Chapter 5
Assessment of Results

5.1. Preamble

This phase in the write-up comprises two significant case studies. One is a specific finding from work during the proposal of the algorithms. It is the evaluation among the two frequently used discrete transformation techniques based on their implication to order reduction of discrete-time interval systems. Other is a summary of the discussions remarked throughout the Chapter 3 and 4 while proposing new algorithms for discrete-time interval systems. It also elaborates the limitations, if any, discovered during the discourse.

5.2. Case 1: Appraisal of Discrete- Transformation Techniques

This section conveys a significant finding through the work of algorithm development. In the literature survey for order reduction of discrete-time non-interval/interval system, two discrete transformation techniques are frequently accessed. They play a substantial role in the advancement of an algorithm from continuous-time domain to the discrete-time domain. In literature, both are approached widely but do not present their importance in the arena of order reduction. This lead to the current work for appraisal between them giving a significant finding based on order reduction of discrete-time interval systems.

Two widely used transformation techniques are notably

a) Tustin or bilinear or trapezoidal method [112]–[117] and

b) Euler's Forward differentiation method [87], [118]–[120].

This work does not provide any new outcome but attempts to offer a justification, in a manner to directly aim which of the two transformation techniques is better to opt.

5.2.1. Introduction

Conversion from discrete-time to the continuous-time domain is essential in a manner to apply the continuous-time algorithms to discrete-time systems. As stated in the introduction, there are many transformation techniques but here only two of them, namely, Tustin and Forward difference transformation techniques are considered. The reason is their broad applicability towards the order reduction of discrete-time systems stating the leading cause for the discussion that establishes their grandness, indicating a significant difference among each other.

Chapter 2, section 2.8., submits the two techniques exclusively with a brief elaboration of their advantages and disadvantages. According to the second disadvantage, most of the traditional reduction techniques either in the frequency domain or in the time domain do not secure a proper fitting in their respective field. Thus, a reduced model may be satisfactory in one domain, but unsatisfactory in another domain. This disadvantage is attempted here to assess between the two transformations techniques for their broad implication through order reduction.

Till date, to the authors' knowledge, no discussion is available that specifies, which of the either transformation is more accurate or preferable for obtaining reduced order models. The unavailability set the motive to attempt for a convincing difference between the two conversion methods based on their simplicity and ease of computation via order reduction of discrete-time interval systems. Moreover, this appraisal of the frequently used conversion techniques would be helpful for the researchers who work on a higher order system for improvement of the system performance.

5.2.2. Order Reduction

This section present the evaluation among the two frequently used discrete transformation techniques as stated in the introduction. It is divided in two subsections *a*) Reduction methodology applied for the approximation and *b*) examples to understand the discovery from the work. The performance analysis is achieved on the basis of error computation and step response.

5.2.2.1. Reduction Methodology

Any of the acknowledged or proposed reduction algorithms can be chosen for deriving the reduced models. Here conceived is Gamma-Delta approximation, which is among the proposed algorithm in this thesis and its procedural steps are elaborated in Chapter 3 under section 3.2. Both the conversion techniques pose all the similarities except for $z = \frac{1+w}{1-w}$ in w-domain and $z = 1 + p$ in p-domain and their respective inversions. Individually both the transformations are considered as:

a) *Tustin or Bilinear transformation* (*w*-domain): This conversion on (2.13) results the higher order interval system as (2.15)

b) *Euler Forward Difference or Linear transformation* (*p*-domain): This conversion on (2.13) consequences (2.16) as the higher order system.

Execution of the techniques is performed as illustrated in section 3.2. This mathematical computation in both the domains is presented below in example subsection. Figure 5.1 depict the algorithmic procedure for the assessment.

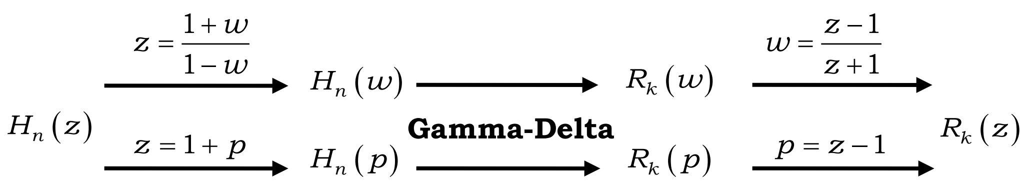

Figure 5.1: Algorithmic procedure for the assessment

5.2.2.2. Example

E.5.2.2.2.1. Consider the third order interval transfer function as

$$H_3(z)=\frac{[3.25,3.35]z^2+[3.5,3.65]z+[2.8,3]}{[5.4,5.5]z^3+[1,1.1]z^2+[1.5,1.6]z+[2.1,2.15]} \quad (5.1)$$

a) The Tustin transformation leads to

$$H_3(w)=\frac{[-2.85,-2.4]w^3+[1.3,2.35]w^2+[-9.05,-8.9]w+[9.55,10]}{[3.65,4]w^3+[19.8,20.45]w^2+[9.15,9.8]w+[10,10.35]} \quad (5.2)$$

$\gamma's$ and $\delta's$ parameters obtained from the above denominator and numerator polynomials are

$$\left[\gamma_1^-,\gamma_1^+\right]=[1.02,1.13] \qquad \left[\gamma_2^-,\gamma_2^+\right]=[0.51,0.69]$$

$$\left[\delta_1^-,\delta_1^+\right]=[0.97,1.09] \qquad \left[\delta_2^-,\delta_2^+\right]=[-0.63,-0.49]$$

b) The forward difference transforms (5.1) to (5.3) and the respective $\gamma's$ and $\delta's$ parameters obtained are

$$H_3(p)=\frac{[3.25,3.35]p^2+[10,10.35]p+[9.55,10]}{[5.4,5.5]p^3+[17.2,17.6]p^2+[19.7,20.3]p+[10,10.35]} \quad (5.3)$$

$$\left[\gamma_1^-,\gamma_1^+\right]=[0.49,0.53] \qquad \left[\gamma_2^-,\gamma_2^+\right]=[1.12,1.18]$$

$$\left[\delta_1^-,\delta_1^+\right]=[0.47,0.51] \qquad \left[\delta_2^-,\delta_2^+\right]=[0.57,0.60]$$

Substituting the above obtained parameters in (3.10), result in the simplified *z*-domain model with varied transformations as

$$R_{2w}(z)=\frac{[-0.137,0.244]z^2+[0.85,1.107]z+[0.995,1.376]}{[2.032,2.464]z^2+[-0.958,-0.447]z+[0.834,1.266]} \tag{5.4}$$

$$R_{2p}(z)=\frac{[0.568,0.601]z+[-0.475,-0.402]}{z^2+[-0.881,-0.820]z+[0.550,0.619]} \tag{5.5}$$

Assessment about the finding is performed in the next sub-section.

E.5.2.2.2.2. Consider an eighth order interval system from real-time world as

$$H_8(z)=\frac{N_7(z)}{D_8(z)} \tag{5.6}$$

where

$$N_7(z)=[1.6484,1.7156]z^7+[1.0937,1.1383]z^6+[-0.2142,-0.2058]z^5$$
$$+[0.1490,0.1550]z^4+[-0.5263,-0.5057]z^3+[-0.2672,-0.2568]z^2$$
$$+[0.0431,0.0449]z+[-0.0061,-0.0059]$$

(5.7)

$$D_8(z)=[23.52,24.48]z^8+[-1.7156,-1.6484]z^7+[-1.1383,-1.0937]z^6$$
$$+[0.2058,0.2142]z^5+[-0.1550,-0.1490]z^4+[0.5057,0.5263]z^3 \tag{5.8}$$
$$+[0.2568,0.2672]z^2+[-0.0449,-0.0431]z+[0.0059,0.0061]$$

Upon respective transformation, the $\gamma-\delta$ parameters are obtained as

	w-domain	*p*-domain
$[\gamma_1^-,\gamma_1^+]$	$[0.1190,0.1313]$	$[0.1189,0.1314]$
$[\gamma_2^-,\gamma_2^+]$	$[0.3257,0.4184]$	$[0.3335,0.4317]$
$[\delta_1^-,\delta_1^+]$	$[0.0107,0.0121]$	$[0.0107,0.0121]$
$[\delta_2^-,\delta_2^+]$	$[0.0297,0.0380]$	$[0.0302,0.0394]$

Above computed parameters, result the reduced order models in *z*-domain as

$$R_{2w}(z)=\frac{[0.0332,0.0400]z^2+[0.0070,0.0102]z+[-0.0345,-0.0246]}{[1.3645,1.4733]z^2+[-1.9224,-1.8902]z+[0.6204,0.7292]} \tag{5.9}$$

$$R_{2p}(z)=\frac{[0.0302,0.0394]z+[-0.0358,-0.0250]}{z^2+[-1.6665,-1.5683]z+[0.6080,0.7232]} \tag{5.10}$$

Finding from this example is discussed in the next sub-section.

5.2.3. Evaluation and Discussion

A brief discussion of the varied transformation techniques based on their analysis through the computation of errors and step responses between the higher-order systems and reduced lower order models is submitted here.
Tables 5.1 and 5.2 show the errors for E.5.2.2.2.1 and E.5.2.2.2.2 respectively.

Table 5.1: Error of 2nd order reduced models for E.5.2.2.2.1

Transformation	Error	
	Lower Limit	Upper Limit
w-domain	0.0845	0.0116
p-domain	0.0011	$6.5463\text{x}10^{-05}$

Table 5.2: Error of 2nd order reduced models for E.5.2.2.2.2

Transformation	Error	
	Lower Limit	Upper Limit
w-domain	$6.2265\text{x}10^{-04}$	$7.3850\text{x}10^{-04}$
p-domain	$9.4157\text{x}10^{-04}$	$9.4137\text{x}10^{-04}$

The step responses of the reduced models and higher systems for E.5.2.2.2.1 are shown in Figures 5.2 and 5.3 for lower and upper limit transfer functions respectively. Later Figures 5.4 and 5.5 depict the step responses for the two limit transfer function for E.5.2.2.2.2 correspondingly.

The errors in Tables 5.1 and 5.2 are minimal as per the requirement of the error computation. Also the Figures 5.2-5.5 present an appreciable tracking of the step response of the higher order systems and the reduced models for both the examples via two transformations.

Thus, the above results and thorough observations of the methodologies, evoke that p-domain discretization technique is quite simple and easy to transform, making it preferable over the w-domain techniques to produce similar reduced models.

A feasible query developed on the basis of this assessment; is it relevant to compare the performance based on the model reduction as the result may depend on the reduction techniques as well as the numerical examples. The answer is; yes, they are a major factor to be considered but on a large scale of simplification, the linear transformation (*p-domain*) can be applied directly to obtain an acceptable result.

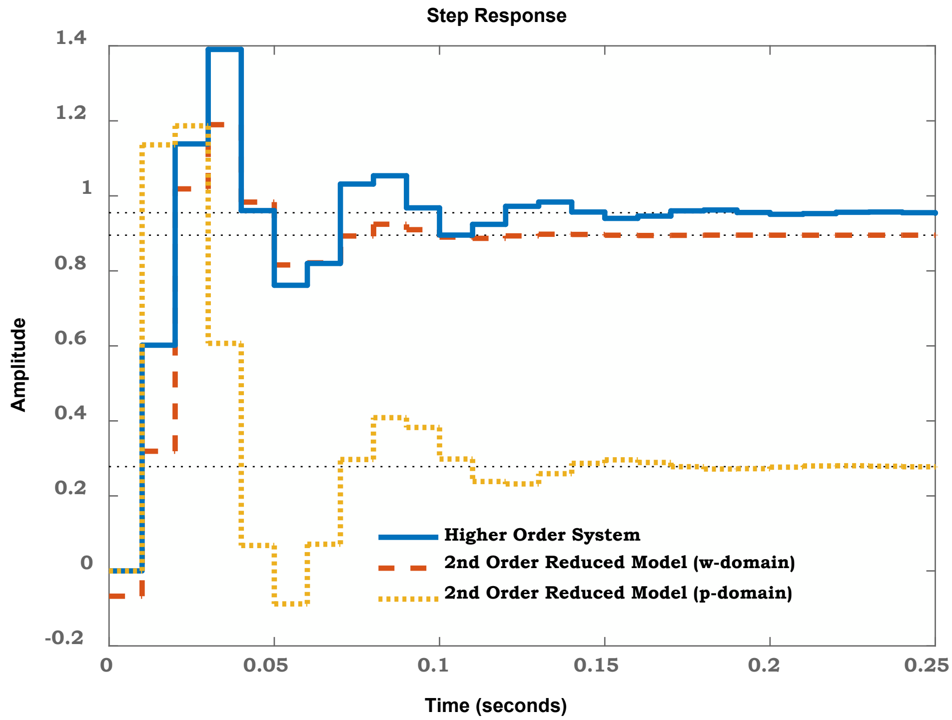

Figure 5.2: Step responses for reduced models (Lower Limit) for E.5.2.2.2.1

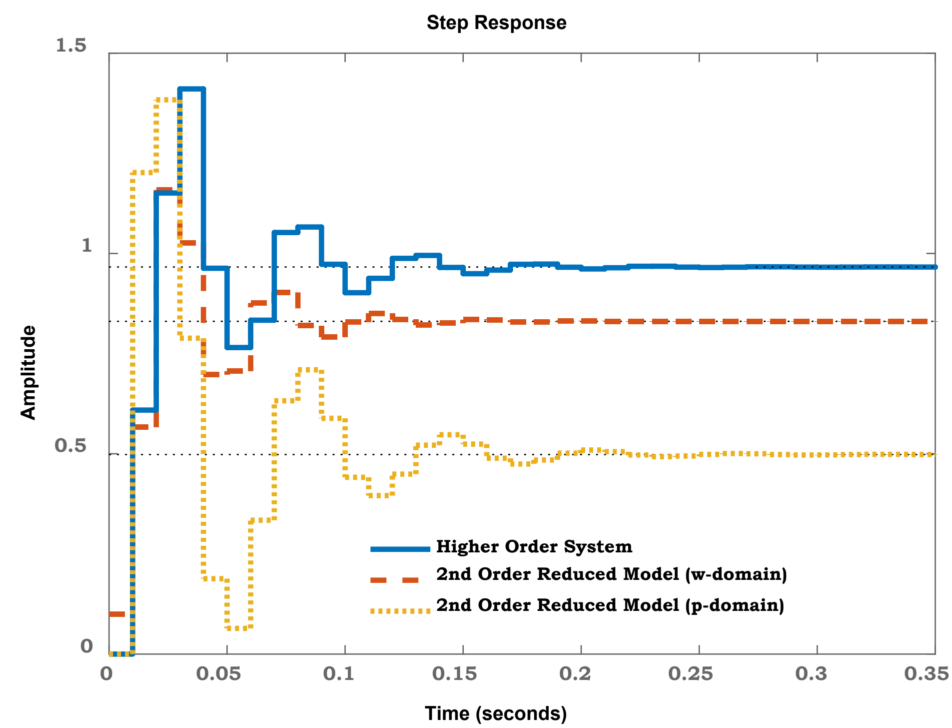

Figure 5.3: Step responses for reduced models (Upper Limit) for E.5.2.2.2.1

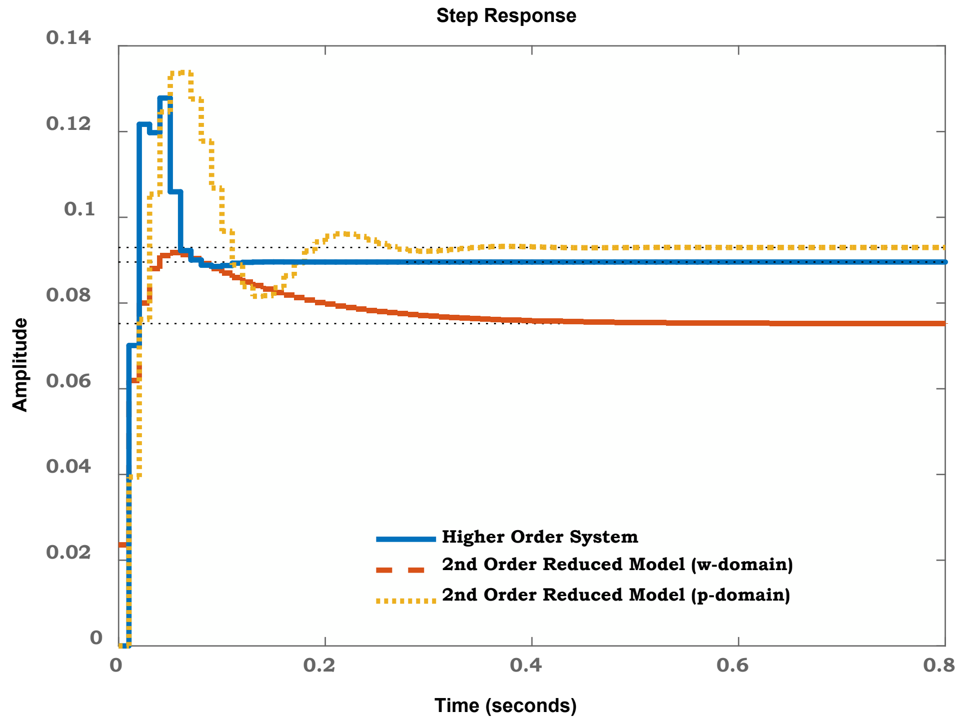

Figure 5.4: Step responses for reduced models (Lower Limit) for E.5.2.2.2.2

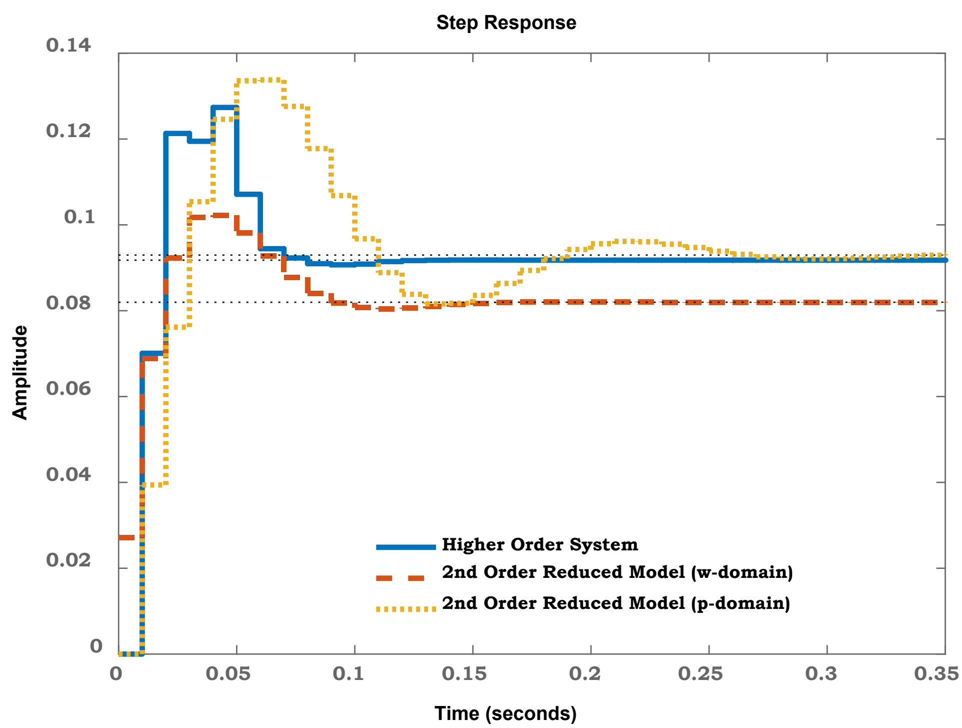

Figure 5.5: Step responses for reduced models (Upper Limit) for E.5.2.2.2.2

5.2.4. Conclusions

An assessment of the two discretization techniques owing their own advantages and disadvantages is outlined here. The main goal to examine which of them supplies more convenient discretization with ease is recognized here. It is found in general that, forward difference achieves the advantage for being easy and simple at every step. The main reason is its linear form *i.e.* $z = 1 + p$ instead of rational form as $z = \frac{1+w}{1-w}$. These observations indicate that any of the two techniques can be used; as both of them result in almost equivalent reduced forms; but for convince and ease of computation, p-domain emerge to be superior. The achieved results are good basis for further work in the area of discrete-time to continuous-time transformation and vice versa. Overall, the conclusion is that the latter technique is much simpler, matches as many moments as the former one does and hence appears to be better to the former technique. Moreover, the assessment seeks to be helpful to the researchers or control engineers or designers who work on higher order system in the discrete-time domain arena.

5.3 Case 2: Analysis of Accumulated Algorithms

This section exhibits the summary of the discussion and conclusions remarked throughout the Chapter 3 and 4, while proposing new algorithms for discrete-time interval systems. The limitations, if any, discovered during the course is also mentioned.

5.3.1. Introduction

A cumulative of 15 algorithms is reported in the thesis under two nomenclatures namely *Routh Approximation* approach and *Assorted* approach. Precisely, the proposed techniques in the thesis are classified as;

I. Routh Approximation approach

01) Gamma Delta Approximation
02) Arithmetic Operator or Multiplicative Approach
03) Novel Arrangement of Routh Array
04) Simplified Interval Structure
05) Advanced Routh Approximation Method (*A-RAM*)
06) Extended Direct Routh Approximation Method (*E-DRAM*)
07) Routh Approximant algorithm
08) Routh Approximation and Pade Approximations

09) Direct Truncation and Routh Approximation

10) Pade Approximation and Routh Approximation

II. Assorted approach

11) Non-computational Technique or Shifting Algorithm

12) Classical Differentiation Method

13) Direct Truncation Method

14) Routh Approximation and Mikhailov Stability Criterion

15) Direct Truncation and Mikhailov Stability Criterion

5.3.2. Routh Approximation Approach

Techniques demonstrated here are stated to be novel for their existences and procedural steps ground on Routh Approximation.

Algorithm 1 presents a remarkable extension to discrete-time interval systems that satisfies the conditions for model stability, step response and minimum error computation.

Contributing to the prevailing techniques, ***Algorithm 2*** formulates a novel algorithm based on *RA* that retains the dynamic characteristic of the higher order system to its lower equivalent, *i.e.* stability. The novelty of the algorithm is in two folds; *a*) implication of multiplicative operator and *b*) finding the best possible arrangement from the varied combinations (mentioned as *cases*) of Routh table for deriving the numerator and denominator polynomials coefficients of the reduced model.

Algorithm 3 presents a novel arrangement of Routh Table array for deriving an approximate model of a higher order discrete-domain interval system. The foresaid new arrangement is accomplished from the arena of varied combinations of numerator and denominator polynomials (mentioned as *cases*) practiced over the prevailing numerical example from the literature.

In the course of attaining the retention of model stability; a limitation is discovered, which is ignored for the algorithm's proficiency. Limitation is the computation of high error sum. From the figures it is observed that the tracking of the responses are not exact, but are taken under consideration as some of the dynamic characteristics depict their improvement in the reduced models responses.

Algorithm 4 consider the interval structure to be the major contributor towards the order reduction of the overall interval system. The proposal presents an analysis through two limits; lower and upper.

Algorithm 5 and ***6***, both are built on the ground of *RA*. The only key difference between them is the algorithmic steps for attaining the reduced models. Former considers the reciprocal of the higher order system to reduction procedure, whereas latter do not. Latter directly employs the steps over the higher order system without any reciprocity involvement. Both the techniques give stable models, minimal error and a considerable step response tracking.

Algorithm 7 uses Routh approximant for order reduction. Examples state the proficiency through the error computed and the step response.

Algorithm 8 is mixed form of two varied approximations namely Routh and Pade. It demonstrates appreciable outcome on each of the grounds whether it be step response, minimal error computation or model stability. It's a substantial algorithm.

Algorithm 9 and ***10*** revisits few noteworthy estimation techniques for simplification of discrete-time interval system. In particular, the denominator polynomial is computed by a new algorithm of reciprocity and numerator by two different prevailing techniques namely *a*) Direct Truncation and *b*) Pade Approximation. These algorithms encounter the limitation of computing high errors but not always; as can be seen through the examples. Thus are disregarded. Same is the case with stability. The techniques do not guarantee to generate stable models always, even if their higher representations are stable.

5.3.3. Assorted Approach

Techniques that submit their procedural step in a different manner than the prevailing techniques are illustrated here.

Algorithm 11 is based on shifting coefficients. It is straightforward and simple with no computation. It poses to be superior, easy and direct method for the order reduction.

Algorithm 12 is grounded on the base of classical approach of calculus *i.e.* Differentiation. The presented technique is better and effortless with straightforward calculation. The algorithm is not commendable if discussed in terms of step response, but is thoughtful based on the minimal error computation.

Algorithm 13 address a computationally simple and intuitively appealing algorithm based on Direct Truncation. It preserves the criterion of minimum error and step response tracking but do not guarantee to give stable models always.

Algorithm 14 and ***15*** delivers two varied algorithms interlaced with the property of Mikhailov Stability Criterion. Precisely, this criterion is employed to derive the reduced order denominator coefficients and the numerator coefficients are computed by *a*) Routh Approximation and *b*) Direct Truncation.

Table 5.3 briefs about the findings for the proposed algorithms from Chapters 3 and 4.

Table 5.3: Findings for the proposed algorithms from Chapters 3 and 4

Algorithms	Minimal Error	Stable Models	Response Tracking	Limitations
01	Obtained	Obtained	Favorable	---
02	Obtained	Obtained	Favorable	---
03	Obtained	Obtained	Not Exactly	Yes
04	Obtained	Obtained	Favorable	
05	Obtained	Obtained	Favorable	---
06	Obtained	Obtained	Favorable	---
07	Obtained	Obtained	Favorable	---
08	Obtained	Obtained	Favorable	---
09	Obtained	Not Always	Not Exactly	Yes
10	Obtained	Not Always	Not Exactly	Yes
11	---	Yet to establish	Favorable	---
12	Obtained	Yet to establish	Not Exactly	---
13	Obtained	Not Always	Favorable	---
14	Obtained	Obtained	Favorable	---
15	Obtained	Obtained	Favorable	---

Among the proposed algorithms for discrete-time interval systems, *few* are appealing for their prolongation to real-time implementation namely

- ✓ **04**: Simplified Interval Structure
- ✓ **05**: Advanced Routh Approximation (*A-RAM*)
- ✓ **06**: Extended Direct Routh Approximation (*E-DRAM*)
- ✓ **12**: Classical Differentiation Approach

This is discovered from an arena of an example available from the literature and all of the proposed techniques been applied over it. This establishment is based on the computation of minimal error as follows;

Example: Consider the third order interval system as

$$H_3(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{5.11}$$

Implication of the proposed algorithms in the thesis result for the computation of the errors as depicted in the Table 5.4.

Table 5.4: Error for 2nd order reduced models by proposed algorithms

Algorithms	Error	
	Lower Limit	Upper Limit
Gamma-Delta Appr.	*0.1292*	*0.0443*
Multiplicative Operator	*0.0442*	*0.1860*
Novel Arrangement	*0.0211*	*0.0233*
Simplified Interval Structure	***3.4294X10⁻⁴***	***0.0018***
A-RAM	***0.0013***	***0.0060***
E-DRAM	***0.0012***	***0.0128***
Routh Approximant	*0.0029*	*0.0235*
Routh Appr. & Pade Appr.	*0.1079*	*0.0342*
Direct Truncation & Routh Appr.	*0.0553*	*0.0033*
Pade Appr. & Routh Appr.	*1.1265*	*0.2183*
Non-Computational	…	…
Differentiation	***0.0031***	***0.0123***
Direct Truncation	*0.0278*	*0.0077*
Routh Appr. & Mikhailov	*0.3154*	*0.0947*
Direct Truncation & Mikhailov	*0.0079*	*0.0643*

5.3.4. Conclusions

All of the proposed algorithms pose some or the other special feature cumulative of minimal error, step response and model stability. This chapter attempted to summaries all the findings from the thesis.

5.4. Summary

This chapter conclude with two varied but significant case studies. One states an appraisal of the discrete transformation techniques for presenting the higher order discrete-time interval systems accessible to continuous time algorithms. Precisely, it procures the linear transformation for being simple and easily accessible for discrete transformation. Another case study demonstrates various properties of the proposed techniques compared among themselves under one table.

Next chapter concludes the thesis with the possible scope for future works.