Chapter 4
Order Reduction Techniques based on Assorted Approach

4.1. Preamble

Another classification of the proposed algorithms in the thesis is entitled *Assorted Approach.* These are grounded on various procedural steps for computation of reduced models. The *five* proposed algorithms are *Non-Computational Technique, Classical Differentiation Method, Direct Truncation Method,* and *amalgamation of Mikhailov Stability Criterion with Routh Approximation* and *Direct Truncation.* All these algorithms are illustrated below.

4.2. Non-Computational Technique

Other appropriate name for this technique is Shifting Algorithm. This is stated to be superior, simple and direct method for the order reduction of an interval system as compared to the existing methods in the literature. The algorithm is understood better in the next sections.

Methodology

As the name suggest, the coefficients of the higher order system in (2.13) is shifted towards right (*n-k*) times, where *k=1, 2…,* in deriving $R_k(z)$ of order *(n-1), (n-2),…* For an instance, the higher order system pose *n=3* and the desired reduced order model is *k=2*; then the higher order system is shifted (*n-k*) times *i.e.* (*3-2=1*). The algorithm is explained through numerical examples.

The shifting results in the desired k^{th} order model, $k < n$ as

$$R_k(z) = \frac{N_k(z)}{D_k(z)} = \frac{\left[n_{k-1}^-, n_{k-1}^+\right]z^{k-1} + \left[n_{k-2}^-, n_{k-2}^+\right]z^{k-2} + \ldots.. + \left[n_0^-, n_0^+\right]}{\left[d_k^-, d_k^+\right]z^k + \left[d_{k-1}^-, d_{k-1}^+\right]z^{k-1} + \ldots.. + \left[d_0^-, d_0^+\right]} \quad (4.1)$$

Example

E.4.2.1. Consider the third order transfer function from [66], [86] as

$$H_3(z) = \frac{[3.25, 3.35]z^2 + [3.5, 3.65]z + [2.8, 3]}{[5.4, 5.5]z^3 + [1, 1.1]z^2 + [1.5, 1.6]z + [2.1, 2.15]} \quad (4.2)$$

The reduced models obtained by shifting the coefficients *(n-k)* times with *n=3* and *k=1, 2* are

Shifting (*3-2=1*) time and (*3-1=2*) times result second and first order models respectively as

$$R_2(z)=\frac{[3.25,3.35]z+[3.5,3.65]}{[5.4,5.5]z^2+[1,1.1]z^1+[1.5,1.6]} \tag{4.3}$$

$$R_1(z)=\frac{[3.25,3.35]}{[5.4,5.5]z+[1,1.1]} \tag{4.4}$$

The step responses of the higher-order and reduced -order model procured by the proposed method are shown in Figures 4.1 and 4.2 for lower and upper limits transfer functions respectively. In addition, Figures 4.3 and 4.4 depict the frequency response for the lower and upper limit models.

E.4.2.2. Consider the digital control system of eighth order as

$$H_8(z)=\frac{\begin{array}{l}[1.6484,1.7156]z^7+[1.0937,1.1383]z^6+[-0.2142,-0.2058]z^5\\+[0.1490,0.1550]z^4+[-0.5263,-0.5057]z^3+[-0.2672,-0.2568]z^2\\+[0.0431,0.0449]z+[-0.0061,-0.0059]\end{array}}{\begin{array}{l}[23.52,24.48]z^8+[-1.7156,-1.6484]z^7+[-1.1383,-1.0937]z^6\\+[0.2058,0.2142]z^5+[-0.1550,-0.1490]z^4+[0.5057,0.5263]z^3\\+[0.2568,0.3672]z^2+[-0.0449,-0.0431]z+[0.0059,0.0061]\end{array}} \tag{4.5}$$

Reduced models using the proposed method where *n=8* and *k=2, 1* are;
By shifting (*8-2=6*) times second order reduced model is

$$R_2(z)=\frac{[1.6484,1.7156]z+[1.0937,1.1383]}{[23.52,24.48]z^2+[-1.7156,-1.6484]z+[-1.1383,-1.0937]} \tag{4.6}$$

Shifting (*8-1=7*) times result first order reduced model as

$$R_1(z)=\frac{[1.6484,1.7156]}{[23.52,24.48]z+[-1.7156,-1.6484]} \tag{4.7}$$

Step responses for the lower and upper limits are shown in Figures 4.5 and 4.6 respectively. Figures 4.7 and 4.8 demonstrate the frequency response of the two limit transfer functions correspondingly.

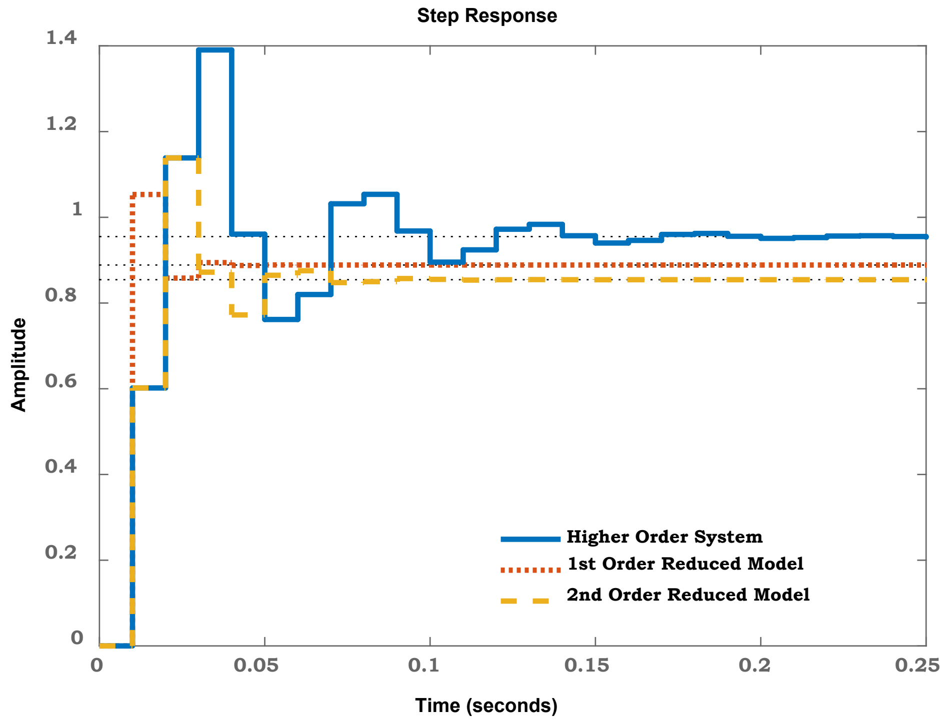

Figure 4.1: Step responses of reduced models (Lower Limit) for E.4.2.1

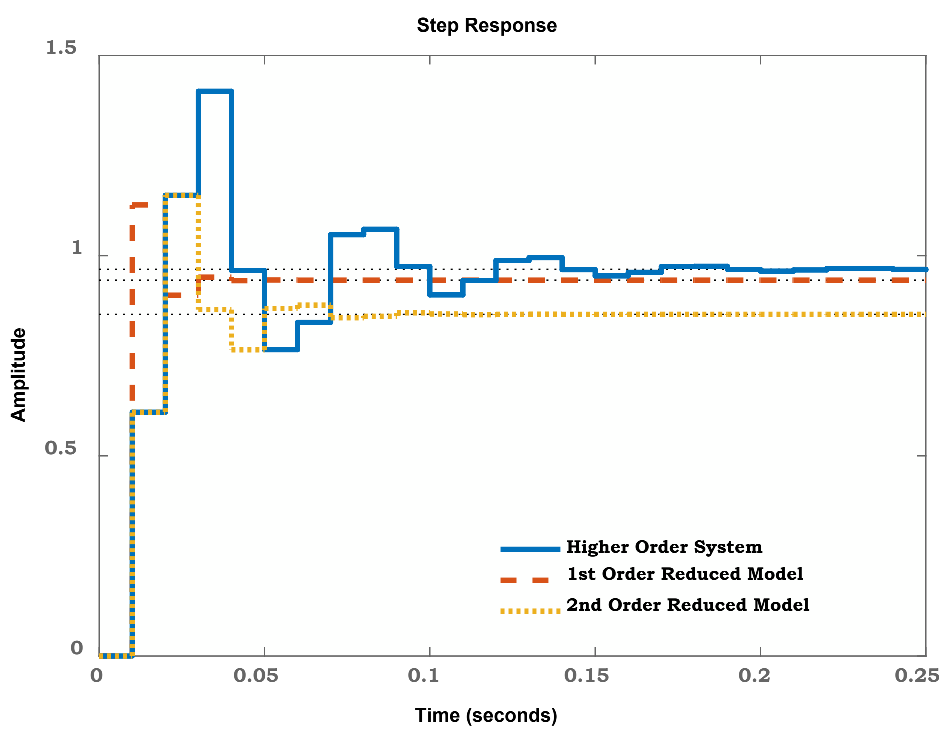

Figure 4.2: Step responses of reduced models (Upper Limit) for E.4.2.1

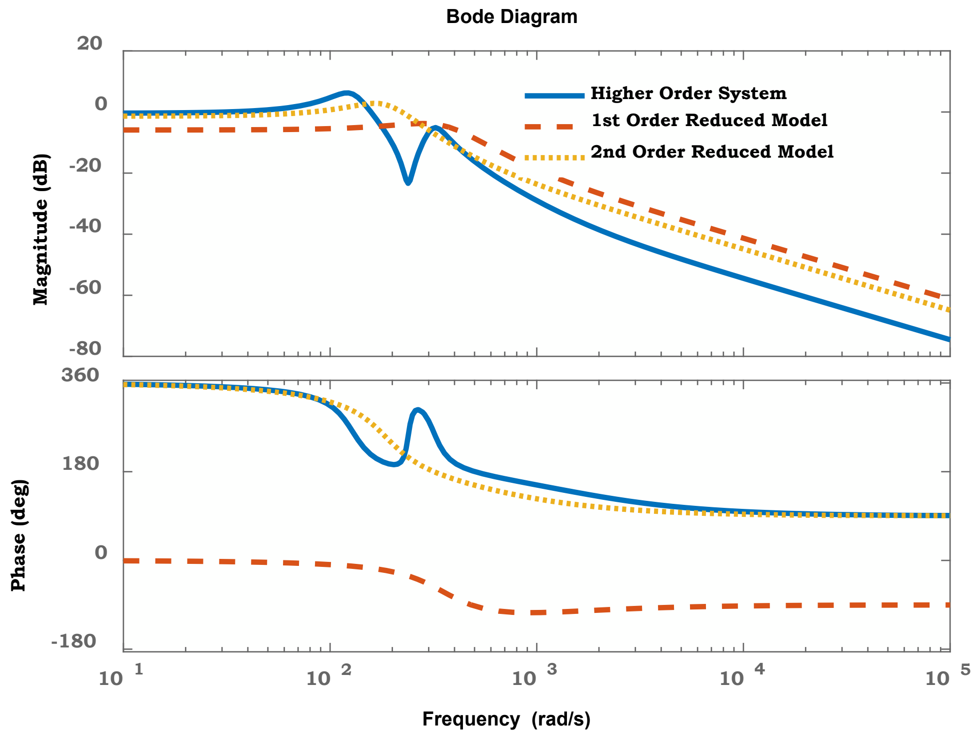

Figure 4.3: Frequency responses of reduced models (Lower Limit) for E.4.2.1

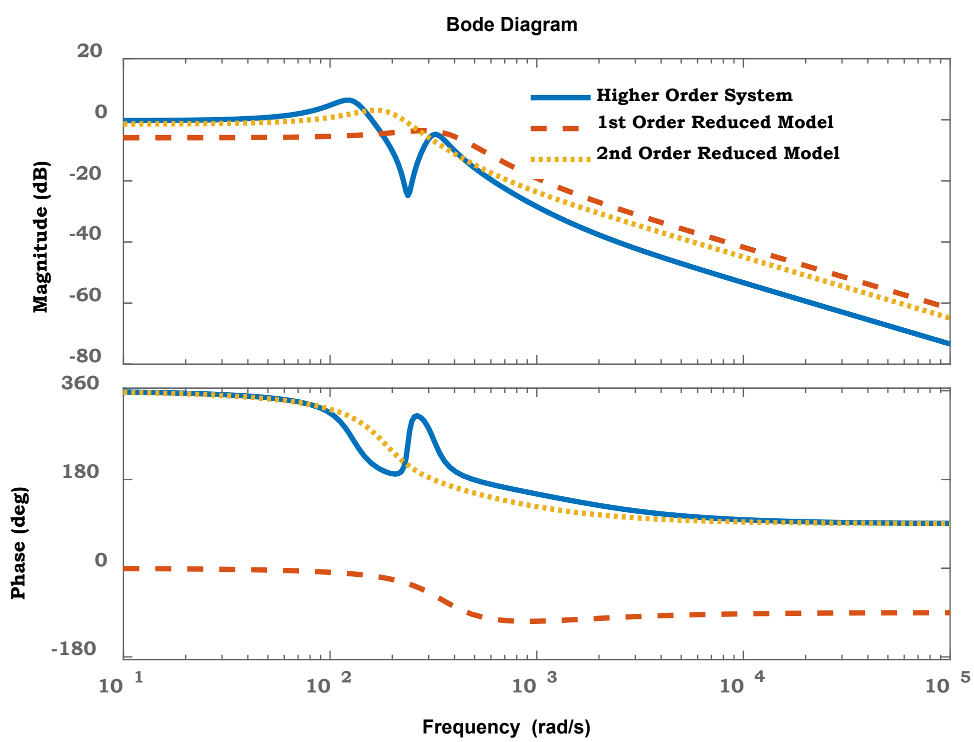

Figure 4.4: Frequency responses of reduced models (Upper Limit) for E.4.2.1

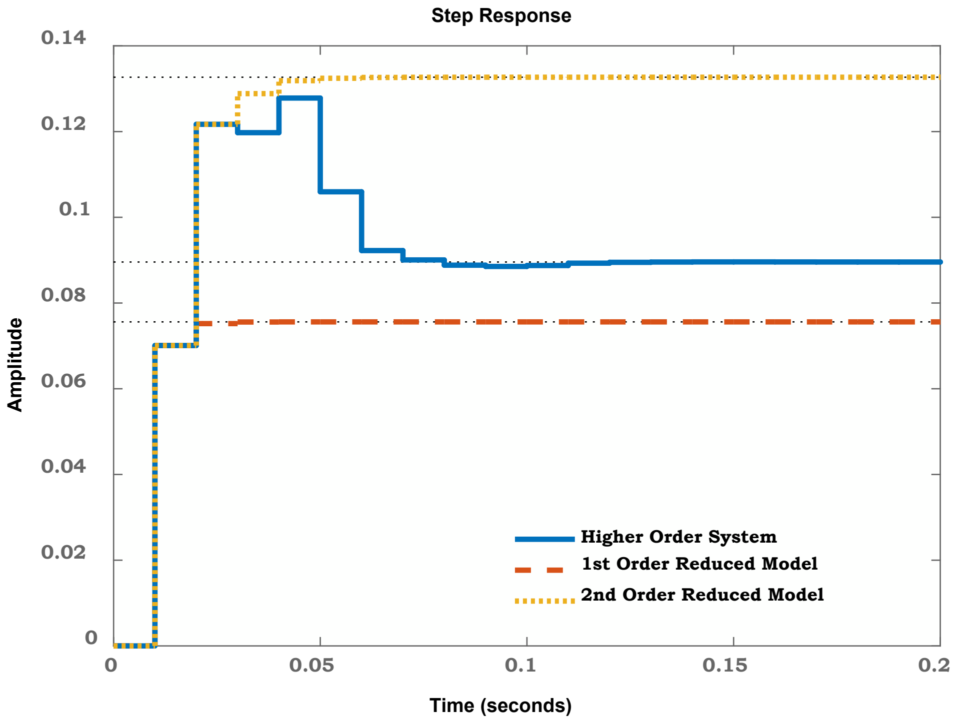

Figure 4.5: Step responses of reduced models (Lower Limit) for E.4.2.2

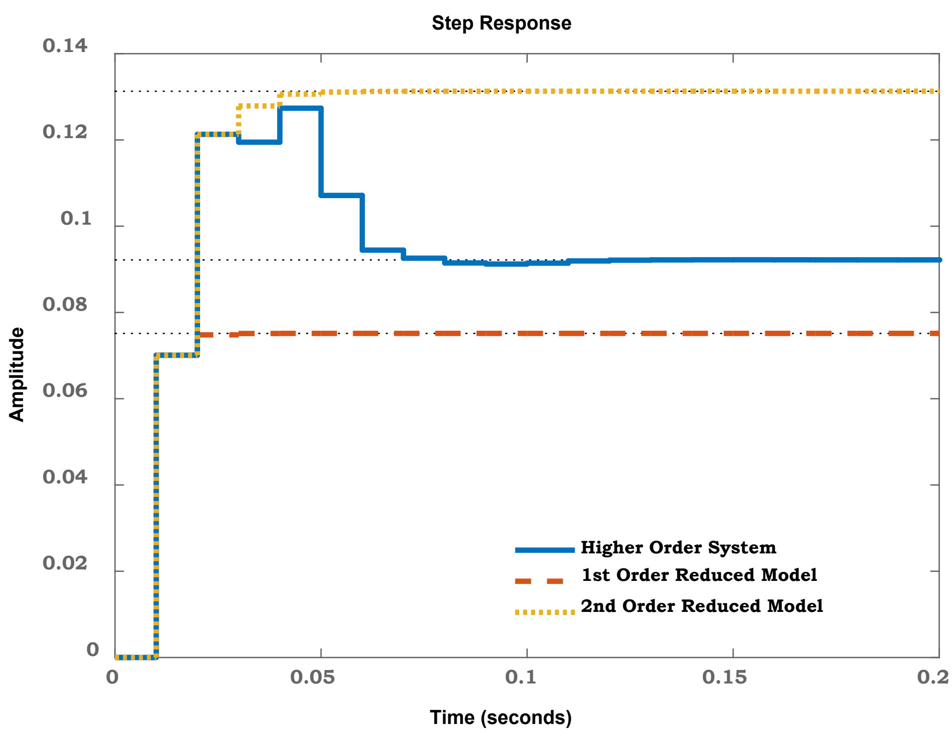

Figure 4.6: Step responses of reduced models (Upper Limit) for E.4.2.2

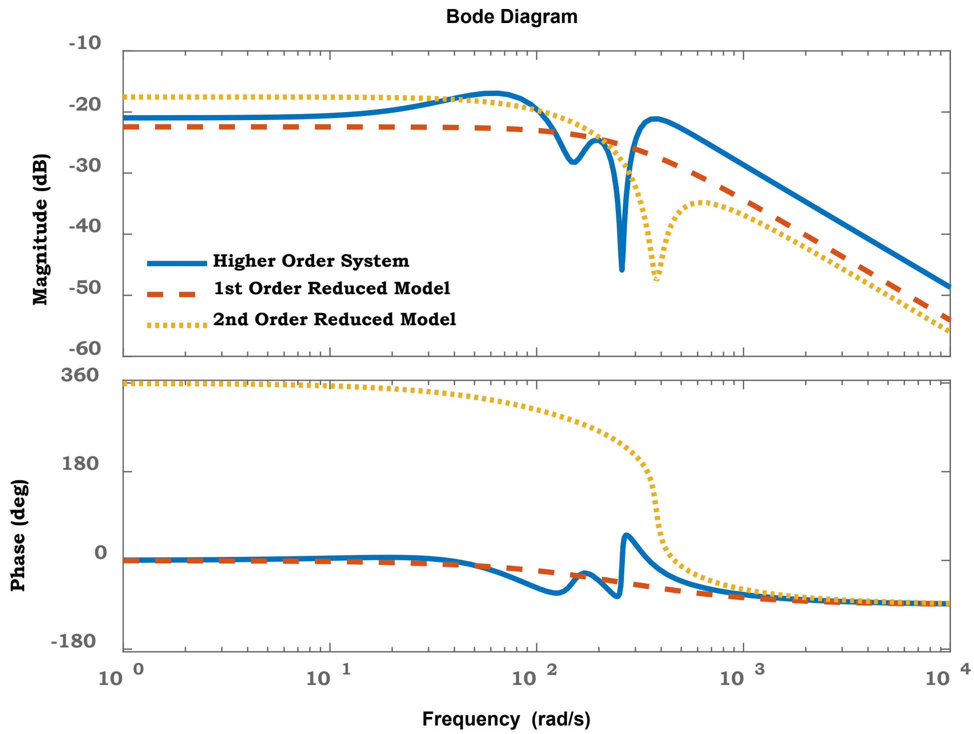

Figure 4.7: Frequency responses of reduced models (Lower Limit) for E.4.2.2

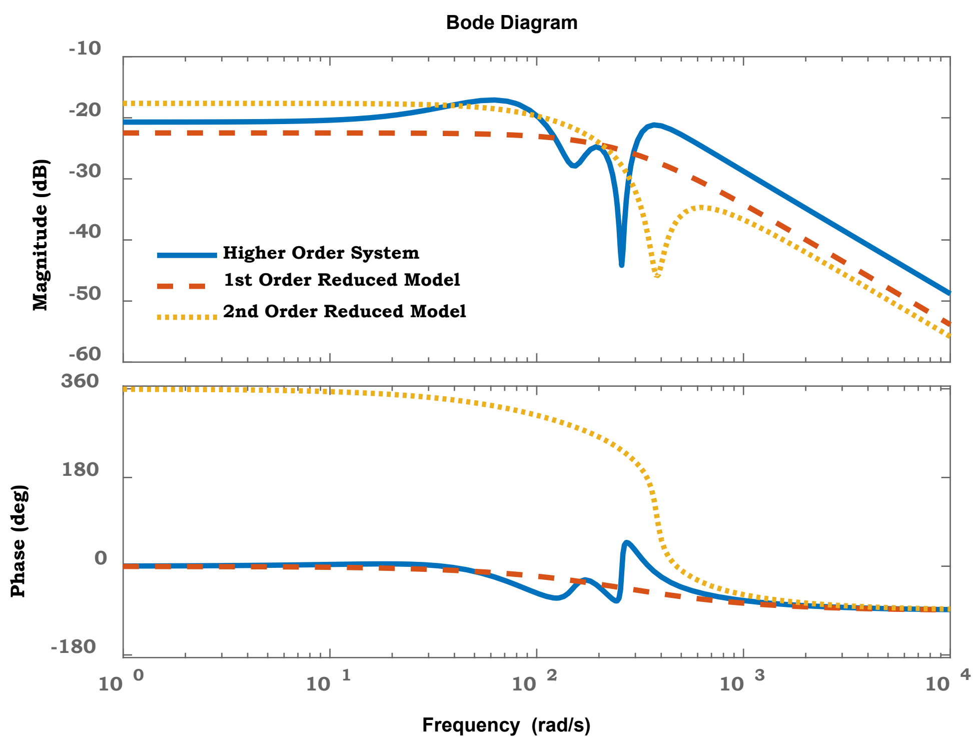

Figure 4.8: Frequency responses of reduced models (Upper Limit) for E.4.2.2

Figures for the above examples deliver an approximate tracking of the step and frequency responses of the reduced order models to the responses of the higher order systems. This lead to the acceptance of the proposed algorithm.

Conclusions

A superior, simple and direct method for the order reduction is proposed.

4.3. Direct Truncation Method

The Truncation algorithm pioneered by Gustafson [109] was for systems with no numerator dynamics in their transfer function. Later on Shamash [110] extended the algorithm to multivariable systems with numerator dynamics and confronted to be simple and computationally superior to the prevailing reduction techniques. Defense for being simple and easily accessible plotted an arena for its consideration towards discrete-time interval system as elaborated below.

Methodology

Exclude the higher order terms and retain the coefficients of desired order transfer function. By the definition of transfer function, the order of numerator should be one less than or equal to that of the denominator.

Precisely, as per the illustration, the denominator polynomial of the k^{th} order reduced model is given as

$$D_k(z) = \left[d_k^-, d_k^+\right] z^k + \left[d_{k-1}^-, d_{k-1}^+\right] z^{k-1} + \ldots + \left[d_0^-, d_0^+\right] \tag{4.8}$$

And the numerator polynomial as

$$N_k(z) = \left[n_{k-1}^-, n_{k-1}^+\right] z^{k-1} + \left[n_{k-2}^-, n_{k-2}^+\right] z^{k-2} + \ldots + \left[n_0^-, n_0^+\right] \tag{4.9}$$

This is better understood by the examples ahead.

Example

E.4.3.1. Consider a third order interval transfer function from [68], [83] be

$$H_3(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{4.10}$$

By the algorithm, the truncated numerator and denominator for second order reduced model are

$$N_2(z) = [3,4]z + [8,10] \tag{4.11}$$

$$D_2(z) = [9,9.5]z^2 + [4.9,5]z + [0.8,0.85] \tag{4.12}$$

This delivers the reduced model as

$$R_2(z) = \frac{[3,4]z + [8,10]}{[9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{4.13}$$

The comparison between errors of the above models with the prevailing models are tabulated in Table 4.1.

Table 4.1: Error for 2nd order reduced models for E.4.3.1

Methods	Error	
	Lower Limit	Upper Limit
Proposed Method	0.0278	0.0077
Pade and Dominant Poles [68]	0.1810	0.0741
Dominant Pole and Direct Series [83]	0.0555	0.0097

The step response of higher order system and lower order models by the proposed method with the other prevailing algorithms are shown in Figure 4.9 (lower limit) and Figure 4.10 (upper limit). The acceptance of the proposed algorithm is affirmed by the followed frequency responses in Figures 4.11 and 4.12 for lower and upper limit transfer functions respectively.

E.4.3.2. Consider a fourth order system from [77] described as

$$H_3(z) = \frac{[12,14]z^3 + [220,240]z^2 + [800,900]z + [1100,1200]}{[1,1.2]z^4 + [16,18]z^3 + [90,100]z^2 + [160,180]z + [110,120]} \tag{4.14}$$

The third order reduced model, by the proposed algorithm is

$$R_3(z) = \frac{[220,240]z^2 + [800,900]z + [1100,1200]}{[16,18]z^3 + [90,100]z^2 + [160,180]z + [110,120]} \tag{4.15}$$

Table 4.2 present the error computed for evaluation.

Table 4.2: Error for 3rd order reduced models for E.4.3.2

Methods	Error	
	Lower Limit	Upper Limit
Proposed Method	3.0625	2.77
Differentiation Tech. [77]	4	5.44

Figures 4.13 and 4.14 depict the frequency response for lower and upper limit reduced models respectively.

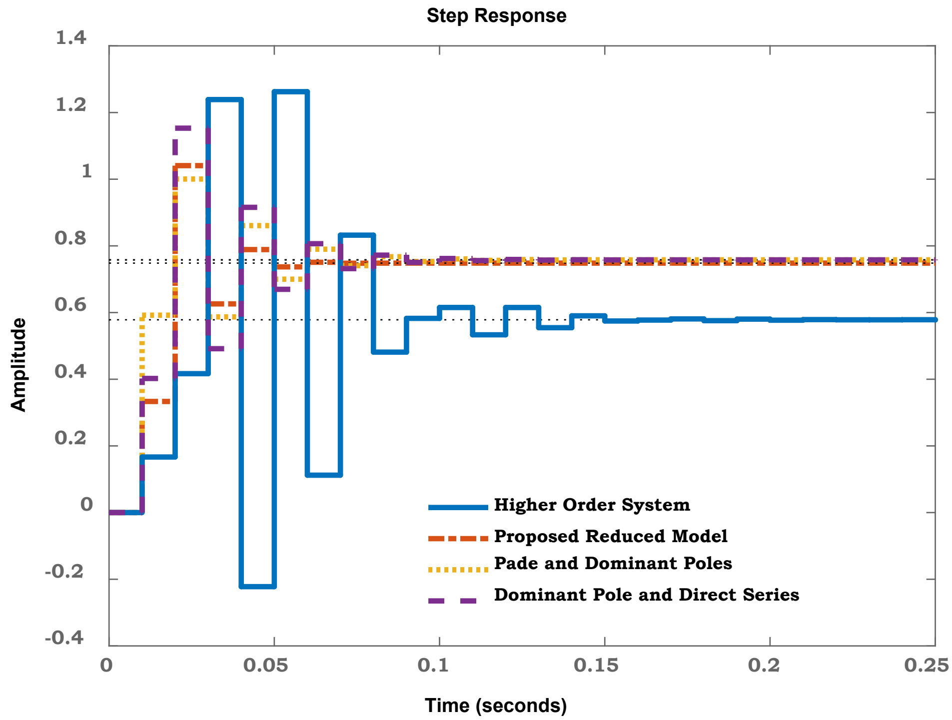

Figure 4.9: Step responses of reduced models (Lower Limit) for E.4.3.1

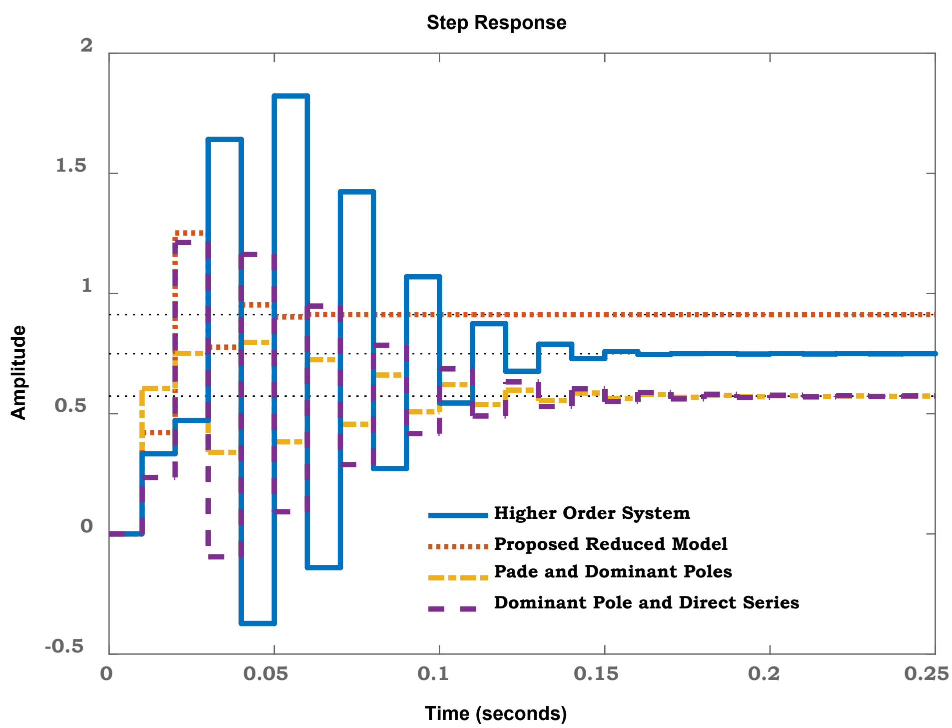

Figure 4.10: Step responses of reduced models (Upper Limit) for E.4.3.1

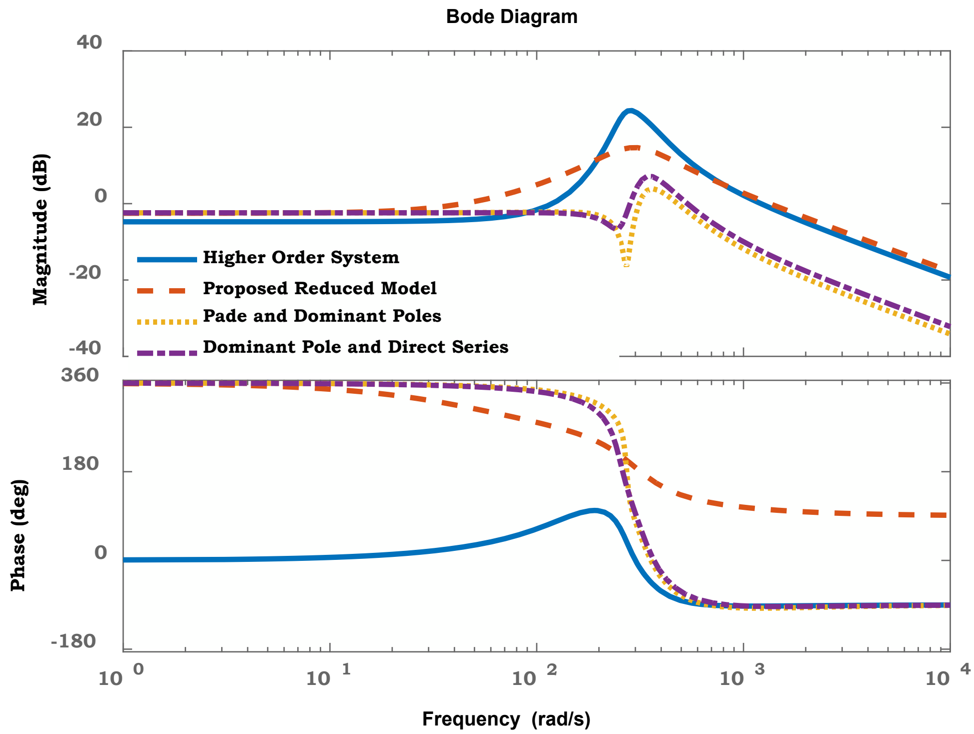

Figure 4.11: Frequency responses of reduced models (Lower Limit) for E.4.3.1

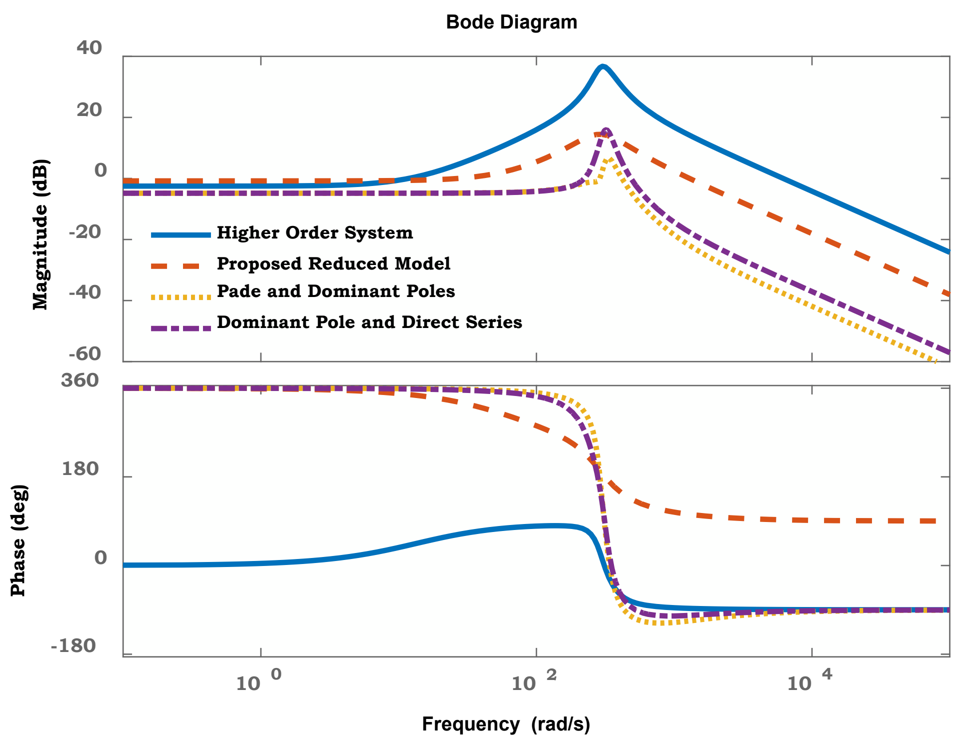

Figure 4.12: Frequency responses of reduced models (Upper Limit) for E.4.3.1

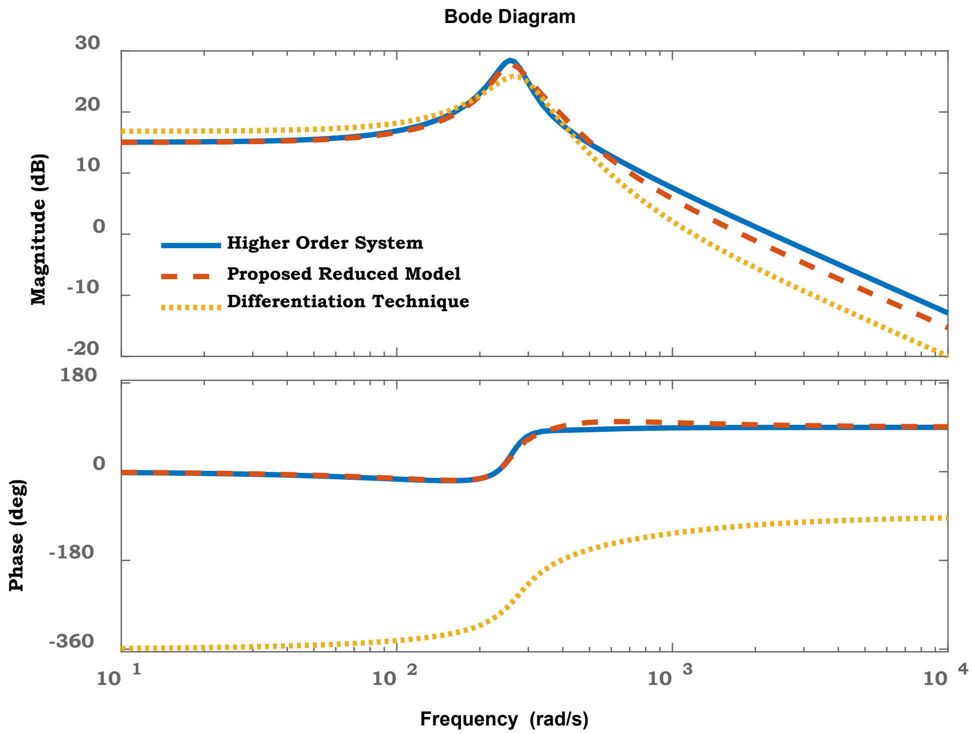

Figure 4.13: Frequency responses of reduced models (Lower Limit) for E.4.3.2

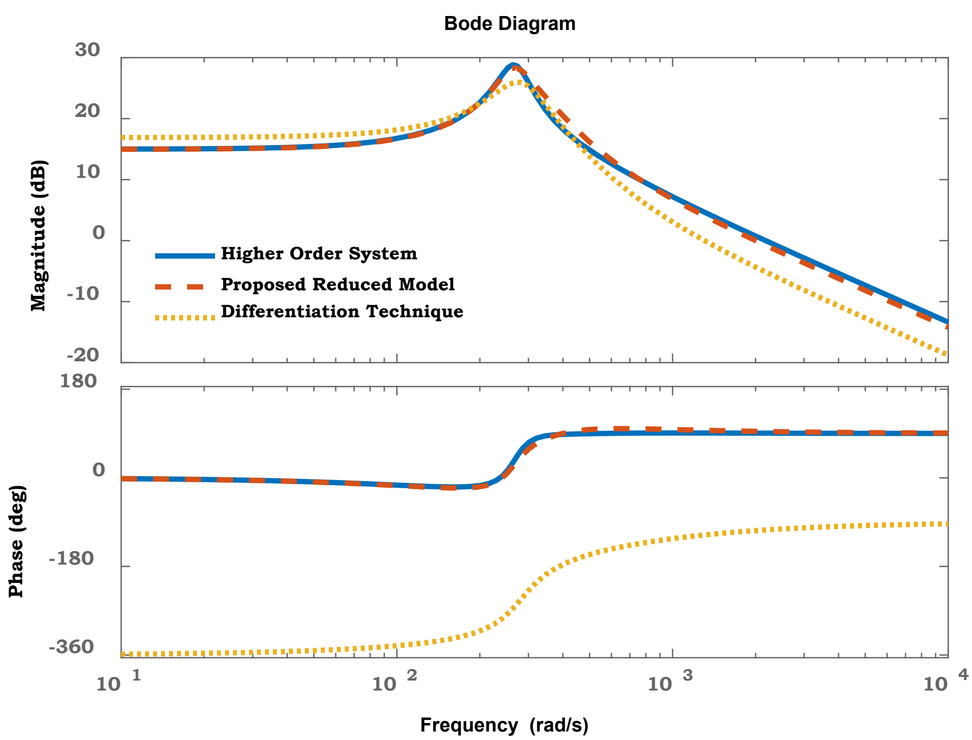

Figure 4.14: Frequency responses of reduced models (Upper Limit) for E.4.3.2

Discussion

Frequency response for E.4.3.1 commit to be stable and E.4.3.2 state to be unstable. This submit that the technique is although simple and acceptable based on error computation, but it lacks the advantage of generating stable models for varied circumstance.

Conclusions

This section elaborates the extension of a well-known and easy to apply method of Direct Truncation to discrete-time interval systems. Also states that the reduced model is not guaranteed to be stable, even if the higher order system is stable.

4.4. Classical Differentiation Technique

In this approach, the fundamental theorem of Calculus *i.e.* Differentiation technique is accessed. Since this theorem is not directly applicable on discrete-time systems, a proper transformation to its equivalent continuous-time domain is performed. As mentioned in Chapter 2, section 2.8, any of the two transformations can be applied, here considered is Euler Forward differentiation technique *i.e.* $z=1+p$ for the ease of computation and simplicity. The algorithm proposed is discussed below in steps.

Methodology

Step 1: Transform $H_n(z)$ in (2.13) to $H_n(p)$ in (2.16).

Step 2: Successive differentiation of $H_n(p)$ gives $R_k(p)$ of order *k*, where *k=n-1, n-2….*

Step 3: Apply inverse transformation $p=z-1$ on $R_k(p)$ to obtain the desired order reduced model $R_k(z)$.

Example

E.4.4.1. Consider the third order transfer function available from [68], [90], [107] as

$$H_3(z)=\frac{[1,2]z^2+[3,4]z+[8,10]}{[6,6]z^3+[9,9.5]z^2+[4.9,5]z+[0.8,0.85]} \tag{4.16}$$

Step 1: $$H_3(p)=\frac{[1,2]p^2+[5,8]p+[12,16]}{[6,6]p^3+[27,27.5]p^2+[40.9,42]p+[20.7,21.35]} \tag{4.17}$$

Step 2: $R_2(p)=\dfrac{[2,4]p+[5,8]}{[18,18]p^2+[54,55]p+[40.9,42]}$ (4.18)

$$R_1(p)=\frac{[2,4]}{[36,36]p+[54,55]} \quad (4.19)$$

Step 3: The second and first order reduced models obtained by proposed algorithm are

$$R_2(z)=\frac{[2,4]z+[1,6]}{[18,18]z^2+[18,19]z+[3,9.6]} \quad (4.20)$$

$$R_1(z)=\frac{[2,4]}{[36,36]z+[18,19]} \quad (4.21)$$

The error for derived $R_1(z)$ and $R_2(z)$ along with errors obtained by other methods is demonstrated in Table 4.3.

Table 4.3: Error for 1st and 2nd order reduced models for E.4.4.1

Methods	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Method	0.0123	0.0094	0.0031	0.0123
Pade/Dominant Poles [68]	0.1398	0.0195	0.1810	0.0741
Gamma-Delta Appr. [90]	0.0157	0.0035	0.1292	0.0443
Direct-Truncation [107]	2.1419	2.7778	0.0278	0.0077

Depiction of the step responses of the higher-order system $H_3(z)$ and the reduced second order models by the proposed and existing algorithms are in Figures 4.15 and 4.16 for lower and upper limits transfer functions respectively. Figures 4.17 and 4.18 present the frequency response for the two limits models correspondingly.

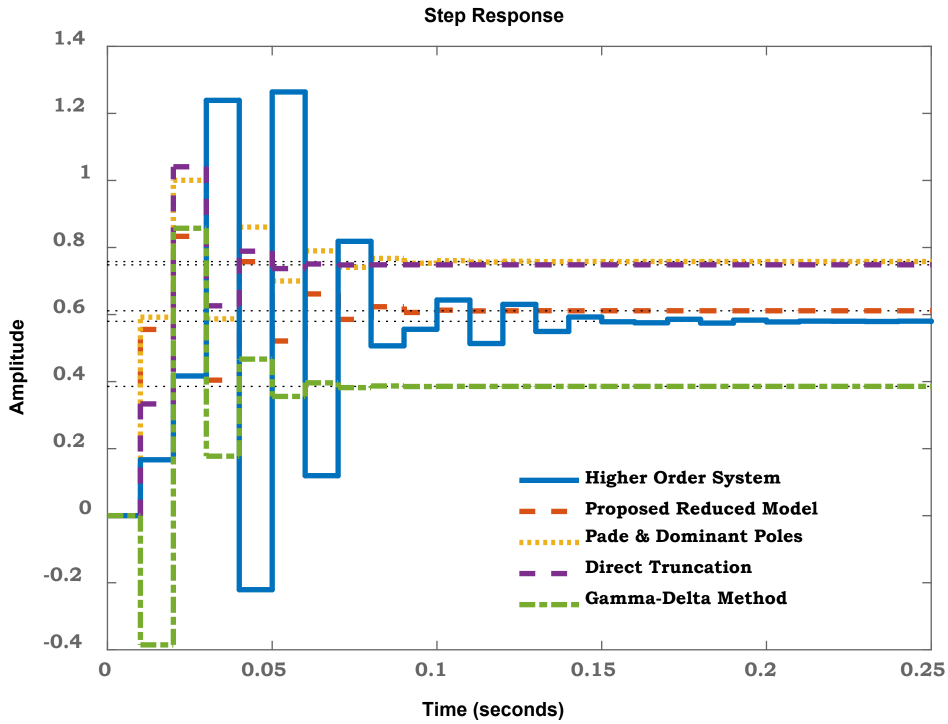

Figure 4.15: Step responses of reduced models (Lower Limit) for E.4.4.1

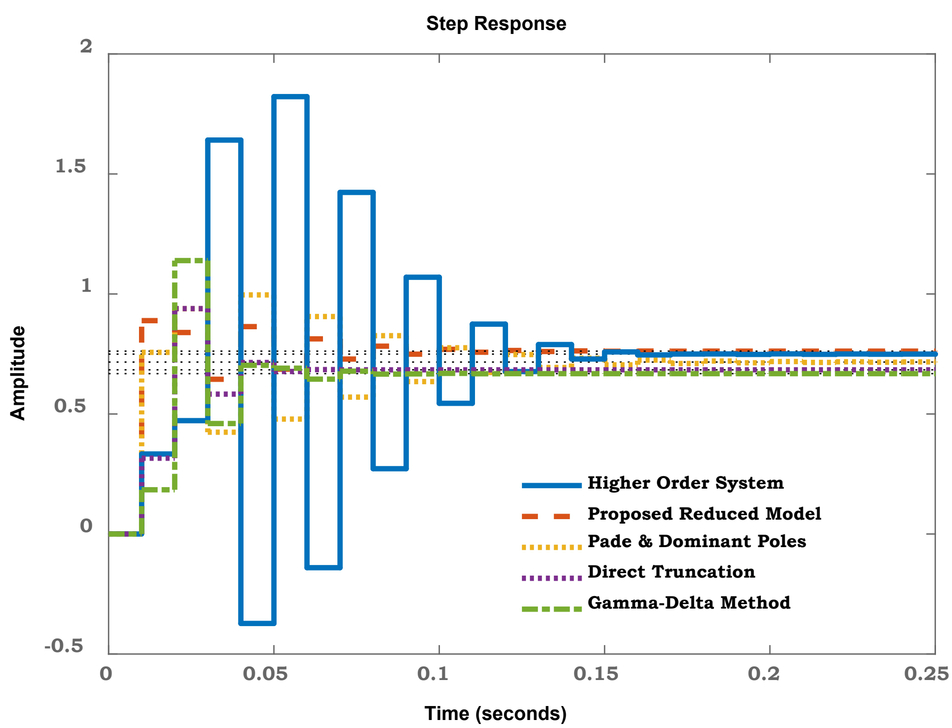

Figure 4.16: Step responses of reduced models (Upper Limit) for E.4.4.1

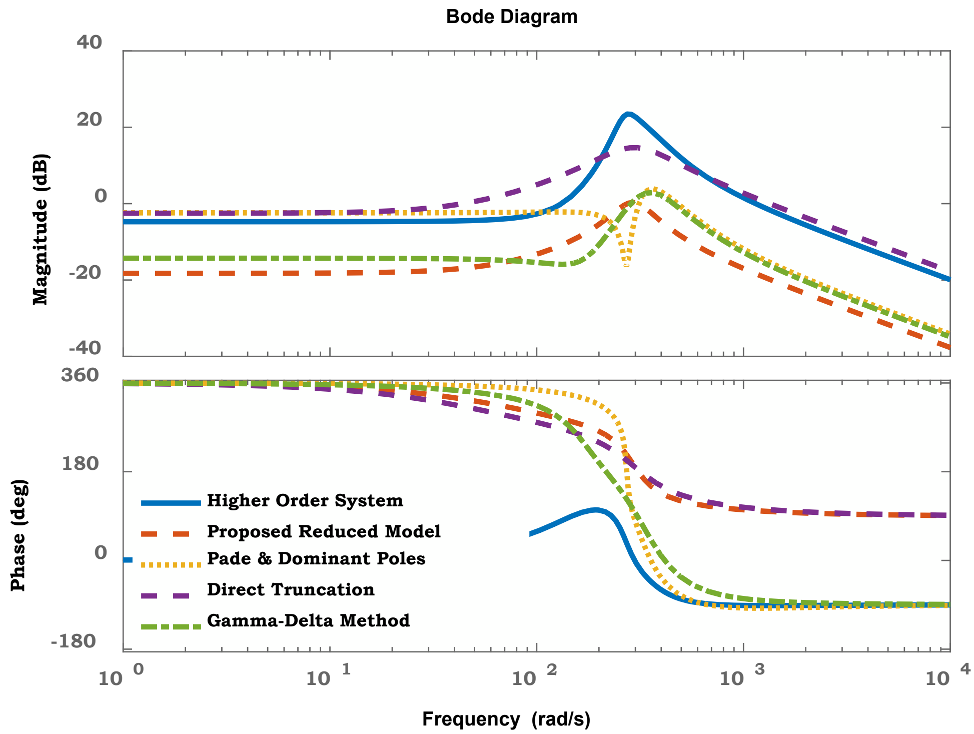

Figure 4.17: Frequency responses of reduced models (Lower Limit) for E.4.4.1

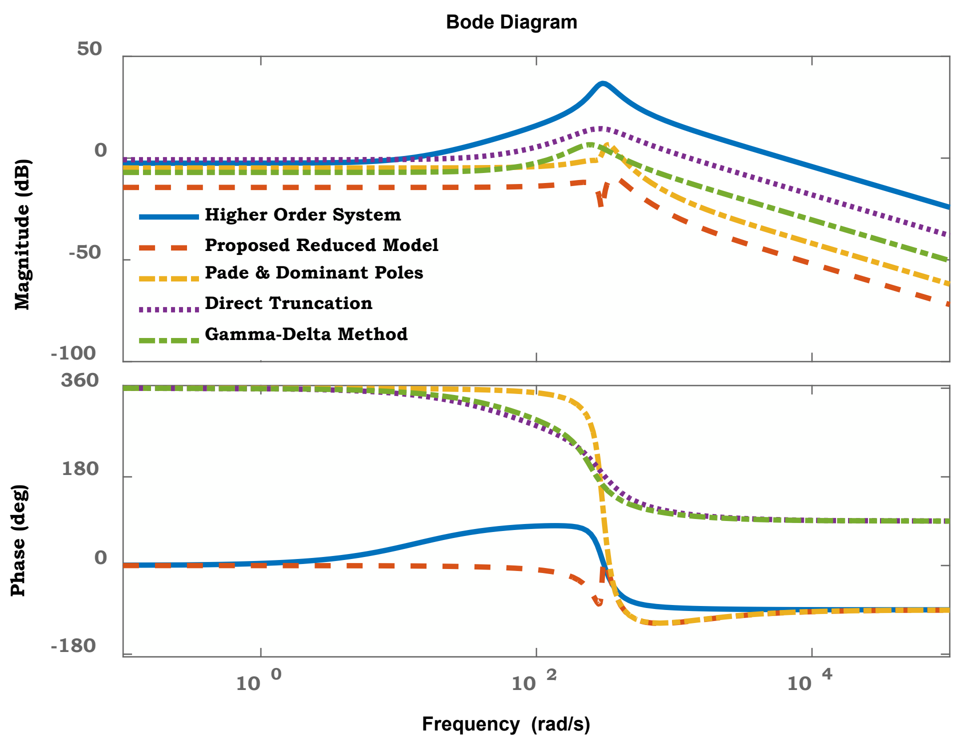

Figure 4.18: Frequency responses of reduced models (Upper Limit) for E.4.4.1

E.4.4.2. Consider the digital control system as

$$H_8(z) = \frac{\begin{matrix}[1.6484,1.7156]z^7 + [1.0937,1.1383]z^6 + [-0.2142,-0.2058]z^5 \\ +[0.1490,0.1550]z^4 + [-0.5263,-0.5057]z^3 + [-0.2672,-0.2568]z^2 \\ +[0.0431,0.0449]z + [-0.0061,-0.0059]\end{matrix}}{\begin{matrix}[23.52,24.48]z^8 + [-1.7156,-1.6484]z^7 + [-1.1383,-1.0937]z^6 \\ +[0.2058,0.2142]z^5 + [-0.1550,-0.1490]z^4 + [0.5057,0.5263]z^3 \\ +[0.2568,0.3672]z^2 + [-0.0449,-0.0431]z + [0.0059,0.0061]\end{matrix}} \tag{4.22}$$

The reduced models of this system are

$$R_2(z) = \frac{[8307.94,8646.62]z + [448.776,1158,264]}{[474566.4,493516.8]z^2 + [-47353.82,29592.86]z + [-39462.26,38258.42]} \tag{4.23}$$

$$R_1(z) = \frac{[8307.936,8646.624]}{[949132.8,987033.6]z + [-47353.824,29592.864]} \tag{4.24}$$

The error for the considered example is in Table 4.4.

Table 4.4: Error for 1st and 2nd order reduced models for E.4.4.2

Methods	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Method	0.0038	0.0038	0.0028	0.0028
Gamma-Delta [90]	0.0021	0.0019	0.0035	0.0034
Direct-Trunc. [107]	0.0096	0.0027	0.0043	0.0045

The step responses of the higher-order system, the respective first and second order reduced models to consolidate the algorithm are shown in Figures 4.19 and 4.20 for the two limits respectively. Later Figures 4.21 and 4.22 depict the frequency responses correspondingly.

Conclusions

Finally the method is submitted to be simple and straightforward in computation. The method is very much different from the earlier existing differential techniques grounded on every aspect. The method employs the differential calculus for order reduction of discrete-time interval system.

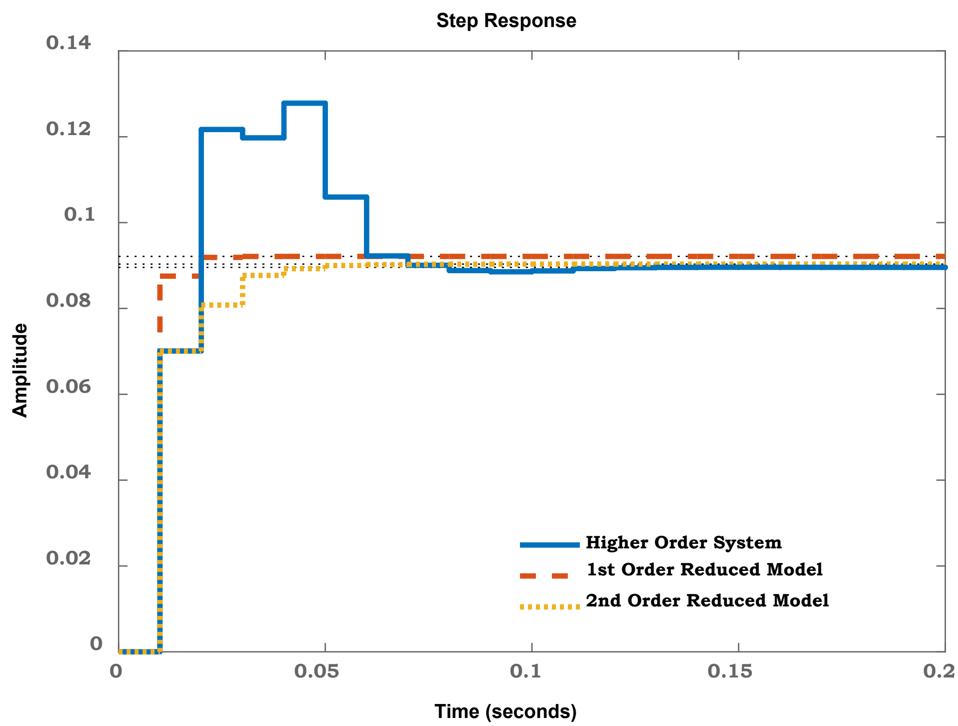

Figure 4.19: Step responses of reduced models (Lower Limit) for E.4.4.2

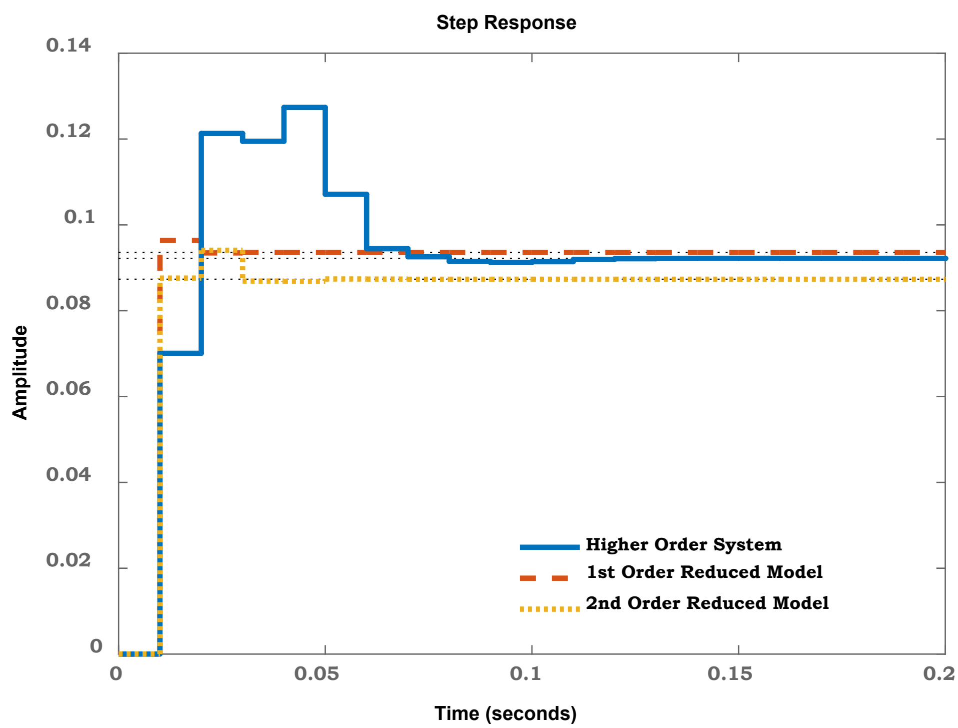

Figure 4.20: Step responses of reduced models (Upper Limit) for E.4.4.2

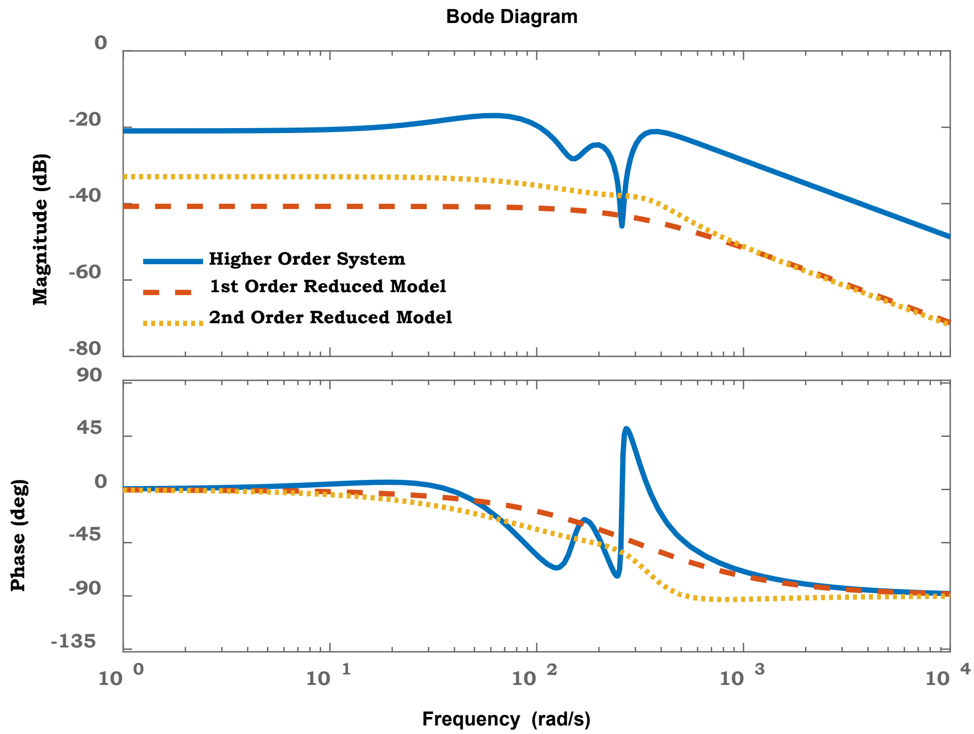

Figure 4.21: Frequency responses of reduced models (Lower Limit) for E.4.4.2

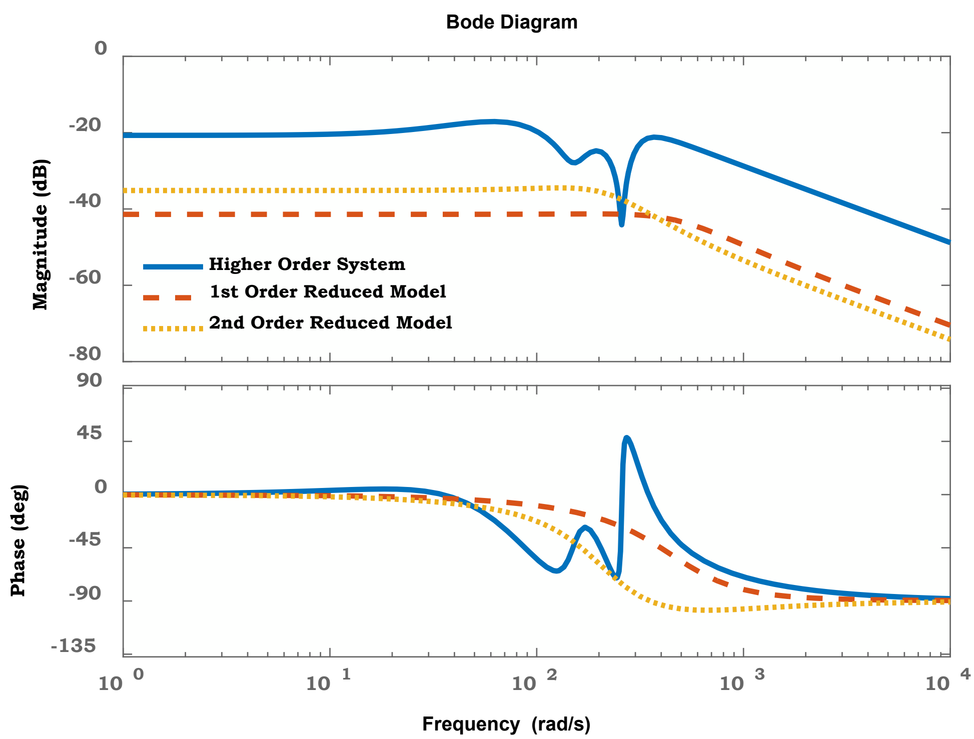

Figure 4.22: Frequency responses of reduced models (Upper Limit) for E.4.4.2

4.5. Using the advantages of Mikhailov Stability Criterion

Mikhailov Stability Criterion is an alternate technique to check stability in conventional control system apart from Routh approximation. Considering the retention of stability characteristic in the reduced model, it is used here interlaced with the other reduction methodologies. Precisely, the reduced order denominator is derived using the Mikhailov criterion and the reduced numerator is by two varied algorithms namely *Routh Approximation* and *Direct Truncation*. Firmly, the present section, confers two different algorithms elaborated ahead.

Methodology

Algorithms discussed under this heading are well known for their significance of retaining stability in continuous-time domain, which would be violated when employed directly to discrete-time domain. Thus, bilinear transformation is used to yield $H_n(w)$ from $H_n(z)$ as in (2.15).

Usage of Mikhailov criterion to compute the reduced order denominator of k^{th} order is as;

Replace $w = j\omega$ in $A_n(w)$ and separate the polynomial in real and imaginary parts as in [111]

$$A_n(j\omega) = \left[\alpha_{1,1}^-, \alpha_{1,1}^+\right] + \left[\alpha_{1,2}^-, \alpha_{1,2}^+\right](j\omega) + \ldots + \left[\alpha_{1,n+1}^-, \alpha_{1,n+1}^+\right](j\omega)^n \tag{4.25}$$

where $\left[\alpha_{1,1}^-, \alpha_{1,1}^+\right] = \left[\alpha_0^-, \alpha_0^+\right]$, ...$\left[\alpha_{1,n+1}^-, \alpha_{1,n+1}^+\right] = \left[\alpha_n^-, \alpha_n^+\right]$

$$= \left(\left[\alpha_{1,1}^-, \alpha_{1,1}^+\right] - \left[\alpha_{1,3}^-, \alpha_{1,3}^+\right]\omega^2 + \ldots\right) + j\left(\left[\alpha_{1,2}^-, \alpha_{1,2}^+\right]\omega - \left[\alpha_{1,4}^-, \alpha_{1,4}^+\right]\omega^3 + \ldots\right) \tag{4.26}$$

$$= \xi(\omega) + j\eta(w) \tag{4.27}$$

with ω as the angular frequency in rad/sec.

From (4.27), equate $\xi(\omega) = 0$ and $\eta(\omega) = 0$, to determine the intersecting frequencies $\omega_i = 0, \pm\left[\omega_1^-, \omega_1^+\right], \ldots, \pm\left[\omega_{n-1}^-, \omega_{n-1}^+\right]$, where $\left\|\left[\omega_1^-, \omega_1^+\right]\right\| < \left\|\left[\omega_2^-, \omega_2^+\right]\right\| < \ldots < \left\|\left[\omega_{n-1}^-, \omega_{n-1}^+\right]\right\|$

Thereafter, consider $A_k(w)$ as the expected approximated denominator and replace $w = j\omega$, that results

$$A_k(j\omega) = \phi(\omega) + j\psi(\omega) \tag{4.28}$$

with $\phi(\omega) = \left[d_{1,1}^-, d_{1,1}^+\right] - \left[d_{1,3}^-, d_{1,3}^+\right]\omega^2 + \ldots$ and $\psi(\omega) = \left[d_{1,2}^-, d_{1,2}^+\right]\omega - \left[d_{1,4}^-, d_{1,4}^+\right]\omega^3 + \ldots$

To defend the stability of the reduced model, its Mikhailov frequency characteristic is intersected with abscissa and ordinate axis's alternatively, k times (the order of the reduced model is k) in the same manner as that of the higher order system. Thus, the first k number of intersecting frequencies $0, \left[\omega_1^-, \omega_1^+\right], \left[\omega_2^-, \omega_2^+\right], \ldots, \left[\omega_{k-1}^-, \omega_{k-1}^+\right]$ are kept unchanged and set to be the roots of $\phi(\omega) = 0$ and $\psi(\omega) = 0$.

Then, $\phi(\omega) = \left[\lambda_1^-, \lambda_1^+\right]\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_3^2\right)\left(\omega^2 - \omega_5^2\right)$ (4.29a)

$$\psi(\omega) = \left[\lambda_2^-, \lambda_2^+\right]\left(\omega^2 - \omega_2^2\right)\left(\omega^2 - \omega_4^2\right)\left(\omega^2 - \omega_6^2\right) \quad (4.29b)$$

for $\omega_i^2 = \left[\omega_i^-, \omega_i^+\right]^2$ $\quad$ *i=1, 2, 3,………*

Values of $\left[\lambda_1^-, \lambda_1^+\right]$ and $\left[\lambda_2^-, \lambda_2^+\right]$ are calculated by equating $\xi(0) = \phi(0)$ and $\eta\left(\left[\omega_1^-, \omega_1^+\right]\right) = \psi\left(\left[\omega_1^-, \omega_1^+\right]\right)$, which on substitution in (4.29), proceed to the derivation of

$$A_k(j\omega) = \phi(\omega) + j\psi(\omega) \quad (4.30)$$

In the above incurred $A_k(j\omega)$, replace $j\omega = w$ to acquire the cut down denominator, which on inverse transformation confers the desired $D_k(z)$.

$$A_k(w) = \left[d_{1,1}^-, d_{1,1}^+\right] + \left[d_{1,2}^-, d_{1,2}^+\right]w + \ldots + \left[d_{1,k+1}^-, d_{1,k+1}^+\right]w^k \quad (4.31)$$

The above description delivers the computation of reduced denominator by Mikhailov criterion. Henceforth, the derivation of the desired order numerator, resulting in the overall reduced order model through two varied algorithms is discussed below.

***Algorithm 1:* Routh Approximation**

For computing the reduced order numerator, the acquainted w-domain transfer function $H_n(w)$ is utilized here. The first two rows of the Routh array shown in Table 4.5 are drafted from the numerator polynomial $B_n(w)$.

The advancement in the table array below third row is by the conventional Routh algorithm described in (4.32) where $i \geq 3$ and $1 \leq j \leq \left[(n - i + 3)/2\right]$

$$\left[b_{i,j}^-, b_{i,j}^+\right] = \left[b_{i-2,j+1}^-, b_{i-2,j+1}^+\right] - \frac{\left[b_{i-2,j}^-, b_{i-2,j}^+\right]\left[b_{i-1,j+1}^-, b_{i-1,j+1}^+\right]}{\left[b_{i-1,1}^-, b_{i-1,1}^+\right]} \quad (4.32)$$

Once the numerator table is put forward, the reduced $B_k(w)$ is derived using it's $(n+1-k)th$ and $(n+2-k)th$ rows as

$$B_k(w)=\left[b^-_{n+1-k,1},b^+_{n+1-k,1}\right]w^{k-1}+\left[b^-_{n+2-k,1},b^+_{n+2-k,1}\right]w^{k-2}+\left[b^-_{n+1-k,2},b^+_{n+1-k,2}\right]w^{k-3}+\ldots \tag{4.33}$$

Table 4.5: Numerator array

$\left[b^-_n,\ b^+_n\right] = \left[b^-_{1,1},\ b^+_{1,1}\right]$	$\left[b^-_{n-2},\ b^+_{n-2}\right] = \left[b^-_{1,2},\ b^+_{1,2}\right]$	$\left[b^-_{n-4},\ b^+_{n-4}\right] = \left[b^-_{1,3},\ b^+_{1,3}\right]$	$\cdots$
$\left[b^-_{n-1},\ b^+_{n-1}\right] = \left[b^-_{2,1},\ b^+_{2,1}\right]$	$\left[b^-_{n-3},\ b^+_{n-3}\right] = \left[b^-_{2,2},\ b^+_{2,2}\right]$	$\left[b^-_{n-5},\ b^+_{n-5}\right] = \left[b^-_{2,3},\ b^+_{2,3}\right]$	$\cdots$
$\left[b^-_{3,1},\ b^+_{3,1}\right]$	$\left[b^-_{3,2},\ b^+_{3,2}\right]$		
.	.		
$\left[b^-_{n,1},\ b^+_{n,1}\right]$			

The computed $B_k(w)$ and the obtained $A_k(w)$ results in the desired $R_k(w)$ by appropriate substitution, which on inverse transformation *i.e.* $w=(z-1/z+1)$ confers $R_k(z)$.

Algorithm 2: **Direct Truncation**

This is another approach to obtain the reduced order numerator. By this algorithm, as elaborated in section 4.3 [107], the desired order numerator polynomial $N_k(z)$ is truncated directly from $N_n(z)$ which on appropriate substitution in $R_k(z)=N_k(z)/D_k(z)$ offer the reduced order model.

$D_k(z)$ is considered from the Mikhailov Stability criterion.

The selection of the order of numerator polynomial is significantly one less than the desired order denominator. Examples hereunder accompany the better understanding of the above algorithms.

Example

E.4.5.1. Consider the higher order system and its w-domain equivalent as

$$H_3(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{4.34}$$

$$H_3(w) = \frac{B_3(w)}{A_3(w)} = \frac{[-9,-5]w^3 + [17,27]w^2 + [-34,-24]w + [12,16]}{[0.55,1.2]w^3 + [5.9,6.65]w^2 + [19.45,20.2]w + [20.7,21.35]} \tag{4.35}$$

Reduced order denominator is derived by the Mikhailov Stability criterion; where substitution of $w = j\omega$ in $A_3(w)$ from (4.35) gives

$$A_3(j\omega) = \left([20.7,21.35] - [5.9,6.65]\omega^2\right) + j\left([19.45,20.2]w - [0.55,1.2]\omega^3\right) \tag{4.36}$$

The roots are obtained as $\left[\omega_i^-, \omega_i^+\right] = 0, \pm[1.76,1.9], \pm[4.02,6.05]$ and values of $\left[\lambda_1^-, \lambda_1^+\right] = -[5.73,6.86]$ and $\left[\lambda_2^-, \lambda_2^+\right] = [15.12,18.49]$ as per the algorithmic rules.

Finally, the desired denominator in *w*-domain and its *z*-domain equivalent after inverse transformation is obtained as

$$A_2(w) = [5.73,6.86]w^2 + [15.12,18.49]w + [17.82,24.76] \tag{4.37}$$

$$D_2(z) = [38.67,50.11]z^2 + [21.92,38.06]z + [5.06,16.5] \tag{4.38}$$

Once, the denominator is received, the numerator polynomials are produced by applying two varied algorithms described below;

Algorithm 1

The Routh approximation rule is applied here. The complete Routh array drafted with the numerator polynomial taken from (3.45) is shown in Table 4.6 offering the coefficients of the reduced order numerator formulated by (4.33) as

$$B_2(w) = [-31.77,-15.52]w + [12,16] \tag{4.39}$$

Table 4.6: Numerator array for E.4.5.1

w^3	$[-9,-5]$	$[-34,-24]$
w^2	$[17,27]$	$[12,16]$
w^1	$[-31.77,-15.52]$	
w^0	$[12,16]$	

Since the reduced order numerator and denominator polynomials are obtained in w-domain, conceive (4.37) and (4.39) for forming the desired reduced order model which on appropriate inverse transformation lead to $R_2(z)$ as

$$R_2(z)=\frac{[-19.77,0.48]z^2+[24,32]z+[27.52,47.77]}{[38.67,50.11]z^2+[21.92,38.06]z+[5.06,16.5]} \tag{4.40}$$

Algorithm 2

The direct truncation truncates the numerator polynomial from $N_3(z)$ to receive the reduced order numerator polynomial as

$$N_2(z)=[3,4]z+[8,10] \tag{4.41}$$

This derived numerator $N_2(z)$ is combined with the above obtained reduced denominator from (4.38) both in z-domain resulting in the reduced model as

$$R_2(z)=\frac{[3,4]z+[8,10]}{[38.67,50.11]z^2+[21.92,38.06]z+[5.06,16.5]} \tag{4.42}$$

For the considered example, the reduced order models are computed by the two different algorithms. Their validation through the error computation is made known in Table 4.7 and is compared to the prevailing techniques. The step responses of the reduced models through different algorithms is depicted in Figures 4.23 and 4.24 for the two limit transfer functions respectively. Figures 4.25 and 4.26 present their frequency responses correspondingly.

Table 4.7: Error for 2nd order reduced models for E.4.5.1

Methods	Error	
	Lower Limit	Upper Limit
Proposed Case 1 (Routh Appr. & Mikhailov)	0.3154	0.0947
Proposed Case 2 (Direct Trunc. & Mikhailov)	0.0079	0.0643
Pade and Dominant Poles [68]	0.1810	0.0741
Dominant poles & Direct Series [83]	0.3237	0.3229
Mikhailov & Factor Division [88]	0.0105	0.3250
Gamma-Delta Approximation [90]	0.1292	0.0443
Direct Truncation [107]	0.0278	0.0077

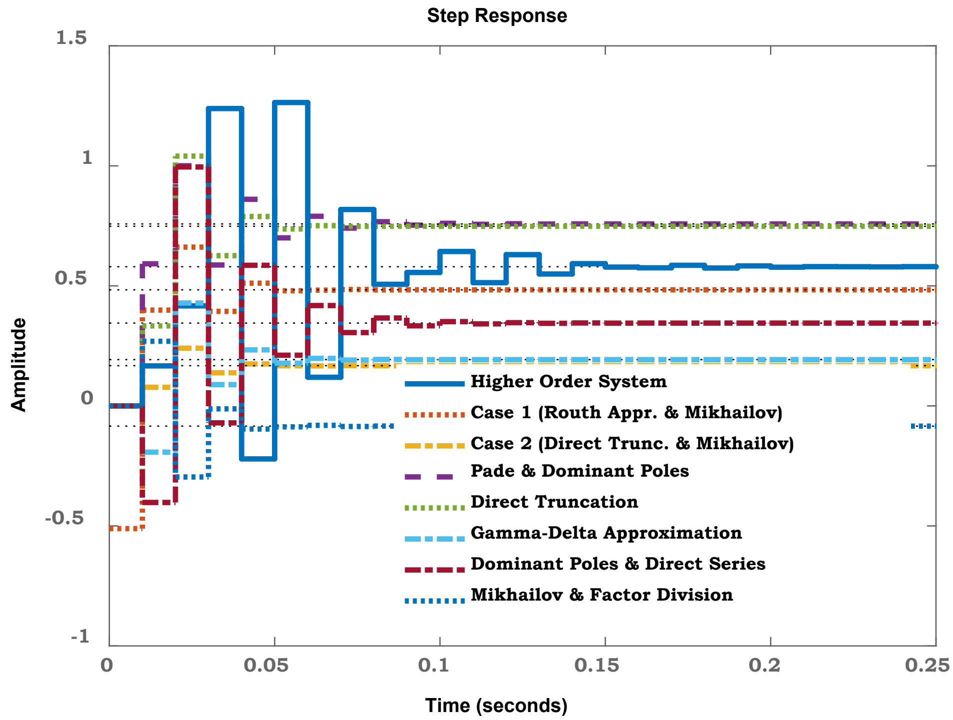

Figure 4.23: Step responses of reduced models (Lower Limit) for E.4.5.1

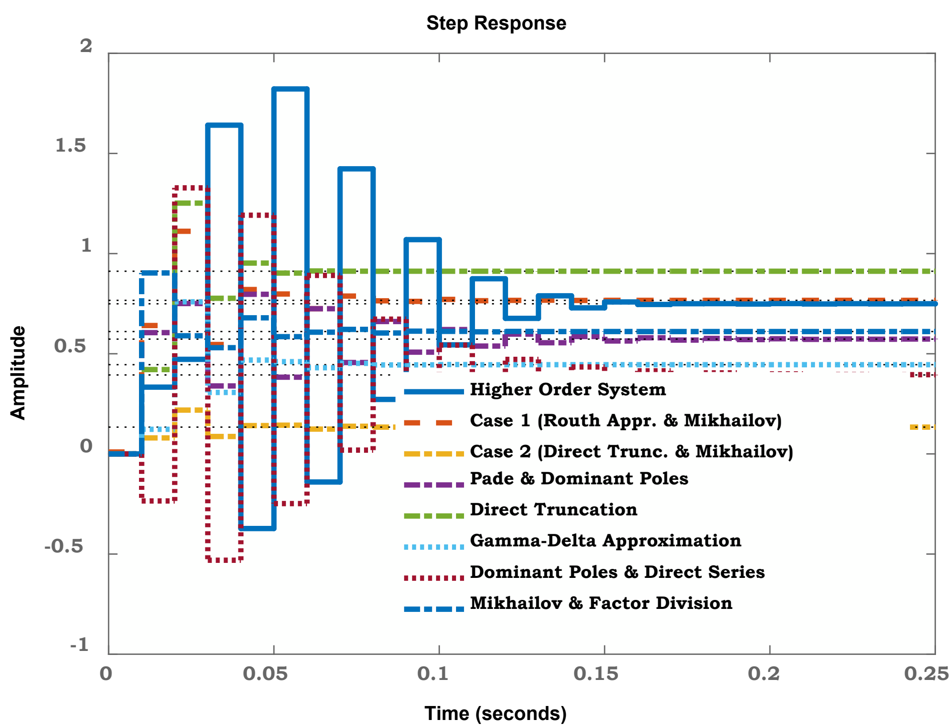

Figure 4.24: Step responses of reduced models (Upper Limit) for E.4.5.1

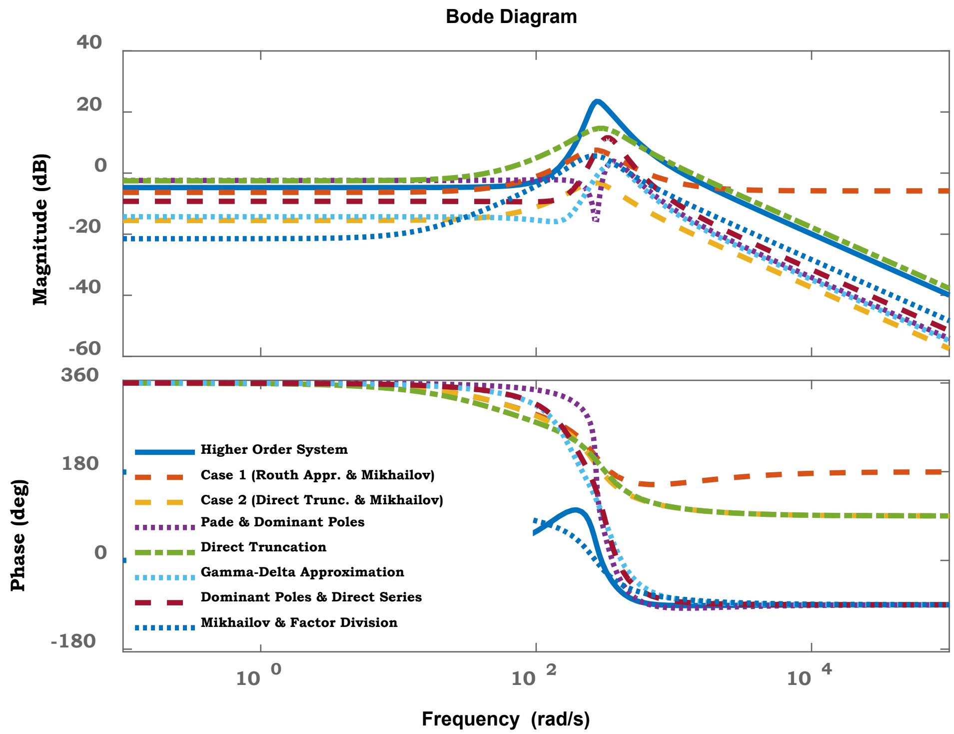

Figure 4.25: Frequency responses of reduced models (Lower Limit) for E.4.5.1

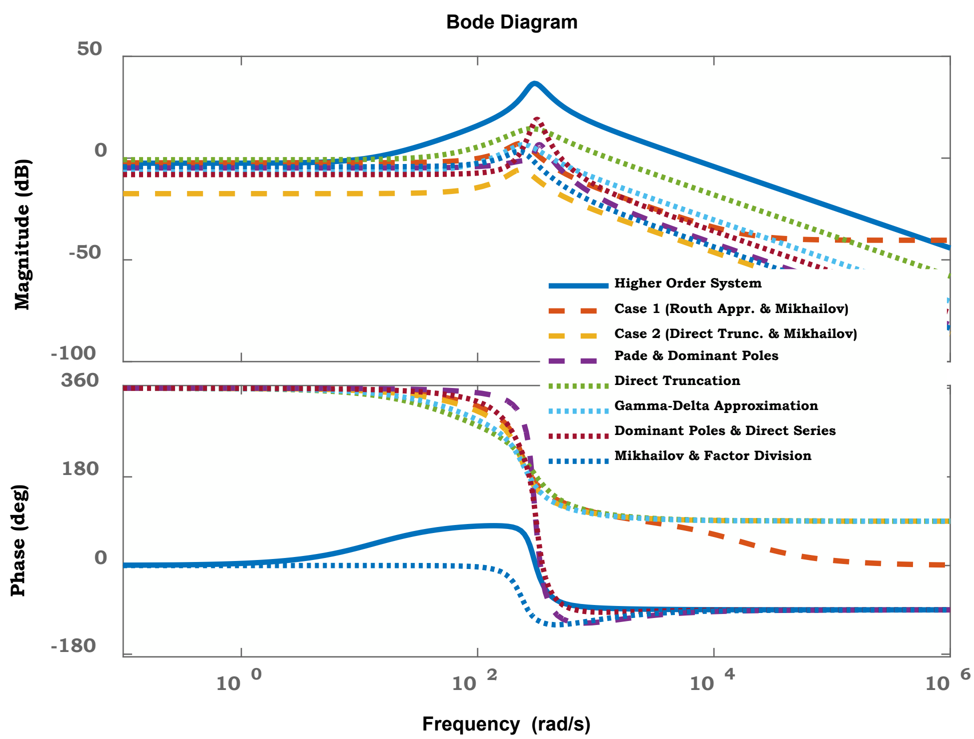

Figure 4.26: Frequency responses of reduced models (Upper Limit) for E.4.5.1.

The minimum error computation and appropriate tracking of the reduced order models to the higher order system response assure for the proposed algorithms over the prevailing techniques. Next example will project the acceptance of the algorithms over the real time systems.

E.4.5.2. Consider the real-time digital control system with transfer function

$$H_8(z)=\frac{\begin{array}{l}[1.6484,1.7156]z^7+[1.0937,1.1383]z^6+[-0.2142,-0.2058]z^5\\+[0.1490,0.1550]z^4+[-0.5263,-0.5057]z^3+[-0.2672,-0.2568]z^2\\+[0.0431,0.0449]z+[-0.0061,-0.0059]\end{array}}{\begin{array}{l}[23.52,24.48]z^8+[-1.7156,-1.6484]z^7+[-1.1383,-1.0937]z^6\\+[0.2058,0.2142]z^5+[-0.1550,-0.1490]z^4+[0.5057,0.5263]z^3\\+[0.2568,0.3672]z^2+[-0.0449,-0.0431]z+[0.0059,0.0061]\end{array}} \tag{4.43}$$

By the above elaborated algorithms, the reduced order model are obtained as

Algorithm 1

$$R_2(z)=\frac{[13.21,21.14]z^2+[3.84,4.14]z+[-17.15,-9.22]}{[714.09,813.60]z^2+[-1264.4,-1094.6]z+[454.38,553.89]} \tag{4.44}$$

Algorithm 2

$$R_2(z)=\frac{[0.0431,0.0449]z+[-0.0061,-0.0059]}{[714.09,813.60]z^2+[-1264.4,-1094.6]z+[454.38,553.89]} \tag{4.45}$$

Table 4.8 present the minimum error for the considered example that showcase the acceptance for the adoption of the proposed algorithms.

Table 4.8: Error for 2nd order reduced models for E.4.5.2

Methods	Error	
	Lower Limit	Upper Limit
Proposed Case 1 (Routh Appr. & Mikhailov)	5.2320×10^{-4}	6.9155×10^{-4}
Proposed Case 2 (Direct Trunc. & Mikhailov)	0.0049	0.0049
Direct-Truncation [107]	0.0043	0.0045
Gamma-Delta Appr. [90]	0.0035	0.0034

Discussion

This section discourse the valuable findings from the algorithms. A firm query that rises is what would be the basis for obtaining the reduced order numerator and denominator polynomials? To answer this is, by the definition of transfer function available in various control system books, the order of the numerator should be equal or one less than that of the denominator. So, the order of the polynomials should be strict and not be hampered. A reasonable explanation for the witnessing the varied orders of the transfer functions is as below;

Equation (2.14) which is the mathematical representation of the desired reduced order model, express the numerator polynomials to be one less than that of the denominator polynomials. But in (2.15), both the polynomials are of same order; the reason behind this equalization is the bilinear transformation of (2.13) for implication of the varied continuous-time domain algorithms namely Mikhailov Criterion and Routh approximation over the discrete-time domain systems. Now, all the simplification as per the proposed algorithms is carried over the (2.15).

Approaching to algorithm 1, the reduced order model is derived in w-domain with one less numerator order than that of the denominator but, upon inverse bilinear transformation to obtain the model in z-domain, the reduced order model increases its order by one, equaling it to the denominator polynomials. And in algorithm 2, since it's a direct approach, the reduced order model is produced as desired with a difference of one between the numerator and denominator polynomials.

Moreover, the stability of the reduced order models is checked and is stated to be preserved as declared by the proposed algorithms interlaced with the advantages of the Mikhailov Stability Criterion technique.

Conclusions

Stability preservation, acknowledged being the major advantage of Mikhailov criterion is considered here for order reduction of discrete-time interval system. This is used to derive the reduced denominator amalgamated with the numerator polynomial being computed by two varied algorithms namely *Routh Approximation* and *Direct Truncation*. The two proposed algorithms justify for the novelty.

4.6. Summary

This chapter conclude with an acceptable proposal of techniques based on *Assorted approach* for order reduction of discrete-time interval systems. Collectively the chapter illustrated *five* algorithms.

Throughout the elaboration of the algorithms in Chapter 3 and 4, a conclusion is gathered for each of the algorithm. The next chapter will summarize all these algorithms with their individual advantages and disadvantages under one roof.