}+31.7s+9.9\\
\Delta^{3}\left(s\right)&=2.6s^{6}+76.7s^{5}+119.1s^{4}+111s^{3}+72.3s^{2}+31.7s+9\\
\Delta^{4}\left(s\right)&=2.1s^{6}+76.1s^{5}+119.6s^{4}+111.6s^{3}+71.8s^{2}+31s+9.9
\end{aligned} \tag{2.16}$$

Apply Routh- Hurwitz criterion to each of these four polynomials.

Table 2.1: Routh table for the first polynomial $\Delta^1(s)$ of example 2.1

2.6	119.1	72.3	9
76.1	111.6	31	
115.28712	71.24086	9	
64.57454	25.05918		
26.50186	9		
3.1294			
9			

Table 2.2: Routh table for the second polynomial $\Delta^2(s)$ of example 2.1

2.1	119.6	71.8	9.9
76.7	111	31.7	
116.56089	70.93207	9.9	
64.32491	25.18355		
25.29423	9.9		
0.00919			
9.9			

Table 2.3: Routh table for the third polynomial $\Delta^3(s)$ of example 2.1

2.6	119.1	72.3	9
76.7	111	31.7	
115.33729	71.22542	9	
63.63466	25.71494		
24.61730	9		
2.45033			
9			

Table 2.4: Routh table for the fourth polynomial $\Delta^4(s)$ of example 2.1

2.1	119.6	71.8	9.9
76.1	111.6	31	
116.52037	70.94455	9.9	
65.26578	24.53426		
27.14301	9.9		
0.72956			
9.9			

The first column of the Routh table for each Kharitonov polynomial has zero sing changes. It shows that all the roots of the given interval polynomial $P_1(s)$ lie in the left half of the s- plane. Therefore the system is stable.

Example 2.2: Consider a numerical example reported by Hwang and Yang [159], determine the stability of the sixth order interval polynomial

$$\begin{aligned} P_2(s) = &[9, 9.5]s^6 + [31, 31.5]s^5 + [71, 71.5]s^4 \\ &+ [111.0, 111.5]s^3 + [119, 119.5]s^2 + [76, 76.5]s + [2, 2.5] \end{aligned} \tag{2.17}$$

The interval polynomial has been divided into four fixed polynomials.

$$\begin{aligned} \Delta^1(s) &= 9.5s^6 + 31s^5 + 71s^4 + 111.5s^3 + 119.5s^2 + 76s + 2 \\ \Delta^2(s) &= 9s^6 + 31.5s^5 + 71.5s^4 + 111s^3 + 119s^2 + 76.5s + 2.5 \\ \Delta^3(s) &= 9.5s^6 + 31.5s^5 + 71s^4 + 111s^3 + 119.5s^2 + 76.5s + 2 \\ \Delta^4(s) &= 9s^6 + 31s^5 + 71.5s^4 + 111.5s^3 + 119s^2 + 76s + 2.5 \end{aligned} \tag{2.18}$$

Apply Routh- Hurwitz criterion to each of these four polynomials.

Table 2.5: Routh table for the first polynomial $\Delta^1(s)$ of example 2.2

9.5	71	119.5	2
31	111.5	76	
36.83065	96.20968	2	
30.52124	74.31662		
6.53019	2		
64.96889			
2			

Table 2.6: Routh table for the second polynomial $\Delta^2(s)$ of example 2.2

9	71.5	119	2.5
31.5	111	76.5	
39.78571	97.14286	2.5	
34.08797	74.52065		
10.16622	2.5		
66.13799			
2.5			

Table 2.7: Routh table for the third polynomial $\Delta^3(s)$ of example 2.2

9.5	71	119.5	2
31.5	111	76.5	
37.52381	96.42857	2	
30.05134	74.82106		
3.00291	2		
54.80624			
2			

Table 2.8: Routh table for the fourth polynomial $\Delta^4(s)$ of example 2.2

9	71.5	119	2.5
31	111.5	76	
39.12903	96.93548	2.5	
34.70280	74.01937		
13.47518	2.5		
67.58109			
2.5			

The first column of the Routh table for each Kharitonov polynomial has zero sign changes. It shows that all the roots of the given interval polynomial $P_2(s)$ lie in the left half of the s- plane. Therefore the system is stable.

Example 2.3: Consider a numerical example reported by Bandyopadhyay et al. [157], determine the stability of the third order interval transfer function.

$$G_3(s) = \frac{[2,3]s^2 + [17.5,18.5]s + [15,16]}{[2,3]s^3 + [17,18]s^2 + [35,36]s + [20.5,21.5]} \tag{2.19}$$

The characteristic polynomial of the given third order transfer function by considering unity feedback

$$P_3(s) = 1 + G_3(s)H(s) = [2,3]s^3 + [19,21]s^2 + [52.5,54.5]s + [35.5,37.5] \tag{2.20}$$

The interval polynomial has been divided into four fixed polynomials.

$$\begin{aligned}
\Delta^1(s) &= 3s^3 + 21s^2 + 52.5s + 35.5 \\
\Delta^2(s) &= 2s^3 + 19s^2 + 54.5s + 37.5 \\
\Delta^3(s) &= 2s^3 + 21s^2 + 54.5s + 35.5 \\
\Delta^4(s) &= 3s^3 + 19s^2 + 52.5s + 37.5
\end{aligned} \tag{2.21}$$

Apply Routh- Hurwitz criterion to each of these four polynomials.

Table 2.9: Routh table for the first polynomial $\Delta^1(s)$ of example 2.3

3	52.5
21	35.5
47.42857	
35.5	

Table 2.10: Routh table for the first polynomial $\Delta^2(s)$ of example 2.3

2	54.5
19	37.5
50.55263	
37.5	

Table 2.11: Routh table for the first polynomial $\Delta^3(s)$ of example 2.3

2	54.5
21	35.5
51.11905	
35.5	

Table 2.12: Routh table for the first polynomial $\Delta^4(s)$ of example 2.3

3	52.5
19	37.5
46.57895	
37.5	

The first column of the Routh table for each Kharitonov polynomial has zero sing changes. It shows that all the roots of the given interval transfer function $G_3(s)$ lie in the left half of the s- plane. Therefore the system is stable.

Example 2.4: Consider a numerical example reported by Selvaganesan [169], determine the stability of the seventh order interval transfer function.

$$G_7(s)=\frac{\begin{array}{c}[1.9,2.1]s^6+[24.7,27.3]s^5+[157.7,174.3]s^4+[542,599]s^3+\\ [930,1028]s^2+[721.8,797.8]s+[187.1,206.7]\end{array}}{\begin{array}{c}[0.95,1.05]s^7+[8.779,9.703]s^6+[52.23,57.73]s^5+[182.9,202.1]s^4+\\ [429.1,474.2]s^3+[572.5,632.7]s^2+[325.3,359.5]s+[57.35,63.93]\end{array}} \tag{2.22}$$

The characteristic polynomial of the given seventh order transfer function by considering unity feedback

$$\begin{aligned}P_4(s)&=1+G_7(s)H(s)\\ &=[0.95,1.05]s^7+[10.679,11.803]s^6+[76.93,85.03]s^5+[340.6,376.4]s^4+\\ &\quad[971.1,1073.2]s^3+[1502.5,1660.7]s^2+[1047.1,1157.3]s+[244.45,270.63]\end{aligned} \tag{2.23}$$

The interval polynomial has been divided into four fixed polynomials.

$$\begin{aligned}\Delta^1(s)&=1.05s^7+11.803s^6+76.9s^5+340.6s^4+1073.2s^3+1660.7s^2+1047.1s+244.5\\ \Delta^2(s)&=0.95s^7+10.679s^6+85.03s^5+376.4s^4+971.1s^3+1502.5s^2+1157.3s+270.63\\ \Delta^3(s)&=0.95s^7+11.803s^6+85.03s^5+340.6s^4+971.1s^3+1660.7s^2+1157.3s+244.5\\ \Delta^4(s)&=1.05s^7+10.679s^6+76.9s^5+376.4s^4+1073.2s^3+1502.5s^2+1047.1s+270.63\end{aligned} \tag{2.24}$$

Apply Routh- Hurwitz criterion to each of these four polynomials.

Table 2.13: Routh table for the first polynomial $\Delta^1(s)$ of example 2.4

1.05	76.93	1073.2	1047.1
11.803	340.6	1660.7	244.45
46.63008	925.46341	1025.35362	
106.34680	1401.1626	244.5	
311.09307	918.16918		
1087.28755	244.5		
848.22752			
244.5			

Table 2.14: Routh table for the first polynomial $\Delta^2(s)$ of example 2.4

0.95	85.03	971.1	1157.3
10.679	376.4	1502.5	270.63
51.54559	837.43814	1133.22485	
202.90307	1267.72321	270.63	
515.38514	1064.47388		
848.64823	270.63		
900.11994			

Table 2.15: Routh table for the first polynomial $\Delta^3(s)$ of example 2.4

0.95	85.03	971.1	1157.3
11.803	340.6	1660.7	244.45
57.61578	837.43356	1137.6247	
169.04582	1427.64955	244.5	
350.84865	1054.30896		
919.66262	244.5		
961.05199			
244.5			

Table 2.16: Routh table for the first polynomial $\Delta^4(s)$ of example 2.4

1.05	76.93	1073.2	1047.1
10.679	376.4	1502.5	270.63
39.92092	925.46847	1020.49063	
128.8336	1229.5148	270.63	
544.48587	936.63208		
1007.89348	270.63		
790.4319			
270.63			

The first column of the Routh table for each Kharitonov polynomial has zero sign changes. It shows that all the roots of the given interval transfer function $G_7(s)$ lie in the left half of the s- plane. Therefore the system is stable.

2.3 KHARITONOV'S THEOREM FOR COMPLEX POLYNOMIALS

Consider the family $\mathbb{F}^*(s)$ of all complex polynomials of degree n of the form

$$\Delta(s) = (\alpha_0 + j\beta_0) + (\alpha_1 + j\beta_1)s + \cdots + (\alpha_n + j\beta_n)s^n \tag{2.25}$$

with $\alpha_i \in \left[\alpha_i^-, \alpha_i^+\right], \beta_i \in \left[\beta_i^-, \beta_i^+\right], i = 0,1,....,n$

The complex polynomials arise in the study of time delay systems and phase margin of control systems. Kharitonov developed his result for real polynomials to the above complex interval family by introducing two sets of complex polynomials as follows

$$\begin{aligned}
\Delta_1^+(s) &= \left(\alpha_0^- + j\beta_0^-\right) + \left(\alpha_1^- + j\beta_1^+\right)s + \left(\alpha_2^+ + j\beta_2^+\right)s^2 + \left(\alpha_3^+ + j\beta_3^-\right)s^3 + \left(\alpha_4^- + j\beta_4^-\right)s^4 + \cdots \\
\Delta_2^+(s) &= \left(\alpha_0^- + j\beta_0^+\right) + \left(\alpha_1^+ + j\beta_1^+\right)s + \left(\alpha_2^+ + j\beta_2^-\right)s^2 + \left(\alpha_3^- + j\beta_3^-\right)s^3 + \left(\alpha_4^- + j\beta_4^+\right)s^4 + \cdots \\
\Delta_3^+(s) &= \left(\alpha_0^+ + j\beta_0^-\right) + \left(\alpha_1^- + j\beta_1^-\right)s + \left(\alpha_2^- + j\beta_2^+\right)s^2 + \left(\alpha_3^+ + j\beta_3^+\right)s^3 + \left(\alpha_4^+ + j\beta_4^-\right)s^4 + \cdots \\
\Delta_4^+(s) &= \left(\alpha_0^+ + j\beta_0^+\right) + \left(\alpha_1^+ + j\beta_1^-\right)s + \left(\alpha_2^- + j\beta_2^+\right)s^2 + \left(\alpha_3^- + j\beta_3^+\right)s^3 + \left(\alpha_4^+ + j\beta_4^+\right)s^4 + \cdots
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
\Delta_1^-(s) &= \left(\alpha_0^- + j\beta_0^-\right) + \left(\alpha_1^+ + j\beta_1^-\right)s + \left(\alpha_2^+ + j\beta_2^+\right)s^2 + \left(\alpha_3^- + j\beta_3^+\right)s^3 + \left(\alpha_4^- + j\beta_4^-\right)s^4 + \cdots\cdot \\
\Delta_2^-(s) &= \left(\alpha_0^- + j\beta_0^+\right) + \left(\alpha_1^- + j\beta_1^-\right)s + \left(\alpha_2^+ + j\beta_2^-\right)s^2 + \left(\alpha_3^+ + j\beta_3^+\right)s^3 + \left(\alpha_4^- + j\beta_4^+\right)s^4 + \cdots\cdot \\
\Delta_3^-(s) &= \left(\alpha_0^+ + j\beta_0^-\right) + \left(\alpha_1^+ + j\beta_1^+\right)s + \left(\alpha_2^- + j\beta_2^+\right)s^2 + \left(\alpha_3^- + j\beta_3^-\right)s^3 + \left(\alpha_4^+ + j\beta_4^-\right)s^4 + \cdots\cdot \\
\Delta_4^-(s) &= \left(\alpha_0^+ + j\beta_0^+\right) + \left(\alpha_1^- + j\beta_1^+\right)s + \left(\alpha_2^- + j\beta_2^+\right)s^2 + \left(\alpha_3^- + j\beta_3^-\right)s^3 + \left(\alpha_4^+ + j\beta_4^-\right)s^4 + \cdots\cdot
\end{aligned} \tag{2.27}$$

Theorem 2.2 The family of polynomials $\mathbb{F}^*(s)$ is Hurwitz if and only if the eight Kharitonov polynomials $\Delta_1^+(s), \Delta_2^+(s), \Delta_3^+(s), \Delta_4^+(s), \Delta_1^-(s), \Delta_2^-(s), \Delta_3^-(s), \Delta_4^-(s)$ are all Hurwitz.

Proof: [203] The requirement of the condition is apparent because the eight Kharitonov polynomials are in $\mathbb{F}^*(s)$. The proof of sufficiency follows again from Hemite- Biehler Theorem for complex polynomials.

Observe that the Kharitonov polynomials in Eqs. (2.26) and (2.27) are composed of the following extremal polynomials.

For the "positive" Kharitonov polynomials define as

$$\begin{aligned}
R_{\max}^+ &= \alpha_0^+ + j\beta_1^- s + \alpha_2^- s^2 + j\beta_3^+ s^3 + \alpha_4^+ s^4 + \cdots\cdot \\
R_{\min}^+ &= \alpha_0^- + j\beta_1^+ s + \alpha_2^+ s^2 + j\beta_3^- s^3 + \alpha_4^- s^4 + \cdots\cdot \\
I_{\max}^+ &= j\beta_0^+ + \alpha_1^+ s + j\beta_2^- s^2 + \alpha_3^- s^3 + j\beta_4^+ s^4 + \cdots\cdot \\
I_{\min}^+ &= j\beta_0^- + \alpha_1^- s + j\beta_2^+ s^2 + \alpha_3^+ s^3 + j\beta_4^- s^4 + \cdots\cdot
\end{aligned}$$

so that

$$\Delta_1^+ = R_{\min}^+(s) + I_{\min}^+(s)$$
$$\Delta_2^+ = R_{\min}^+(s) + I_{\max}^+(s)$$
$$\Delta_3^+ = R_{\max}^+(s) + I_{\min}^+(s)$$
$$\Delta_4^+ = R_{\max}^+(s) + I_{\max}^+(s)$$

For the "negative" Kharitonov polynomials define as

$$R_{\max}^- = \alpha_0^+ + j\beta_1^+ s + \alpha_2^- s^2 + j\beta_3^- s^3 + \alpha_4^+ s^4 + \cdots$$
$$R_{\min}^- = \alpha_0^- + j\beta_1^- s + \alpha_2^+ s^2 + j\beta_3^+ s^3 + \alpha_4^- s^4 + \cdots$$
$$I_{\max}^- = j\beta_0^+ + \alpha_1^- s + j\beta_2^- s^2 + \alpha_3^+ s^3 + j\beta_4^+ s^4 + \cdots$$
$$I_{\min}^- = j\beta_0^- + \alpha_1^+ s + j\beta_2^+ s^2 + \alpha_3^- s^3 + j\beta_4^- s^4 + \cdots$$

so that

$$\Delta_1^- = R_{\min}^-(s) + I_{\min}^-(s)$$
$$\Delta_2^- = R_{\min}^-(s) + I_{\max}^-(s)$$
$$\Delta_3^- = R_{\max}^-(s) + I_{\min}^-(s)$$
$$\Delta_4^- = R_{\max}^-(s) + I_{\max}^-(s)$$

$R_{\max}^+(j\omega), R_{\max}^-(j\omega), R_{\min}^+(j\omega)$ and $R_{\min}^-(j\omega)$ are real and $I_{\max}^+(j\omega), I_{\max}^-(j\omega), I_{\min}^+(j\omega)$ and $I_{\min}^-(j\omega)$ are imaginary. Let $\mathrm{Re}\left[\Delta(j\omega)\right] = \Delta^r(\omega)$ and $\mathrm{Im}\left[\Delta(j\omega)\right] = \Delta^i(\omega)$ denote the real and imaginary parts of $\Delta(s)$ evaluated at $s = j\omega$. Then we have

$$\Delta^r(\omega) = \left[\alpha_0^-, \alpha_0^+\right] - \left[\beta_1^-, \beta_1^+\right]\omega - \left[\alpha_2^-, \alpha_2^+\right]\omega^2 + \left[\beta_3^-, \beta_3^+\right]\omega^3 + \cdots,$$
$$\Delta^i(\omega) = \left[\beta_0^-, \beta_0^+\right] + \left[\alpha_1^-, \alpha_1^+\right]\omega - \left[\beta_2^-, \beta_2^+\right]\omega^2 - \left[\alpha_3^-, \alpha_3^+\right]\omega^3 + \cdots.$$

It is easy to verify that

$$\left.\begin{aligned} & R_{\min}^+(j\omega) \le \Delta^r(\omega) \le R_{\max}^+(j\omega), \text{ for all } \omega \in [0,\infty] \\ & \frac{I_{\min}^+(j\omega)}{j} \le \Delta^i(\omega) \le \frac{I_{\max}^+(j\omega)}{j}, \text{ for all } \omega \in [0,\infty] \end{aligned}\right\} \qquad (2.28)$$

$$\left.\begin{aligned} & R_{\min}^-(j\omega) \le \Delta^r(\omega) \le R_{\max}^-(j\omega), \text{ for all } \omega \in [0,\infty] \\ & \frac{I_{\min}^-(j\omega)}{j} \le \Delta^i(\omega) \le \frac{I_{\max}^-(j\omega)}{j}, \text{for all } \omega \in [0,\infty] \end{aligned}\right\} \qquad (2.29)$$

The proof of the theorem is now completed as follows. The stability of the four positive Kharitonov polynomials guarantees interlacing of the "real tube" (bounded by $R^{+}_{\max}(j\omega)$ and $R^{+}_{\min}(j\omega)$) with the "imaginary tube" (bounded by $I^{+}_{\max}(j\omega)$ and $I^{+}_{\min}(j\omega)$) for $\omega \geq 0$. The relation in Eq. (2.28) then guarantees that the real and imaginary parts of an arbitrary polynomial in $\mathbb{F}^{*}(s)$ are forced to interlace for $\omega \leq 0$. Analogous arguments, using the bounds in Eq. (2.29) and the "negative" Kharitonov polynomials forces interlacing for $\omega \geq 0$. Thus by the Hermite-Biehler Theorem for complex polynomials $\Delta(s)$ is Hurwitz. Since $\Delta(s)$ was arbitrary, it follows that each and every polynomial in $\mathbb{F}^{*}(s)$ is Hurwitz.

Corollary 3.1: For low-order uncertain systems, robust stability can be checked by testing only one, two, and three Kharitonov polynomials for order $n = 3, 4$ and 5 respectively.

The arrangements of Kharitonov polynomials $\Delta^{1}(s), \Delta^{2}(s)$ and $\Delta^{3}(s)$ in terms of lower and upper bounds for order $n = 3, 4$ and 5 are as given below

For $n = 3$;

$$\Delta^{1}(s) = x_0^{+}s^{3} + x_1^{-}s^{2} + x_2^{-}s + x_3^{+}.$$

For $n = 4$;

$$\Delta^{1}(s) = x_0^{+}s^{4} + x_1^{-}s^{3} + x_2^{-}s^{2} + x_3^{+}s + x_4^{+},$$
$$\Delta^{2}(s) = x_0^{+}s^{4} + x_1^{+}s^{3} + x_2^{-}s^{2} + x_3^{-}s + x_4^{+}.$$

For $n = 5$;

$$\Delta^{1}(s) = x_0^{+}s^{5} + x_1^{-}s^{4} + x_2^{-}s^{3} + x_3^{+}s^{2} + x_4^{+}s + x_5^{-},$$
$$\Delta^{2}(s) = x_0^{+}s^{5} + x_1^{+}s^{4} + x_2^{-}s^{3} + x_3^{-}s^{2} + x_4^{+}s + x_5^{+},$$
$$\Delta^{3}(s) = x_0^{-}s^{5} + x_1^{+}s^{4} + x_2^{+}s^{3} + x_3^{-}s^{2} + x_4^{-}s + x_5^{+}.$$

2.4 KHARITONOV'S THEOREM FOR DISCRETE TIME INTERVAL SYSTEMS.

The stability of a polynomial with uncertain coefficients. The polynomial, which may be the characteristic polynomials of a linear system, is given by

$$D(z) = c_n z^{n} + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0 \tag{2.30}$$

where the coefficients are unknown, except for box bounds of the form $c_i^- \le c_i \le c_i^+$ for $0 \le i \le n$, with c_i^- and c_i^+ are known constants.

The Kharitonov polynomial of an n^{th} order discrete time interval system, $D(z)$ are handled by mapping the z -plane to the w -plane by $w = \dfrac{z+1}{z-1}$. This transforms $D(z)$ to $D\left(\dfrac{w+1}{w-1}\right)$ which is multiplied by $(w-1)^n$ to get the polynomial $Q(w)$,

$$Q(w) = \sum_{i=0}^{n} \left[c_i^-, c_i^+\right] \cdot (w+1)^i (w-1)^{n-i} \tag{2.31}$$

Note that $w = 1$ is the image of the point at infinity, which is not a root of $D(z)$

The terms $(w+1)^i (w-1)^{n-i}$ can be rewritten as

$$(w+1)^i (w-1)^{n-i} = \sum_{r=0}^{n} s_{(i,r)} w^r \tag{2.32}$$

giving

$$Q(w) = \sum_{m=0}^{n} \left[C_m^-, C_m^+\right] \cdot (w)^m \tag{2.33}$$

where

$$\left[C_m^-, C_m^+\right] = \sum_{v=0}^{n} \left[c_v^-, c_v^+\right] \cdot s_{(v,m)} \tag{2.34}$$

The bounds on c_i transform into bounds on C_i according to

$$\sum_{v=0}^{n} \min\left[c_v^- \cdot s_{(v,m)}, c_v^+ b \cdot s_{(v,m)}\right] \le \left[C_m^-, C_m^+\right] \le \sum_{v=0}^{n} \max\left[c_v^- \cdot s_{(v,m)}, c_v^+ \cdot s_{(v,m)}\right] \tag{2.35}$$

Using the transformed bounds, it is easy to obtain the Kharitonov polynomials, $Q_i(w)$. Each $Q_i(w)$ is then tested by the Routh criterion. Because the function $w = \dfrac{z+1}{z-1}$ maps the unit circle $|z|\langle 1$ onto the left half of the w - plane, if all the roots of each $Q_i(w), 1 \le i \le 4$, are in the left half plane, then all the roots $D(z)$ of are within the unit circle in the plane. This is a sufficiency condition for the stability of $D(z)$.

Because of the way the bounds c_i^- and c_i^+ are mapped into bounds on the coefficients of $Q[w]$, if one or more of the $Q_i(w)$ are unstable, then it is not possible to decide whether $D(z)$ is stable or unstable. This is because the bounds on C_i do not represent a hyper rectangle but rather a "wrapped" one. Apply Rouche's theorem to find the stability of the higher order discrete polynomial.

The stability test of the uncertain polynomials can also be verified by using SYSTEMS PACKAGE software.

Example 2.5: Consider a characteristic fourth order polynomial

$$D_4(z)=[1,1]z^4+[0.077,0.354]z^3+[0.077,0.354]z^2+[0.077,0.354]z+[0.077,0.354] \tag{2.36}$$

The interval polynomial has been divided into four fixed polynomials.

$$\begin{aligned}
\Delta^1(z) &= z^4+0.354z^3+0.354z^2+0.077z+0.077\\
\Delta^2(z) &= z^4+0.077z^3+0.077z^2+0.354z+0.354\\
\Delta^3(z) &= z^4+0.077z^3+0.354z^2+0.354z+0.077\\
\Delta^4(z) &= z^4+0.354z^3+0.077z^2+0.077z+0.354
\end{aligned} \tag{2.37}$$

$$P_4(s)=[1,1]w^4+[0.077,0.354]w^3+[0.077,0.354]w^2+[0.077,0.354]w+[0.077,0.354] \tag{2.38}$$

The interval polynomial in Eq. (2.37) has been divided into four fixed polynomials.

$$\begin{aligned}
P_1(w) &= 1.308w^4+4.246w^3+7.97w^2+2.03w+0.446\\
P_2(s) &= 2.416w^4+2.03w^3+5.754w^2+4.246w+1.554\\
P_3(s) &= 1.308w^4+2.03w^3+7.97w^2+4.246w+0.446\\
P_4(s) &= 2.416w^4+4.246w^3+5.754w^2+2.03w+1.554
\end{aligned} \tag{2.39}$$

Apply Routh- Hurwitz criterion to each of these four polynomials.

Table 2.17: Routh table for the first polynomial $P_1(w)$ of example 2.5

1.308	7.97	0.446
4.246	2.03	
7.33465	0.446	
1.77216		
0.446		

Table 2.18: Routh table for the first polynomial $P_2(w)$ of example 2.5

2.416	5.75	1.554
2.03	4.246	
0.70063	1.554	
-0.25653		
1.554		

Table 2.19: Routh table for the first polynomial $P_3(w)$ of example 2.5

1.308	7.97	0.446
2.03	4.246	
5.23415	0.446	
4.07302		
0.446		

Table 2.20: Routh table for the first polynomial $P_4(w)$ of example 2.5

2.416	5.754	1.554
4.246	2.03	
4.59892	1.554	
0.59525		
1.554		

The first column of the Routh Table 2.18 has two sign changes. Because of the way of the higher order interval parameters transformed to the w-plane, the instability of the transformed polynomial $P_4(z)$ does not give any information about the stability of $D_4(z)$. The stability of $P_4(w)$ is only sufficient to ensure the stability of $D_4(z)$.

Using Rouche's theorem for stability, $D_4(z) = f_4(z) + g_4(z)$

$$f_4(z) = 0.5 \times \left\{ \left(c_4^- + c_4^+\right) z^4 + \left(c_3^- + c_3^+\right) z^3 + \left(c_2^- + c_2^+\right) z^2 + \left(c_1^- + c_1^+\right) z + \left(c_0^- + c_0^+\right) \right\} \quad (2.40)$$

$$g_4(z) = 0.5 \times \left\{ t_4 \left(c_4^+ - c_4^-\right) z^4 + t_3 \left(c_3^+ - c_3^-\right) z^3 + t_2 \left(c_2^+ - c_2^-\right) z^2 + t_1 \left(c_1^+ - c_1^-\right) z + t_0 \left(c_0^+ - c_0^-\right) \right\} \quad (2.41)$$

$$f_4(z) = z^4 + 0.2155z^3 + 0.2155z^2 + 0.2155z + 0.2155 \tag{2.42}$$

$$F_4(w) = 1.862w^4 + 3.138w^3 + 6.862w^2 + 3.138w + 1 \tag{2.43}$$

Table 2.21: Routh table for the polynomial $F_4(w)$ of example 2.5

1.862	6.862	1
3.138	3.138	
5.0	1	
2.5104		
1.0		

Number of sign changes in Table 2.21 is zero. All the roots of $f_4(z)$ lie inside the unit disk in the z-plane. $f_4(z)$ is stable.

Applying Rouche's theorem

$$L = \frac{Min\left|f_4\left(e^{j\theta}\right)\right|}{Max\left|g_4\left(e^{j\theta}\right)\right|} = 5.6613 \tag{2.44}$$

All the roots of $D_4(z)$ lie inside the unit circle in the z-plane. The system is stable.

$D_4(z)$ is also stable if $0 \le t_k < 5.6613$,where $0 \le k \le 4$.