Chapter 3
Order Reduction Techniques based on Routh Approximation Approach

3.1. Preamble

This chapter illustrates reduction methodologies, for discrete-time interval systems grouped under *Routh Approximation* approach. The procedural steps considers *RA* for computation of reduced order numerator and/or denominator coefficients. Cumulative of *ten* methodologies are described here. They are coined as *Gamma Delta Approximation, Arithmetic Operator or Multiplicative Approach, Novel Arrangement of Routh Array, Simplified Interval Structure, Advanced Routh Approximation Method (A-RAM), Extended Direct Routh Approximation Method (E-DRAM), Routh Approximant algorithm, Routh Approximation and Pade Approximations,* and *combination of Direct Truncation and Pade Approximation with Routh Approximation*. Algorithms termed above are illustrated below.

A quick revision to *RA* is demonstrated below for better understanding of the proposed algorithm. Consider a polynomial of the form

$$c = c_n p^n + c_{n-1} p^{n-1} + c_{n-2} p^{n-2} + c_{n-3} p^{n-3} + \ldots + c_0 \tag{3.1}$$

The conventional Routh table drafted from the above polynomial is depicted in Table 3.1. All the entries in the table are of interval form as $c_{0,0} = \left[c_{0,0}^-, c_{0,0}^+\right]$, or in general as, $c_{i,j} = \left[c_{i,j}^-, c_{i,j}^+\right]$ with *i=0,1,2,…* and *j=0,1,2,….*

The first row of the table consists of odd coefficients (*i.e.*, first, third, fifth, etc.) and the second row comprises the even coefficients (*i.e.*, second, fourth, sixth, etc.). Entries down the third row is by

$$c_{i,j} = \frac{\left(c_{i-2,j+1}.c_{i-1,j}\right) - \left(c_{i-2,j}.c_{i-1,j+1}\right)}{\left(c_{i-2,j}.c_{i-1,j}\right)} \tag{3.2}$$

where $i \geq 2$ and $j = 0,1,2,\ldots$

It should be noted that the effects of all coefficients of the first two rows are taken into consideration while computing the coefficients below the third rows.

Table 3.1: Conventional Routh table

$c_n = c_{0,0}$	$c_{n-2} = c_{0,1}$	$c_{n-4} = c_{0,2}$	$c_{n-6} = c_{0,3}$	$\cdots$
$c_{n-1} = c_{1,0}$	$c_{n-3} = c_{1,1}$	$c_{n-5} = c_{1,2}$	$c_{n-7} = c_{1,3}$	$\cdots$
$c_{2,0}$	$c_{2,1}$	$c_{2,2}$		
$\cdots$	$\cdots$	$\cdots$		
$c_{n-1,0}$	$c_{n-1,1}$			
$c_{n,0}$				

3.2. Gamma-Delta Approximation

Advantage of *RA* for stability preservation affirm its employment in the proposed algorithm for order reduction of discrete-time interval system.

Methodology

Consider the transfer function of higher order interval systems (2.13) and its derived reduced model as (2.14). The algorithm elaborates below;

To apply Routh algorithm, here used is bilinear transformation on (2.13) that result

$$H_n(w) = \frac{B_n(w)}{A_n(w)} = \frac{\left[b_n^-, b_n^+\right]w^n + \left[b_{n-1}^-, b_{n-1}^+\right]w^{n-1} + ... + \left[b_0^-, b_0^+\right]}{\left[a_n^-, a_n^+\right]w^n + \left[a_{n-1}^-, a_{n-1}^+\right]w^{n-1} + ... + \left[a_0^-, a_0^+\right]} \tag{3.3}$$

The desired γ's and δ's parameters are obtained from the denominator and numerator polynomials drafted in array according to the Tables 3.2 and 3.3. Entries of the first two rows of the tables are from (3.3) and the entries down the third row is by

For Table 3.2

$$\left[a_{i,j}^-, a_{i,j}^+\right] = \frac{\left[a_{i-2,j+1}^-, a_{i-2,j+1}^+\right]\left[a_{i-1,j}^-, a_{i-1,j}^+\right] - \left[a_{i-2,j}^-, a_{i-2,j}^+\right]\left[a_{i-1,j+1}^-, a_{i-1,j+1}^+\right]}{\left[a_{i-1,j}^-, a_{i-1,j}^+\right]} \tag{3.4}$$

with $i \geq 2$ and $j = 0,1,2,...$

And for Table 3.3

$$\left[b_{i,j}^-, b_{i,j}^+\right] = \frac{\left[b_{i-2,j+1}^-, b_{i-2,j+1}^+\right]\left[a_{i-2,j}^-, a_{i-2,j}^+\right] - \left[b_{i-2,j}^-, b_{i-2,j}^+\right]\left[a_{i-2,j+1}^-, a_{i-2,j+1}^+\right]}{\left[a_{i-2,j}^-, a_{i-2,j}^+\right]} \tag{3.5}$$

with $i \geq 2$ and $j = 0,1,2,...$

Table 3.2: Routh table for denominator (γ's parameter)

$\left[a_0^-, a_0^+\right] = \left[a_{0,0}^-, a_{0,0}^+\right]$	$\left[a_2^-, a_2^+\right] = \left[a_{0,1}^-, a_{0,1}^+\right]$	$\left[a_4^-, a_4^+\right] = \left[a_{0,2}^-, a_{0,2}^+\right]$	$\cdots$
$\left[a_1^-, a_1^+\right] = \left[a_{1,0}^-, a_{1,0}^+\right]$	$\left[a_3^-, a_3^+\right] = \left[a_{1,1}^-, a_{1,1}^+\right]$	$\left[a_5^-, a_5^+\right] = \left[a_{1,2}^-, a_{1,2}^+\right]$	$\cdots$
.			
$\left[a_{n-1,0}^-, a_{n-1,0}^+\right]$			
$\left[a_{n,0}^-, a_{n,0}^+\right]$			

Table 3.3: Routh table for numerator (δ's parameters)

$\left[b_0^-, b_0^+\right] = \left[b_{1,0}^-, b_{1,0}^+\right]$	$\left[b_2^-, b_2^+\right] = \left[b_{1,1}^-, b_{1,1}^+\right]$	$\left[b_4^-, b_4^+\right] = \left[b_{1,2}^-, b_{1,2}^+\right]$	$\cdot\cdot$
$\left[b_1^-, b_1^+\right] = \left[b_{2,0}^-, b_{2,0}^+\right]$	$\left[b_3^-, b_3^+\right] = \left[b_{2,1}^-, b_{2,1}^+\right]$	$\left[b_5^-, b_5^+\right] = \left[b_{2,2}^-, b_{2,2}^+\right]$	$\cdot\cdot$
.			
$\left[b_{n-1,0}^-, b_{n-1,0}^+\right]$			
$\left[b_{n,0}^-, b_{n,0}^+\right]$			

From both the tables, the γ's and δ's parameters are computed as

$$\gamma_k = \frac{\left[a_{k-1,0}^-, a_{k-1,0}^+\right]}{\left[a_{k,0}^-, a_{k,0}^+\right]} \quad \text{where } k=1,2,3.... \tag{3.6}$$

$$\delta_k = \frac{\left[b_{k,0}^-, b_{k,0}^+\right]}{\left[a_{k,0}^-, a_{k,0}^+\right]} \quad \text{where } k=1,2,3.... \tag{3.7}$$

The desired order of k, *i.e.* equivalent to the required number of $\gamma - \delta$ parameters are retained allowing the reduced model of k^{th} order as

$$R_k(w) = \frac{B_k(w)}{A_k(w)} \tag{3.8}$$

where $A_k(w) = w^2 A_{k-2}(w) + \left[\gamma_k^-, \gamma_k^+\right] A_{k-1}(w)$ (3.9)

$$B_k(w) = \left[\delta_k^-, \delta_k^+\right] w^{k-1} + w^2 B_{k-2}(w) + \left[\gamma_k^-, \gamma_k^+\right] B_{k-1}(w) \tag{3.10}$$

with $A_{-1}(w)=\frac{1}{w}$, $A_0(w)=1$, $B_{-1}(w)=0$, $B_0(w)=0$

The above combinations result in the first and second order reduced models as

$$R_1(w)=\frac{\left[\delta_1^-,\delta_1^+\right]}{w+\left[\gamma_1^-,\gamma_1^+\right]} \tag{3.11}$$

$$R_2(w)=\frac{\left[\delta_2^-,\delta_2^+\right]w+\left[\gamma_2^-,\gamma_2^+\right]\left[\delta_1^-,\delta_1^+\right]}{w^2+\left[\gamma_2^-,\gamma_2^+\right]w+\left[\gamma_2^-,\gamma_2^+\right]\left[\gamma_1^-,\gamma_1^+\right]} \tag{3.12}$$

Once, the reduced model is obtained in w-domain, application of inverse transformation offer the same in z-domain.

The algorithm is better understood in the next section through examples.

Example

E.3.2.1. Consider the higher order interval transfer function from [68] be

$$H_3(z)=\frac{[1,2]z^2+[3,4]z+[8,10]}{[6,6]z^3+[9,9.5]z^2+[4.9,5]z+[0.8,0.85]} \tag{3.13}$$

Transformation to w-domain result

$$H_3(w)=\frac{[5,9]w^2+[-18,-12]w+[12,16]}{[0.55,1.2]w^3+[5.9,6.65]w^2+[19.45,20.2]w+[20.7,21.35]} \tag{3.14}$$

γ's-δ's parameters obtained from the denominator and numerator tables are

$$\left[\gamma_1^-,\gamma_1^+\right]=[1.02,1.09],\ \left[\gamma_2^-,\gamma_2^+\right]=[2.92,3.42]$$

$$\left[\delta_1^-,\delta_1^+\right]=[0.59,0.82],\ \left[\delta_2^-,\delta_2^+\right]=[-3.05,-1.80]$$

Substitution of the above parameters in (3.12) gives

$$R_2(w)=\frac{[-3.05,-1.80]w+[1.722,2.80]}{w^2+[2.92,3.42]w+[2.97,3.72]} \tag{3.15}$$

which on appropriate inverse transformation results in the desired z-domain reduced models as

$$R_2(z)=\frac{[-1.328,1]z+[3.522,5.85]}{[6.89,8.14]z^2+[3.94,5.44]z+[0.55,1.8]} \tag{3.16}$$

Table 3.4 brings out the comparative study of the errors of the proposed and existing method.

The step response of higher order system and the derived reduced order model and the model by prevailing method is presented in Figure 3.1 (lower limit) and Figure 3.2 (upper limit). Figures 3.3 and 3.4 demonstrate the frequency response of the derived lower and upper limit reduced models respectively.

Table 3.4: Error for 2nd order reduced models for E.3.2.1

Methods	Error	
	Lower Limit	Upper Limit
Proposed Method	0.1292	0.0443
Pade and Dominant Poles [68]	0.1810	0.0741

E.3.2.2. Consider a real-time digital control system with $H_8(z)$ as

$$H_8(z)=\frac{\begin{array}{l}[1.6484,1.7156]z^7+[1.0937,1.1383]z^6+[-0.2142,-0.2058]z^5\\+[0.1490,0.1550]z^4+[-0.5263,-0.5057]z^3+[-0.2672,-0.2568]z^2\\+[0.0431,0.0449]z+[-0.0061,-0.0059]\end{array}}{\begin{array}{l}[23.52,24.48]z^8+[-1.7156,-1.6484]z^7+[-1.1383,-1.0937]z^6\\+[0.2058,0.2142]z^5+[-0.1550,-0.1490]z^4+[0.5057,0.5263]z^3\\+[0.2568,0.3672]z^2+[-0.0449,-0.0431]z+[0.0059,0.0061]\end{array}} \tag{3.17}$$

Using algorithmic steps and computed parameters, the reduced models are as

$$R_1(z)=\frac{[0.01,0.01]}{z+[-0.88,-0.86]} \tag{3.18}$$

and

$$R_2(z)=\frac{[0.02,0.02]z+[-0.02,-0.02]}{z^2+[-1.73,-1.70]z+[0.73,0.76]} \tag{3.19}$$

Error for $R_1(z)$ and $R_2(z)$ are shown in Table 3.5.

Table 3.5: Error for 1st and 2nd order reduced models for E.3.2.2

Method	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Algorithm	0.0035	0.0034	0.0021	0.0019

A strict question arises from the above examples is to why the numerator coefficient is one less that the denominator coefficient, when the bilinear transformation is employed. Its implication should result in the same order of numerator and denominator polynomials. The reason for this query is, here numerator and denominator polynomials are considered separately equal to zero *i.e.* $N_n(z)=0$ and $D_n(z)=0$. Thereafter transformation is applied.

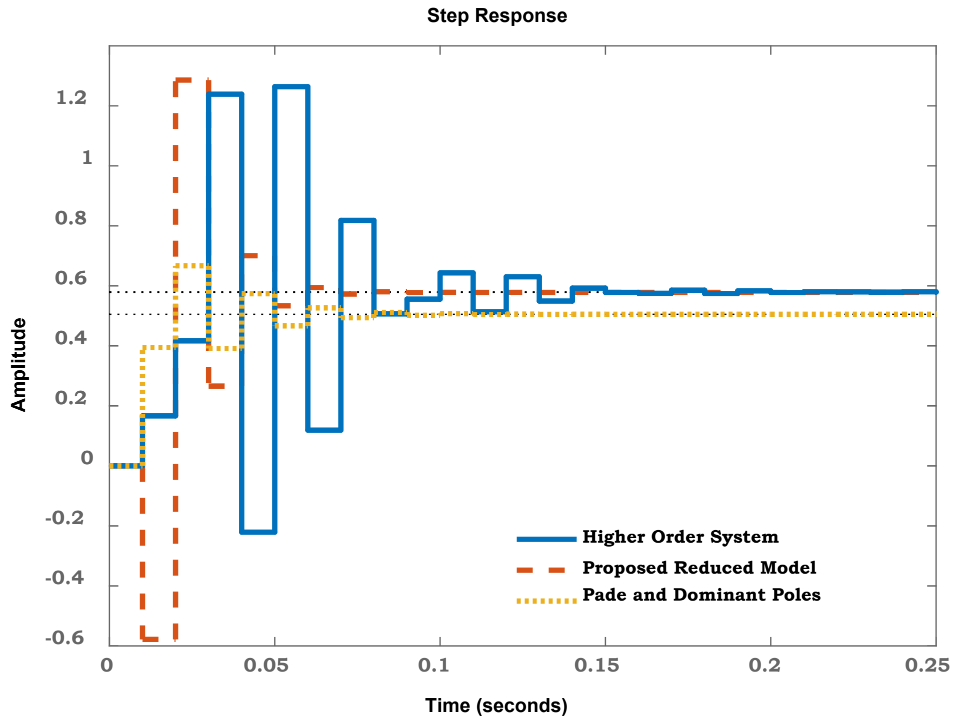

Figure 3.1: Step responses of reduced models (Lower Limit) for E.3.2.1

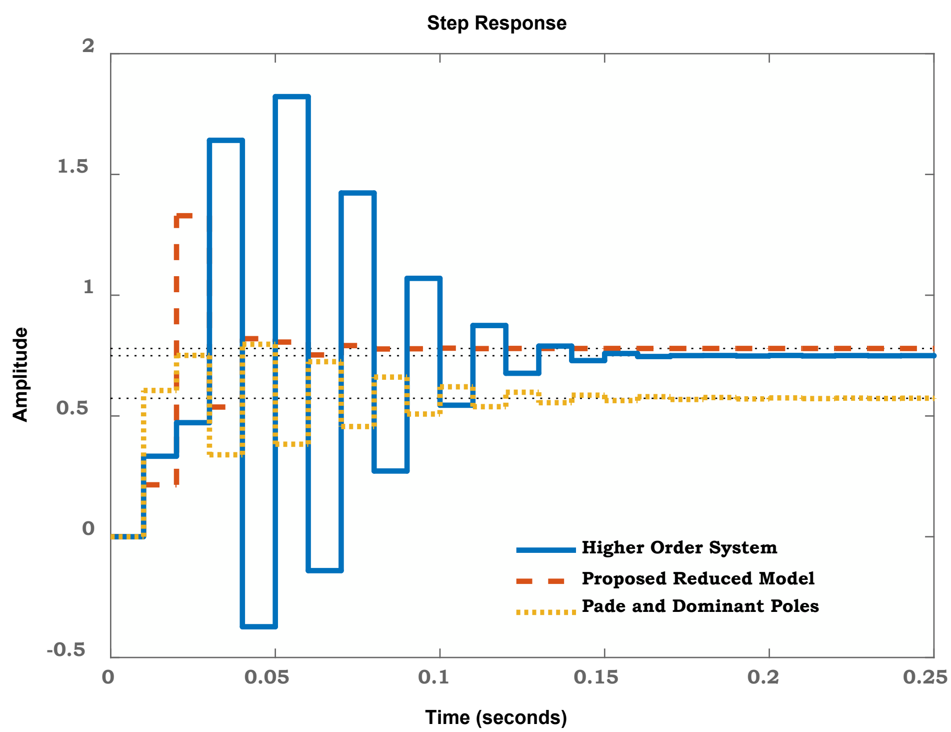

Figure 3.2: Step responses of reduced models (Upper Limit) for E.3.2.1

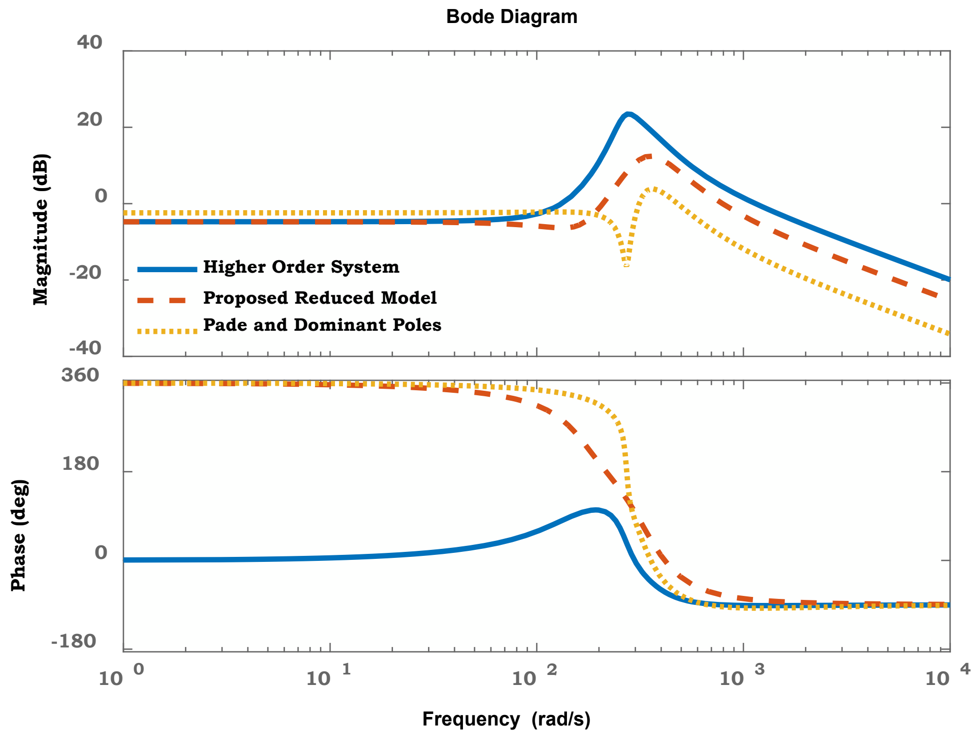

Figure 3.3: Frequency responses of reduced models (Lower Limit) for E.3.2.1

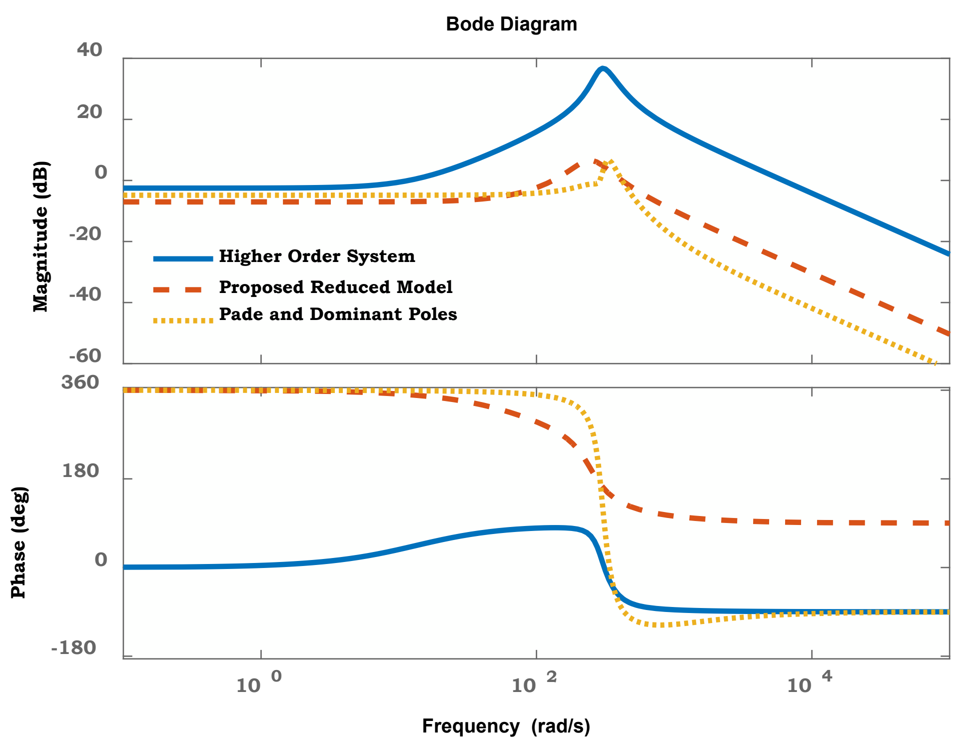

Figure 3.4: Frequency responses of reduced models (Upper Limit) for E.3.2.1

Conclusions

An acceptable $\gamma-\delta$ approximation is conferred for discrete-time interval systems. The examples present the retention of stability through the frequency responses. The proposed algorithm can be stated as an extension from the work of *Bandyopadhyay et. al.* [25] that guarantee to deliver an asymptotically stable model. The earlier technique existed for continuous-time interval system.

3.3. An Arithmetic Operator Approximation

Under this technique the basic arithmetic operator is engaged on the ground of Routh algorithm, retaining the dynamic characteristic of the higher order system to its lower equivalent, *i.e.* stability retention. The proposal here is novel in two folds; one is the application of an arithmetic operator and other is finding the best possible arrangement from the varied combinations (mentioned as *cases* hereunder) of Routh table for deriving the numerator and denominator polynomials coefficients for the reduced model. For use of *RA*, here employed is Euler Forward differentiation technique *i.e.* $z=1+p$. Below elaborated is the algorithm.

Methodology

Consider the higher order system be represented by (3.20) with $m \le n$ equivalent to (2.13)

$$H_{m,n}(z)=\frac{N_m(z)}{D_n(z)}=\frac{N_m z^m+N_{m-1}z^{m-1}+\ldots+N_0}{D_n z^n+D_{n-1}z^{n-1}+\ldots+D_0} \tag{3.20}$$

where $N_i=\left[N_i^-,N_i^+\right]$ and $D_i=\left[D_i^-,D_i^+\right]$ *i=0, 1,2,…,m, n.*

Linear transformation alter (3.20) to

$$H_{m,n}(p)=\frac{B_m(p)}{A_n(p)}=\frac{b_m p^m+b_{m-1}p^{m-1}+\ldots+b_0}{a_n p^n+a_{n-1}p^{n-1}+\ldots+a_0} \tag{3.21}$$

where $b_i=\left[b_i^-,b_i^+\right]$ and $a_i=\left[a_i^-,a_i^+\right]$ *i=0, 1,2,…,m, n.*

As stated about the novelty of the algorithm, here comes in the first one;

In literature, the division operator is applied on (3.21) to compute either time-moments or Markov parameters. In this work, multiplication between the numerator and denominator polynomials offering a larger polynomial retaining the characteristics of both is used. It is termed as π operator over the numerator and denominator polynomials. It is obtained as;

$$T_q(p) = B_m(p) A_n(p) = c_q p^q + c_{q-1} p^{q-1} + c_{q-2} p^{q-2} + \ldots + c_1 p^1 + c_0 \tag{3.22}$$

where $c_i = \left[c_i^-, c_i^+\right]$, i=0,1,2.....,q and $q = m + n$.

Coefficients of the reduced order transfer function are computed from varied combinations of μ- and ν-tables described in Table 3.6 and Table 3.7 respectively. Entries for the μ- table is from $T_q(p)$ and for ν-table is from $\hat{T}_q(p)$. The latter is the inverse of former one defined as

$$\hat{T}_q(p) = \frac{1}{p} T_q\left(\frac{1}{p}\right) = c_0 p^q + c_1 p^{q-1} + c_2 p^{q-2} + \ldots + c_{q-1} p^1 + c_q \tag{3.23}$$

The tables represent the conventional Routh Array (drafted in Table 3.1) and its corresponding rules (3.1) to obtain the below entries of the entries down the third row where $c_{i,j} = \left[c_{i,j}^-, c_{i,j}^+\right]$, i=2,3,4,...,q-1,q and j=0,1,2...q.

The two tables (Table 3.6 and Table 3.7) are considered accordingly for obtaining the reduced order transfer function $R_k(p)$, where $k(<q)$ or $k < m \le n$. The tables are considered under the following four *cases*;

Case 1:

Use only **μ-table** for both numerator and denominator polynomials

Case 2:

Use only **ν-table** for both numerator and denominator polynomials

Case 3:

Use **μ-table** for numerator and **ν-table** for denominator polynomials

Case 4:

Use **ν-table** for numerator and **μ-table** for denominator polynomials

The tables via four *cases* offer the reduced polynomial coefficients considered accordingly to [106]. Thus, reduced transfer function of order $k(<q)$ or $k < m \le n$ is constructed with $(q+1)th$ and $(q+2-k)th$ rows for numerator and $(q+1-k)th$ and $(q+2-k)th$ rows for denominator coefficients. Accordigly, the generalized transfer function through the *cases* is expressed as

$$R_k(p) = \frac{B_k(p)}{A_k(p)} = \frac{b_{(q+1),0} p^{k-1} + b_{(q+2-k),0} p^{k-2} + b_{(q+1),1} p^{k-3} + \ldots}{a_{(q+1-k),0} p^k + a_{(q+2-k),0} p^{k-1} + a_{(q+1-k),1} p^{k-2} + \ldots} \tag{3.24}$$

The obtained $R_k(p)$ transforms to the desired $R_k(z)$ by p=z-1 transformation resulting in (2.14).

Table 3.6: μ-(Routh) Table

$c_{0,0} = c_q$	$c_{0,1} = c_{q-2}$	$c_{0,2} = c_{q-4}$	$c_{0,3} = c_{q-6}$	$\cdots$
$c_{1,0} = c_{q-1}$	$c_{1,1} = c_{q-3}$	$c_{1,2} = c_{q-5}$	$c_{1,3} = c_{q-7}$	$\cdots$
$c_{2,0}$	$c_{2,1}$	$c_{2,2}$		
$\cdots$	$\cdots$	$\cdots$		
$c_{q-1,0}$	$c_{q-1,1}$			
$c_{q,0}$				

Table 3.7: ν-(Routh) Table

$c_{0,0} = c_0$	$c_{0,1} = c_2$	$c_{0,2} = c_4$	$c_{0,3} = c_6$	$\cdots$
$c_{1,0} = c_1$	$c_{1,1} = c_3$	$c_{1,2} = c_5$	$c_{1,3} = c_7$	$\cdots$
$c_{2,0}$	$c_{2,1}$	$c_{2,2}$		
$\cdots$	$\cdots$	$\cdots$		
$c_{q-1,0}$	$c_{q-1,1}$			
$c_{q,0}$				

Finally, the *case* that offers new result based on the performance tools and preserve the dynamic characteristics of stability justifies for novelty of the proposed algorithm.

Figure 3.5 demonstrate the flow chart for the arithmetic operator approximation.

The algorithm is better illustrated and recognized through numerical examples in the next section.

Example

The practice to obtain the reduced model under the discussed *cases* is performed in this section. The results of the examples are compared and justified through error sum computed for reduced models using proposed algorithm *cases*.

E.3.3.1. Consider a higher order interval system from [66] with its equivalent *p*-domain representation as

$$H_{2,3}(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{3.25}$$

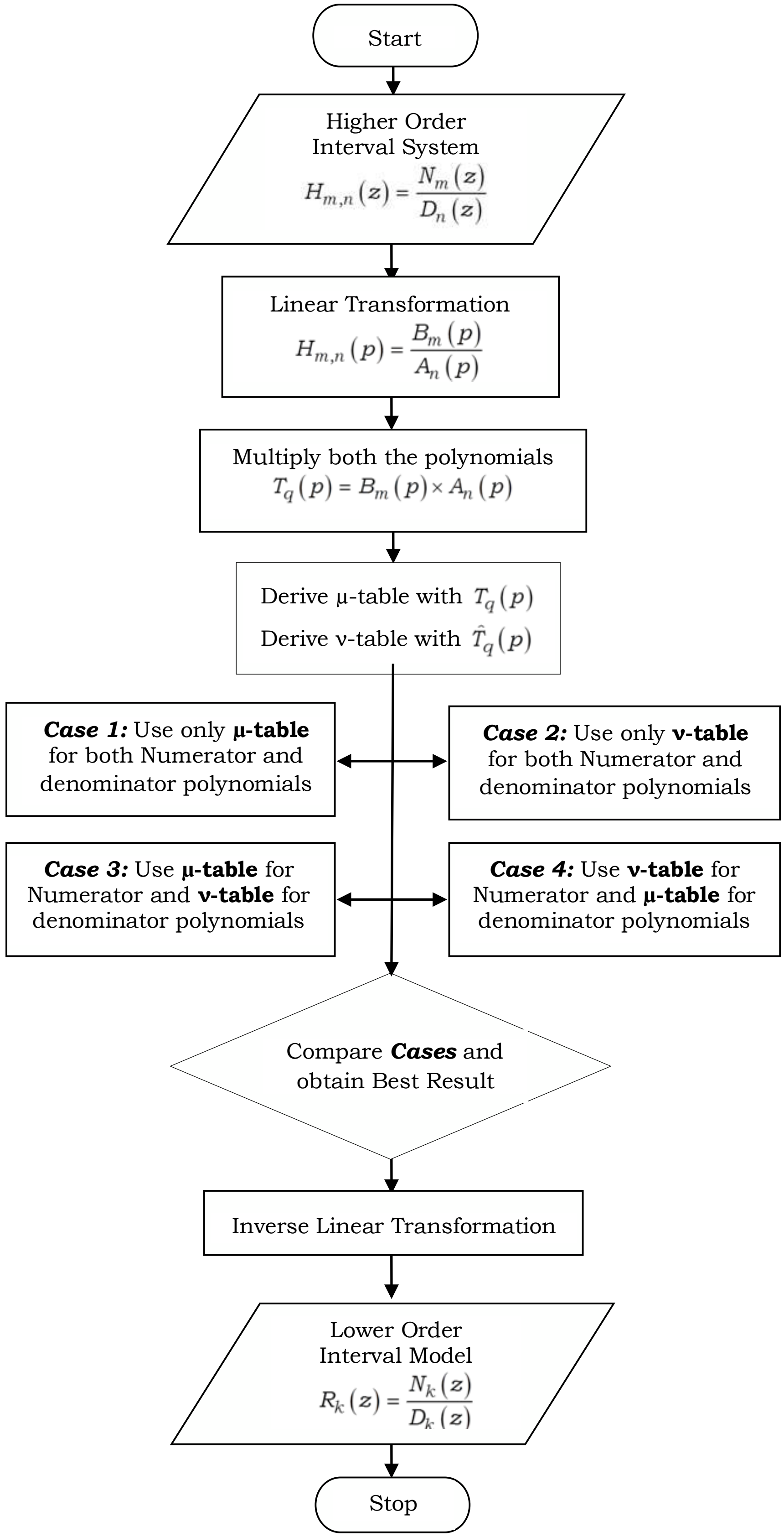

Figure 3.5: Flow chart for the arithmetic operator approximation

$$H_3(p)=\frac{[1,2]p^2+[5,8]p+[12,16]}{[6,6]p^3+[27,27.5]p^2+[40.9,42]p+[20.7,21.35]} \tag{3.26}$$

Multiplying the two polynomials (numerator and denominator), result $T_5(p)$ as

$$\begin{aligned}T_5(p)=&[6,12]p^5+[57,103]p^4+[247.9,400]p^3\\&+[549.2,818.7]p^2+[594.3,842.8]p+[248.4,341.6]\end{aligned} \tag{3.27}$$

Entries for Table 3.8 and Table 3.9 are conceived from (3.27) according to (3.22) and (3.23) respectively.

Table 3.8: μ-Table

p^5	$[6,12]$	$[247.9,400]$	$[594.3,842.8]$
p^4	$[57,103]$	$[549.2,818.7]$	$[248.4,341.6]$

Table 3.9: ν-Table

p^5	$[248.4,341.6]$	$[549.2,818.7]$	$[57,103]$
p^4	$[594.3,842.8]$	$[247.9,400]$	$[6,12]$

Reduced models allowing the four *cases* are derived as:

Case 1

$$R_2(z)=\frac{[352,1045.04]z+[-796.64,-10.4]}{[-580.1,737.79]z^2+[-1123.58,2205.24]z+[-1376.74,727.39]} \tag{3.28}$$

Case 2

$$R_2(z)=\frac{[25.27,564.13]z+[-558.13,-13.27]}{[-19.33,360.07]z^2+[-694.87,602.79]z+[-577.46,346.8]} \tag{3.29}$$

Case 3

$$R_2(z)=\frac{[352,1045.04]z+[-796.64,-10.4]}{[-19.33,360.07]z^2+[-694.87,602.79]z+[-577.46,346.8]} \tag{3.30}$$

Case 4

$$R_2(z)=\frac{[25.27,564.13]z+[-558.13,-13.27]}{[-580.1,737.79]z^2+[-1123.58,2205.24]z+[-1376.74,727.39]} \tag{3.31}$$

Table 3.10 showcase the computed error for the above *cases* comprising comparison with existing techniques. It present a fair indication toward *Case 4* replacing the other methods. Discussion over the explicit finding is done in next section.

Figure 3.6 and Figure 3.7 present the frequency domain response for all the *cases* for lower and upper limits transfer functions respectively. The figures depict an appropriate tracking of the responses affirming the preservation of higher order characteristics.

Table 3.10: Error for 2nd order reduced model for E.3.3.1

Methods	Error	
	Lower Limit	Upper Limit
Proposed Case 1	0.5982	1.1731
Proposed Case 2	2.1726	1.5213
Proposed Case 3	337.7032	6.5997
Proposed Case 4	***0.0442***	***0.1860***
Pade & Dominant Poles [68]	0.1810	0.0741
Dominant Pole and Direct Series [83]	0.3237	0.3229
Gamma-Delta Appr. [90]	0.1292	0.0443
Direct Truncation [107]	0.0278	0.0077

From the example, it is clear that the proposed *Case 4* is good enough for deriving the reduced order model. One probable query rises from the example and *cases* included in the tables as, why are other *cases* considered in tables when *Case 4* is quiet good and acceptable? The reason for this inclusion is to justify that among all the cases, *Case 4* stands the best. This justification would not be liable without any evidence.

E.3.3.2. Consider the automatic voltage regulator problem with a perturbation in the system.

$$H_5(z)=\frac{[0.95,1.05]z^4+[11.10,12.27]z^3+[11.56,112.78]z^2+[1.76,1.95]z+[0.031,0.03]}{[168.22,185.93]z^5+[-473.21,-428.14]z^4+[349.91,386.74]z^3+[-94.12,85.15]z^3+[-4.07,-3.68]z+[-1.24,-1.13]} \tag{3.32}$$

Reduced order model obtained by *Case 4* is

$$R_2(z)=\frac{[-3740.8,5834]z+[-5674.2,3936]}{[-41060,45979]z^2+[-121011,111415]z+[-71886,76564]} \tag{3.33}$$

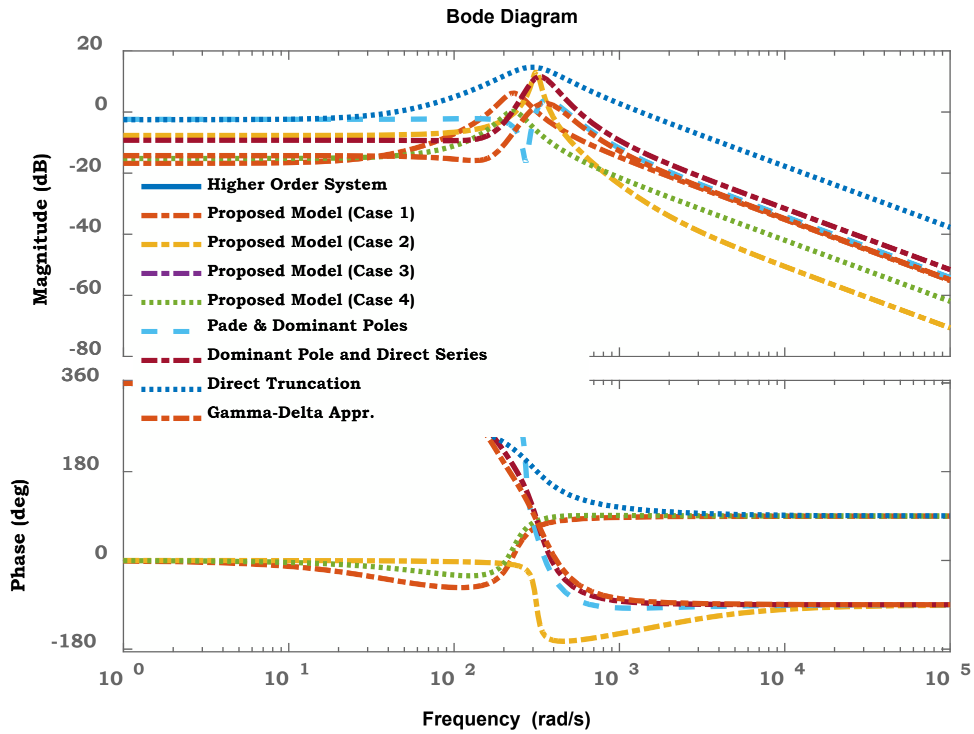

Figure 3.6: Frequency responses of reduced models (Lower Limit) for E.3.3.1

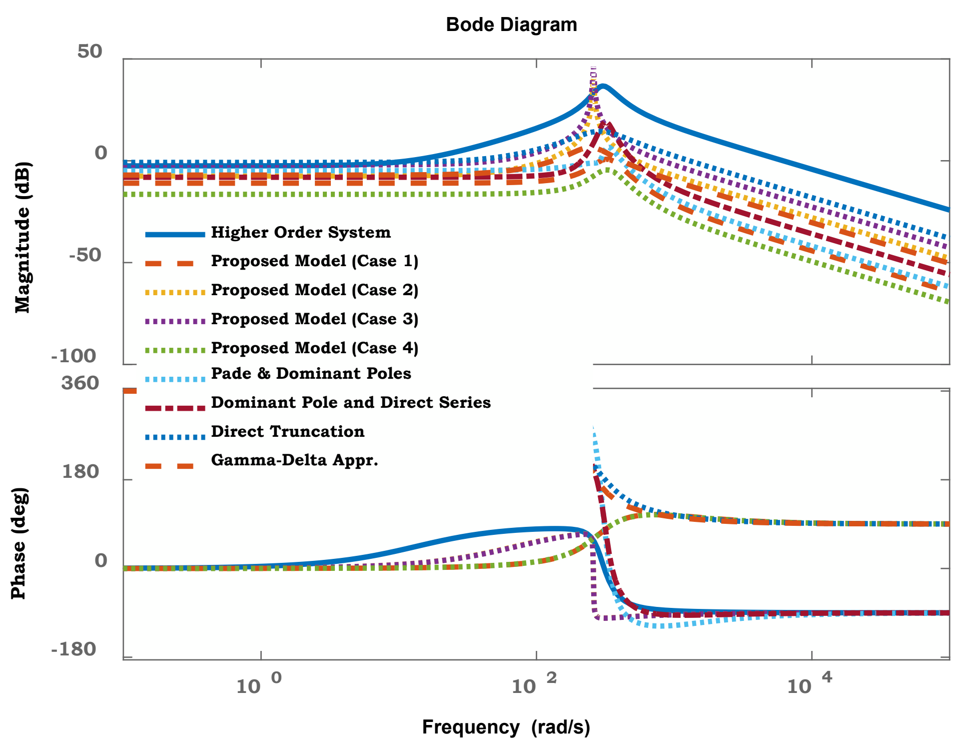

Figure 3.7: Frequency responses of reduced models (Upper Limit) for E.3.3.1

Errors of the computed reduced model, depicted in Table 3.11 indicate the superiority of the accepted *Case 4*.

Table 3.11: Error for 2^{nd} order reduced model for E.3.3.2

Methods	Error	
	Lower Limit	Upper Limit
Proposed Case 1	0.4927	0.3988
Proposed Case 2	0.4091	0.0137
Proposed Case 3	24.2955	0.3731
Proposed Case 4	**0.0073**	**0.0147**

Discussion

Examples in prior section confer the proposed *cases* that closely approximate higher order system by their reduced model. It is observed that the different *cases* offer variations in error computation, making it difficult to choose the best *case*. One observations is as; in Table 3.10, Direct Truncation and Gamma-Delta approximation offers minimum error. After a purposeful consideration of both the error tables, the proposed *Case 4* appears to be the best offering minimum error. The prime advantage of *Case 4* over the others listed in tables is the stability retention which others do not assure. This check is performed by the highly recognized Kharitonov theorem for interval systems. Thus, regardless of comparatively higher error computation, *Case 4* is superior to the exiting techniques.

As a future work, the reduced-order model may be derived by considering numerator and denominator polynomials respectively in different tables. It may yield better result as they individually incur the respective polynomials characteristics. In this proposed algorithm, the π operator deals a larger polynomial that have the characteristics of both the numerator and the denominator polynomials. Disadvantage of considering individual polynomials is being lengthy that requires involving more tables and hence more computation, whereas the proposed one is competitively small in computation with only formation of two μ- and ν-tables.

Conclusions

The motive of proposing an efficient algorithm that preserves the system stability is achieved. Basis of the contributed novelty is through the implication of multiplicative operator and the particular *case* of deriving the reduced order numerator and denominator polynomials.

3.4. Novel Combination of Routh Array

This is another algorithm that poses the importance of preserving stability of the derived reduced models based on *RA*. The realm for attaining novelty is finding the possible arrangements (mentioned as *cases*) of the numerator and denominator polynomials for computing the Routh array. For application of *RA*, here employed is bilinear transformation $\left(z = 1 + w/1 - w\right)$. Algorithm is illustrated below.

Methodology

Consider (2.13) as higher order system which on transformation results in (2.15). For computing the conventional Routh array for discovering the novel arrangement of the numerator and denominator polynomials of the reduced models, consider Table 3.1.

Entries of the first two rows in the above Routh table for numerator and denominator polynomials is drafted by considering (2.15) through the following two *cases*;

Case 1

For numerator

1st Row; $c_{i,j} = b_k$ where *i=0; j=0,1,2,3,…; k=n,n-2,n-4,…*

2nd Row; $c_{i,j} = b_k$ where *i=1;j=0,1,2,3,…; k=n-1,n-3,n-5,…*

For denominator

1st Row; $c_{i,j} = a_k$ where *i=0;j=0,1,2,3,…; k=n,n-2,n-4,…*

2nd Row; $c_{i,j} = a_k$ where *i=1;j=0,1,2,3,…;k=n-1,n-3,n-5,…*

Case 2: interchange the 2nd row entries of numerator and denominator tables in *Case 1*

For numerator

1st Row; $c_{i,j} = b_k$ where *i=0;j=0,1,2,3,…; k=n,n-2,n-4,…..*

2nd Row; $c_{i,j} = a_k$ where *i=1; j=0,1,2,3,…;k=n-1,n-3,n-5,…*

For denominator

1st Row; $c_{i,j} = a_k$ where *i=0; j=0,1,2,3,…; k=n,n-2,n-4,…*

2nd Row; $c_{i,j} = b_k$ where *i=1;j=0,1,2,3,…;k=n-1,n-3,n-5,…*

For other two *cases*, reciprocate (2.15) to (3.34) as

$$\hat{H}_n(w)=\frac{B_n(w)}{A_n(w)}=\frac{\left[b_0^-,b_0^+\right]w^n+\left[b_1^-,b_1^+\right]w^{n-1}+\ldots+\left[b_{n-1}^-,b_{n-1}^+\right]w+\left[b_n^-,b_n^+\right]}{\left[a_0^-,a_0^+\right]w^n+\left[a_1^-,a_1^+\right]w^{n-1}+\ldots+\left[a_{n-1}^-,a_{n-1}^+\right]w^n+\left[a_n^-,a_n^+\right]} \quad (3.34)$$

Case 3

For numerator

1st Row; $c_{i,j}=b_k$ where *i=0; j=0, 1, 2, 3,...... k=0, 2, 4....*

2nd Row; $c_{i,j}=b_k$ where *i=1; j=0, 1, 2, 3,...... k=1, 3, 5...*

For denominator

1st Row; $c_{i,j}=a_k$ where *i=0; j=0, 1, 2, 3,...... k=0, 2, 4.....*

2nd Row; $c_{i,j}=a_k$ where *i=1; j=0, 1, 2, 3,...... k=1, 3, 5,....*

Case 4: interchange the 2nd row entries of numerator and denominator tables in *Case 3*

For numerator

1st Row; $c_{i,j}=b_k$ where *i=0; j=0, 1, 2, 3,...... k=0, 2, 4....*

2nd Row; $c_{i,j}=a_k$ where *i=1; j=0, 1, 2, 3,...... k=1, 3, 5,....*

For denominator

1st Row; $c_{i,j}=a_k$ where *i=0; j=0, 1, 2, 3,...... k=0, 2, 4.....*

2nd Row; $c_{i,j}=b_k$ where *i=1; j=0, 1, 2, 3,...... k=1, 3, 5...*

Once, the first two rows are available from the above stated *cases*, the entries down the table from third row is computed by the conventional Routh algorithm as in (3.1).

From the above computed tables, the reduced order model $R_k(w)$ where $k<n$ is constructed with $(n+1-k)th$ and $(n+2-k)th$ rows of denominator table; along with $(n+1)th$ and $(n+2-k)th$ rows of numerator table, represented as

$$R_k(p)=\frac{B_k(w)}{A_k(w)}=\frac{b_{(n+1),0}w^{k-1}+b_{(n+2-k),0}w^{k-2}+b_{(n+1),1}w^{k-3}+\ldots}{a_{(n+1-k),0}w^k+a_{(n+2-k),0}w^{k-1}+a_{(n+1-k),1}w^{k-2}+\ldots} \quad (3.35)$$

with the respective interval coefficients.

$R_k(w)$ result in the required $R_k(z)$ after appropriate inverse transformation.

Example

The *best arrangement* which form the novelty of the proposal is better understood in this section through numerical examples available from literature. The result from the mentioned *cases* is compared with the prevailing techniques for assessment on the basis of error sum. The step responses also verify the obtained results. In the course of attaining the prime focus of the proposed algorithm *i.e.* the retention of stability; a limitation is discovered, which on later stage is ignored for the algorithm's proficiency. This limitation is discussed in later part of this section.

E.3.4.1. Consider third order interval system available from [68], [90], [107] as

$$H_3(z)=\frac{[1,2]z^2+[3,4]z+[8,10]}{[6,6]z^3+[9,9.5]z^2+[4.9,5]z+[0.8,0.85]} \tag{3.36}$$

$$H_3(w)=\frac{[-9,-5]w^3+[17,27]w^2+[-34,-24]w+[12,16]}{[0.55,1.2]w^3+[5.9,6.65]w^2+[19.45,20.2]w+[20.7,21.35]} \tag{3.37}$$

First two rows of the Routh tables for *Case 1* and *Case 2* are fetched from above $H_3(w)$ as

Case 1*:* Table 3.12 and Table 3.13 for the denominator and numerator array.

Table 3.12: Denominator array for Case 1

w^3	$[0.55,1.2]$	$[19.45,20.2]$
w^2	$[5.9,6.65]$	$[20.7,21.35]$

Table 3.13: Numerator array for Case 1

w^3	$[-9,-5]$	$[-34,-24]$
w^2	$[17,27]$	$[12,16]$

The obtained reduced order models from the above entries are

$$R_1(z)=\frac{[12,16]z+[12,16]}{[35.8,39.83]z+[2.22,6.25]} \tag{3.38}$$

$$R_2(z)=\frac{[-19.77,0.48]z^2+[24,32]z+[27.52,47.77]}{[41.7,46.48]z^2+[28.1,30.9]z+[8.12,12.9]} \tag{3.39}$$

Case 2: Table 3.14 and Table 3.15 state the denominator and numerator array and below are the reduced models.

$$R_1(z)=\frac{[20.7,21.35]z+[20.7,21.35]}{[30.32,35.95]z+[-7.95,-2.32]} \tag{3.40}$$

$$R_2(z)=\frac{[2.27,29.91]z^2+[41.4,42.7]z+[12.14,39.78]}{[47.32,62.95]z^2+[-30,-2]z+[9.05,24.68]} \tag{3.41}$$

Table 3.14: Denominator array for Case 2

w^3	$[0.55, 1.2]$	$[19.45, 20.2]$
w^2	$[17, 27]$	$[12, 16]$

Table 3.15: Numerator array for Case 2

w^3	$[-9, -5]$	$[-34, -24]$
w^2	$[5.9, 6.65]$	$[20.7, 21.35]$

Now for *Case 3* and *Case 4*, reciprocate $H_3(w)$ to $\hat{H}_3(w)$ and draft the two *cases* as

$$\hat{H}_3(w) = \frac{[12,16]w^3 + [-34,-24]w^2 + [17,27]w + [-9,-5]}{[20.7,21.35]w^3 + [19.45,20.2]w^2 + [5.9,6.65]w + [0.55,1.2]} \quad (3.42)$$

Case 3*:* Underneath Table 3.16 and Table 3.17 state the denominator and numerator array.

Table 3.16: Denominator array for Case 3

w^3	$[20.7, 21.35]$	$[5.9, 6.65]$
w^2	$[19.45, 20.2]$	$[0.55, 1.2]$

Table 3.17: Numerator array for Case 3

w^3	$[12, 16]$	$[17, 27]$
w^2	$[-34, -24]$	$[-9, -5]$

The reduced models are

$$R_1(z) = \frac{[-9,-5]z + [-9,-5]}{[5.13,7.28]z + [3.38,5.53]} \quad (3.43)$$

$$R_2(z) = \frac{[2,20.23]z^2 + [22,50.46]z + [16,34.23]}{[24.58,27.48]z^2 + [36.5,39.3]z + [13.92,16.82]} \quad (3.44)$$

Case 4*:* Below Table 3.18 and Table 3.19 state the denominator and numerator array.

The computed reduced models are

$$R_1(z) = \frac{[0.55,1.2]z + [0.55,1.2]}{[-11.1,-1.4]z + [2.9,12.6]} \quad (3.45)$$

Table 3.18: Denominator array for Case 4

w^3	$[20.7, 21.35]$	$[5.9, 6.65]$
w^2	$[-34, -24]$	$[-9, -5]$

Table 3.19: Numerator array for Case 4

w^3	$[12, 16]$	$[17, 27]$
w^2	$[19.45, 20.2]$	$[0.55, 1.2]$

$$R_2(z) = \frac{[16.56, 27.87]z^2 + [32.02, 53.34]z + [14.81, 26.12]}{[-45.1, -25.4]z^2 + [-58, -30]z + [-46.6, -26.9]} \qquad (3.46)$$

The computed error for the varied *cases* illustrated is made known in Table 3.20. It is observed that the result for *Case 2* is minimum, differentiating it from the other *cases* and the prevailing techniques and submitting its acceptance replacing the others. Limitation observed during the computation is explained in the next section.

Another practice to authenticate the algorithm through step response is depicted in Figures 3.8 and 3.9 for lower and upper limits reduced models respectively. Figures emphasize that the response pattern of the higher order system is favourably preserved in the reduced order model.

Table 3.20: Error for 1st and 2nd order reduced models for E.3.4.1

Methods	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Case 1	0.3456	0.3271	0.2894	0.1287
Proposed Case 2	***0.3644***	***0.1497***	***0.0211***	***0.0233***
Proposed Case 3	9.4259	1.8765	0.4812	1.9492
Proposed Case 4	0.0801	96.0295	0.7302	6.1975
Pade/Dominant Poles [68]	0.1398	0.0195	0.1810	0.0741
Gamma-Delta Appr. [90]	0.0157	0.0035	0.1292	0.0443
Direct Truncation [107]	2.1491	2.7778	0.0278	0.0077

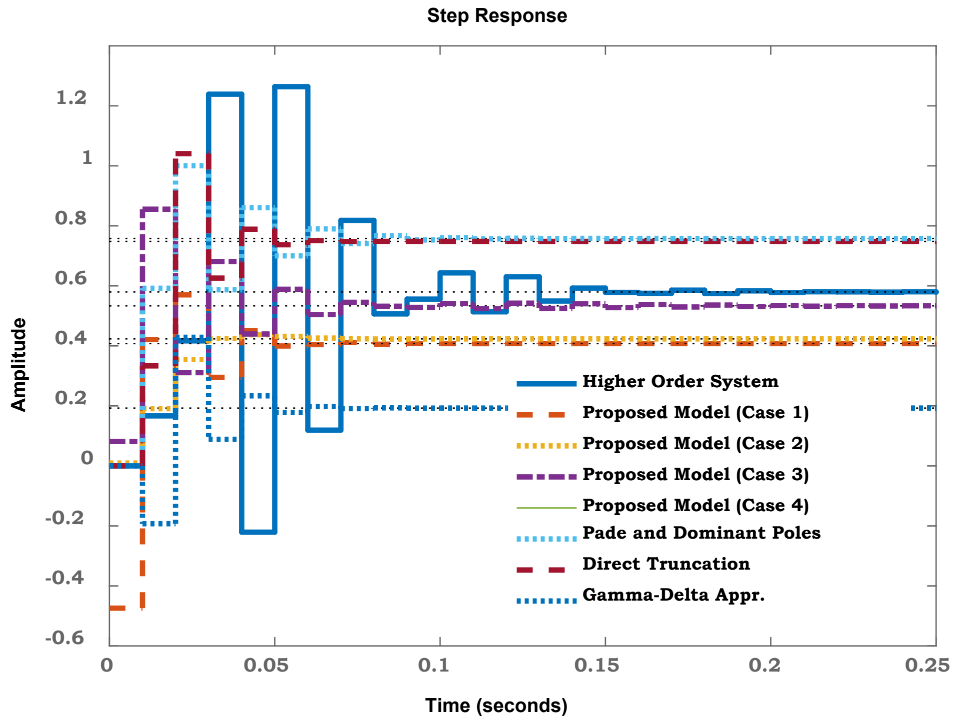

Figure 3.8: Step responses of reduced models (Lower Limit) for E.3.4.1

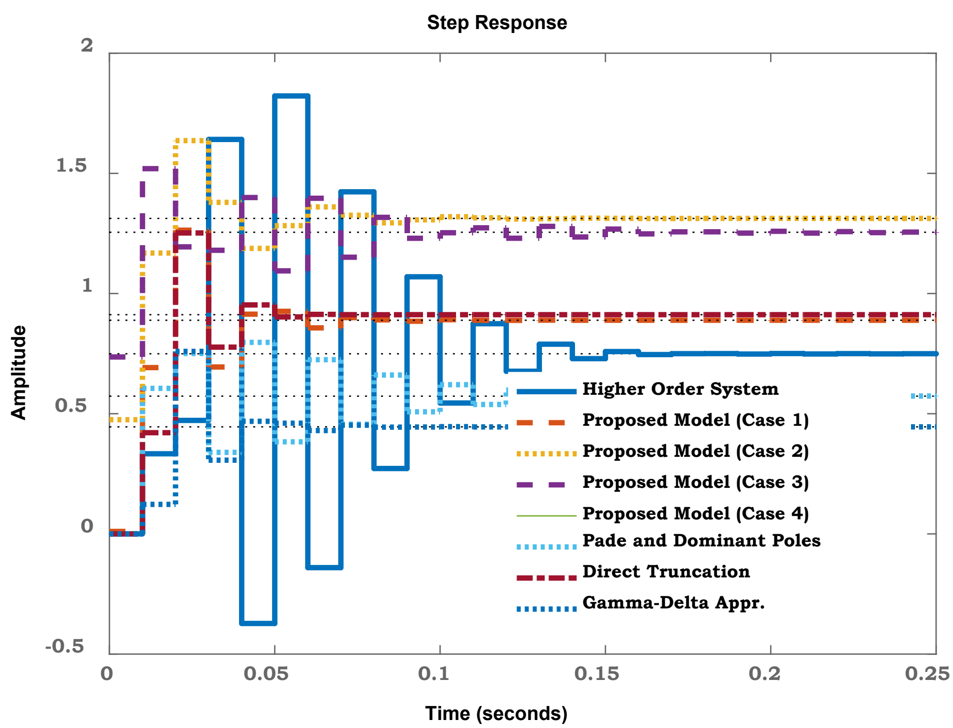

Figure 3.9: Step responses of reduced models (Upper Limit) for E.3.4.1

From the above numerical example, the acceptable arrangement of the Routh Tables is *Case 2*, offering the minimum error and an acceptable step response.

E.3.2.4.2. Consider the real-time digital control system

$$H_8(z)=\frac{\begin{array}{l}[1.6484,1.7156]z^7+[1.0937,1.1383]z^6+[-0.2142,-0.2058]z^5\\+[0.1490,0.1550]z^4+[-0.5263,-0.5057]z^3+[-0.2672,-0.2568]z^2\\+[0.0431,0.0449]z+[-0.0061,-0.0059]\end{array}}{\begin{array}{l}[23.52,24.48]z^8+[-1.7156,-1.6484]z^7+[-1.1383,-1.0937]z^6\\+[0.2058,0.2142]z^5+[-0.1550,-0.1490]z^4+[0.5057,0.5263]z^3\\+[0.2568,0.3672]z^2+[-0.0449,-0.0431]z+[0.0059,0.0061]\end{array}} \tag{3.47}$$

By the proposed algorithm *cases*, the reduced order model are obtained as

Case 1:

$$R_1(z)=\frac{[1.92,2.07]z+[1.92,2.07]}{[110.83,291.57]z+[-247.58,-66.84]} \tag{3.48}$$

$$R_2(z)=\frac{[13.21,21.14]z^2+[3.84,4.14]z+[-17.15,-9.22]}{[-49.49,739.4]z^2+[-852.78,365.74]z+[-407.9,380.99]} \tag{3.49}$$

Case 2:

$$R_1(z)=\frac{[1.92,2.07]z+[1.92,2.07]}{[36.43,39.85]z+[4.14,7.63]} \tag{3.50}$$

$$R_2(z)=\frac{[14.94,182.65]z^2+[3.84,4.14]z+[-178.66,-10.95]}{[-1222.74,886.15]z^2+[-1649.72,2563.3]z+[-1254.96,853.93]} \tag{3.51}$$

Case 3:

$$R_1(z)=\frac{[-0.07,0.07]z+[-0.07,0.07]}{[129.23,279.82]z+[81.24,231.83]} \tag{3.52}$$

$$R_2(z)=\frac{[-5.69,-4.03]z^2+[-11.24,-8.2]z+[-5.69,-4.03]}{[-130.46,762.05]z^2+[-568.48,917.58]z+[-491.52,400.92]} \tag{3.53}$$

Case 4:

$$R_1(z)=\frac{[-0.07,0.07]z+[-0.07,0.07]}{[17.24,22.39]z+[-30.25,-25.6]} \tag{3.54}$$

$$R_2(z)=\frac{[181.98,211.74]z^2+[364.1,423.34]z+[181.98,211.74]}{[-1724.76,589.39]z^2+[-3534.1,1087.12]z+[-1716.9,597.25]} \tag{3.55}$$

Error for $R_1(z)$ and $R_2(z)$ of E.3.4.2 is presented in Table 3.21.

From the Table 3.21, for E.3.4.2., it is clear that the *Case 2* gives a commendable result for real-time system also. This tends to believe that the arrangement in (*Case 2*) is acceptable to the real world.

Table 3.21: Error for 1st and 2nd order reduced models for E.3.4.2

Methods	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Case 1	3.1075×10^{-4}	0.0030	17.5841	0.0033
Proposed Case 2	***0.0036***	***0.0033***	***0.0049***	***0.0250***
Proposed Case 3	0.0050	0.0040	0.0189	0.0064
Proposed Case 4	0.0073	0.0036	0.0402	0.2480

Discussion

As stated earlier about the limitation discovered; from Table 3.20 it is observed that the error are sometimes more than that obtained by the other cases (*eg. Error of 1st order lower limit is more than the other in the same column*). Similarly in Table 3.21, few of the error computed by other *cases* are less as compared to the considered *case*. Since, the key focus to yield a stable reduced model is attained successfully in both the examples, the computation of high error sum can be neglected.

Conclusions

The novelty is explored from the domain of various *cases* constructed by the varied arrangements of numerator and denominator polynomials. A limitation discovered during the course of exploration of the best arrangement for approximation is also discussed.

3.5. Simplified Interval Structure

Interval systems are acquainted to comprise two limits within a boundary namely, lower and upper. In this proposal, these limits are considered to be the major contributor towards the order reduction of interval system. Linear transformation is applied here for application of *RA*.

Methodology

Consider a higher order discrete-time interval system be (2.13) which on linear transformation yield (2.16).

Split the higher order interval system into two transfer functions as depicted below with lower and upper limits coefficients respectively where subscript L and U represent the lower and upper limits correspondingly. Splitting allows consideration of lower and upper coefficients properties individually.

$$H_{nL}(p)=\frac{B_n^-(p)}{A_n^-(p)}=\frac{b_{n-1}^-p^{n-1}+b_{n-2}^-p^{n-2}+\ldots+b_0^-}{a_n^-p^m+a_{n-1}^-p^{n-1}+\ldots+a_0^-} \tag{3.56}$$

$$H_{nU}(p)=\frac{B_n^+(p)}{A_n^+(p)}=\frac{b_{n-1}^+p^{n-1}+b_{n-2}^+p^{n-2}+\ldots+b_0^+}{a_n^+p^m+a_{n-1}^+p^{m-1}+\ldots+a_0^+} \tag{3.57}$$

Assume each of the numerator and denominator polynomials separately from the above transfer functions resulting in four characteristic polynomials (3.58-3.61). These polynomials are used for computation of reduced model coefficients via *RA*.

$$B_n^-(p)=b_{m-1}^-p^{m-1}+b_{m-2}^-p^{m-2}+\ldots+b_0^- \tag{3.58}$$

$$A_n^-(p)=a_n^-p^n+a_{n-1}^-p^{n-1}+\ldots+a_0^- \tag{3.59}$$

$$A_n^-(p)=a_n^-p^n+a_{n-1}^-p^{n-1}+\ldots+a_0^- \tag{3.60}$$

$$A_n^+(p)=a_n^+p^n+a_{n-1}^+p^{n-1}+\ldots+a_0^+ \tag{3.61}$$

The first two polynomials are from (3.56), and next two are from (3.57). The above polynomials at this moment are considered separately for completing the Routh table depicted in Table 3.1.

Completion of the tables permits for finding the desired coefficients for the reduced model transfer functions as stated in [106]. The numerator coefficients are obtained from the combinations of $(n+1)th$ & $(n+1-k)th$ rows of B_n^-, B_n^+ polynomial tables. Correspondingly, the $(n+1-k)th$ & $(n+2-k)th$ rows of A_n^-, A_n^+ Routh tables offer the denominator coefficients. Altogether, the expression for reduced interval transfer function is (3.62).

$$R_k(p)=\frac{B_k(p)}{A_k(p)}=\frac{b_{(n+1),0}p^{k-1}+b_{(n+2-k),0}p^{k-2}+b_{(n+1),1}p^{k-3}+\ldots}{a_{(n+1-k),0}p^k+a_{(n+2-k),0}p^{k-1}+a_{(n+1-k),1}p^{k-2}+\ldots} \tag{3.62}$$

The factors of the above transfer function take the range structure when the anticipated elements from the tables are merged as represented by (3.63) and (3.64). This depiction seizes the interval coefficients for numerator and denominator respectively.

$$B_i=\left[b_i^-,b_i^+\right] \qquad i=0,\ 1\ldots\ k \tag{3.63}$$

$$A_i=\left[a_i^-,a_i^+\right] \qquad i=0,\ 1\ldots\ k \tag{3.64}$$

Inverse transformation on $R_k(p)$ lead to the desired reduced model as (2.14).

Figure 3.10 present the flowchart for the simplified interval structure approximation.

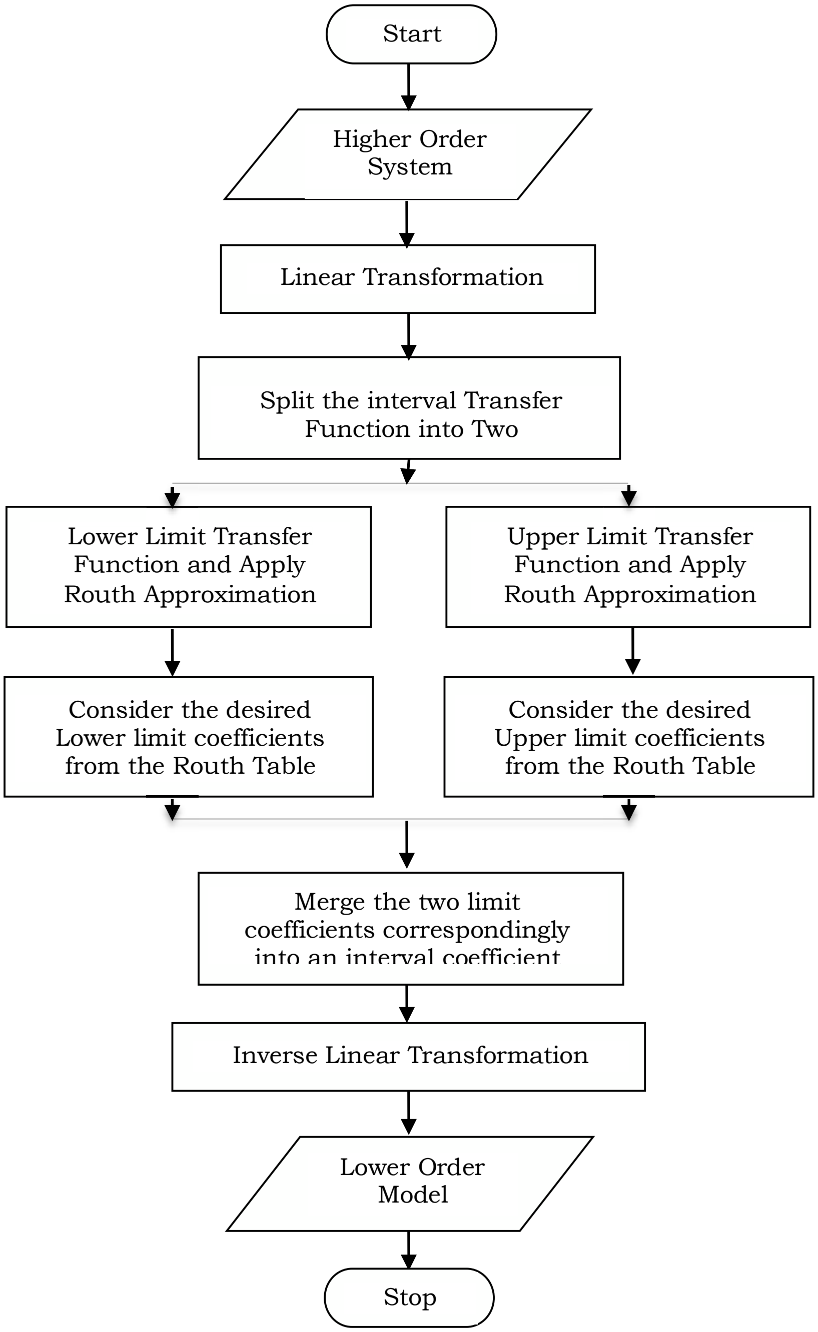

Figure 3.10: Flow chart for the simplified interval structure approximation

Example

E.3.5.1: Consider the higher order discrete-time interval system available from [68], [83], [88], [90], [107], [108] be (3.65) whose *p*-domain representation is (3.66)

$$H_3(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{3.65}$$

$$H_3(p)=\frac{[1,2]p^2+[5,8]p+[12,16]}{[6,6]p^3+[27,27.5]p^2+[40.9,42]p+[20.7,21.35]} \quad (3.66)$$

Split $H_3(p)$ into two transfer functions through lower and upper limits respectively as

$$H_{3L}(p)=\frac{1p^2+5p+12}{6p^3+27p^2+40.9p+20.7} \quad (3.67)$$

$$H_{3U}(p)=\frac{2p^2+8p+16}{6p^3+27.5p^2+42p+21.35} \quad (3.68)$$

The four polynomials from above transfer function are

$$B_3^-=1p^2+5p+12 \quad (3.69)$$

$$A_3^-=6p^3+27p^2+40.9p+20.7 \quad (3.70)$$

$$B_3^+=2p^2+8p+16 \quad (3.71)$$

$$A_3^+=6p^3+27.5p^2+42p+21.35 \quad (3.72)$$

Below drafted are the Routh tables from the above polynomials. Tables 3.22 and 3.23 present the lower limit transfer function (3.67) or (3.69) and (3.70). Correspondingly, Tables 3.24 and 3.25 show the upper limit transfer function (3.68) or (3.71) and (3.72). Precisely, first, two tables depict the numerator and denominator of the lower limit coefficient, and latter two describe the upper limit factors.

Table 3.22: Numerator coefficient for Lower Limit B_3^-

p^2	1	12
p^1	5	
p^0	12	

Table 3.23: Denominator coefficient for Lower Limit A_3^-

p^3	6	40.9
p^2	27	20.7
p^1	36.3	
p^0	20.7	

Table 3.24: Numerator coefficient for Upper Limit B_3^+

p^2	2	16
p^1	8	
p^0	16	

Table 3.25: Denominator coefficient for Upper Limit A_3^+

p^3	6	42
p^2	27.5	21.35
p^1	37.34	
p^0	21.35	

Upon completion of the tables, the coefficients from the lower and upper limits tables merge as stated in (3.63) and (3.64) respectively. Specifically, Tables 3.22 and 3.24 give the reduced numerator coefficients, and Tables 3.23 and 3.25 provide reduced denominator coefficients. Overall the reduced interval model is obtained as

$$R_2(z) = \frac{[5,8]z + [4,11]}{[27,27.5]z^2 + [34.3,35.34]z + [10.36,12.55]} \quad (3.73)$$

Table 3.26 displays the error computed between the higher order system and the reduced models by the proposed algorithm and other prevailing techniques. The result offers an acceptable algorithm.

Table 3.26: Error for 2nd order reduced models for E.3.5.1

Methods	Error	
	Lower Limit	Upper Limit
Proposed Algorithm	3.4294×10^{-4}	0.0018
Pade & Dominant poles [68]	0.1810	0.0741
Dominant poles & Direct Series [83]	0.3237	0.3229
Mikhailov & Factor Division [88]	0.0105	0.0250
Gamma Delta Appr. [90]	0.0278	0.0077
Direct truncation [107]	0.1292	0.0443
Routh-Pade Appr. [108]	0.1079	0.0342

To affirm the algorithm strength, the tracking of the step response of the higher order system to their reduced approximates by different algorithms is shown in Figures 3.11 and 3.12 for lower and upper limits respectively. Figures

3.13 and 3.14 check the stability through frequency response for lower and upper limits correspondingly.

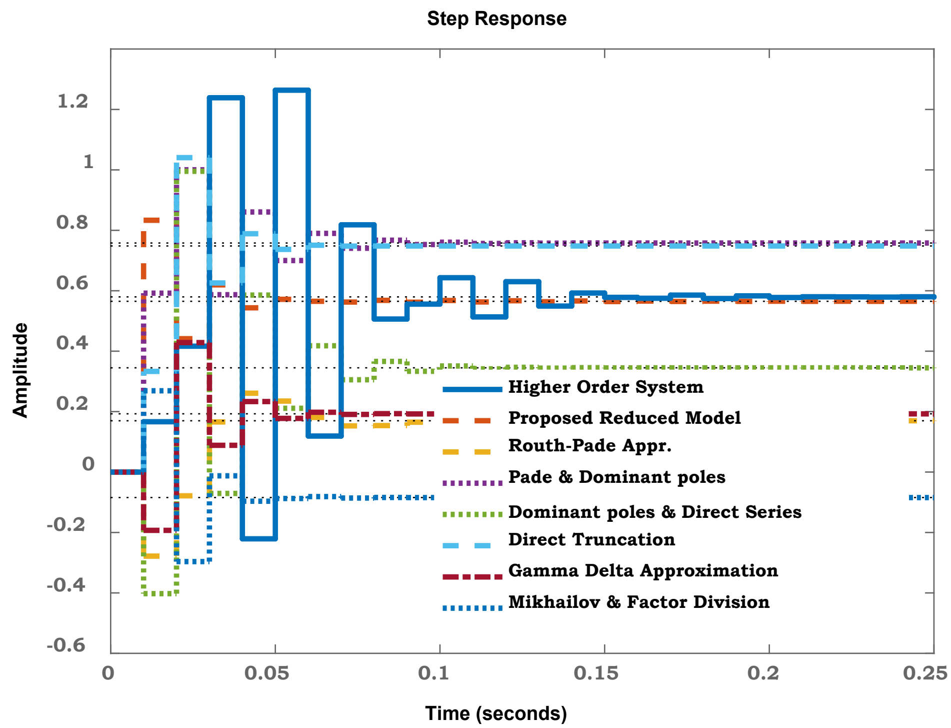

Figure 3.11: Step responses of reduced models (Lower Limit) for E.3.5.1

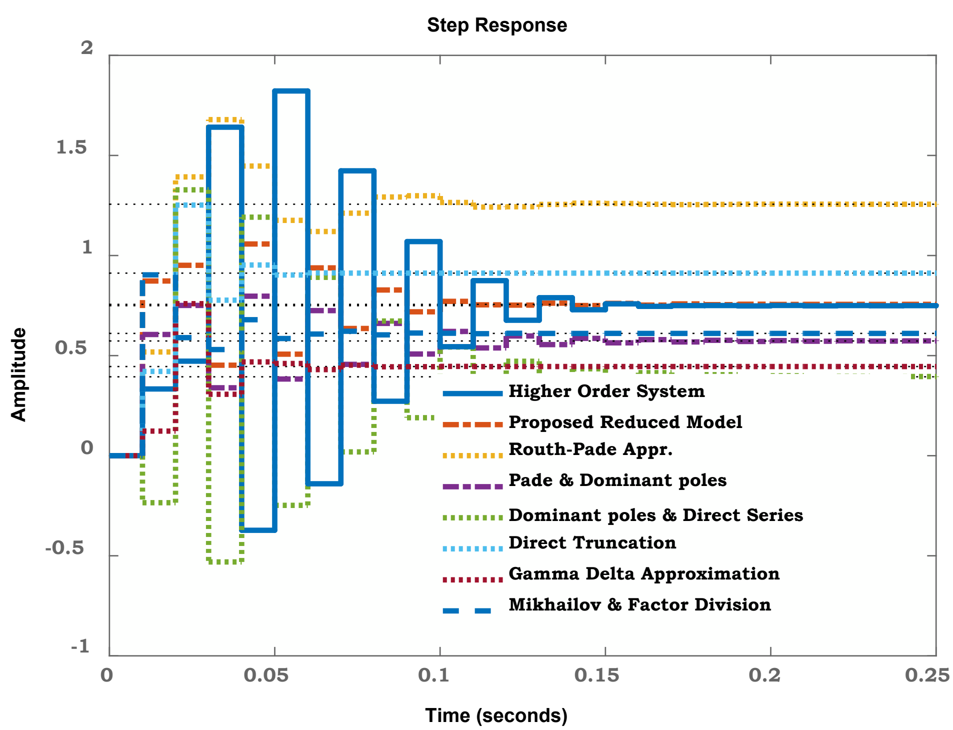

Figure 3.12: Step responses of reduced models (Upper Limit) for E.3.5.1

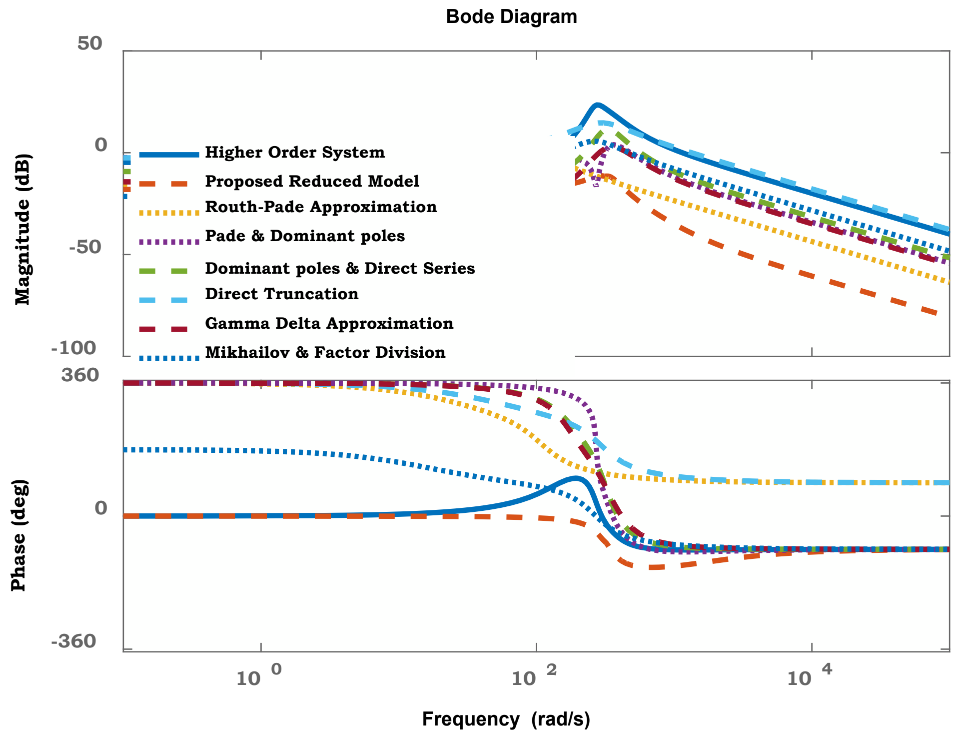

Figure 3.13: Frequency responses of reduced models (Lower Limit) for E.3.5.1

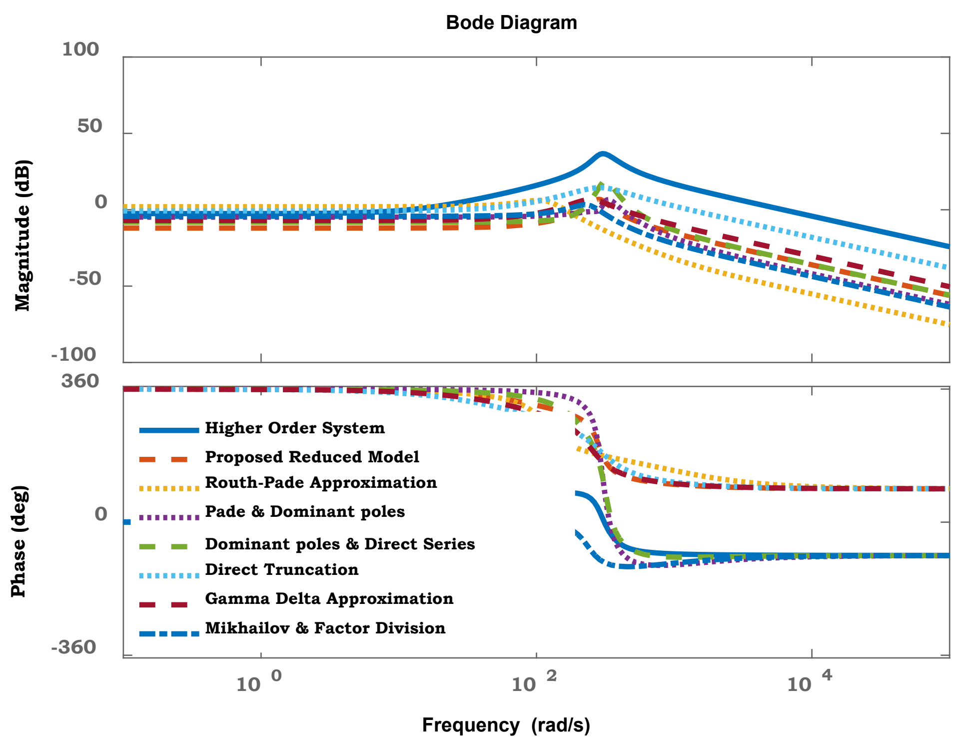

Figure 3.14: Frequency responses of reduced models (Upper Limit) for E.3.5.1

E.3.5.2. Consider another discrete-time interval system from [66], [88] be

$$H_3(p)=\frac{[3.25.3.35]p^2+[10,10.35]p+[9.55,10]}{[5.4,5.5]p^3+[17.2,17.6]p^2+[19.7,20.3]p+[10,10.35]} \quad (3.74)$$

The transfer functions with lower and upper limits are

$$H_{3L}(p)=\frac{3.25p^2+10p+9.55}{5.4p^3+17.2p^2+19.7p+10} \quad (3.75)$$

$$H_{3U}(p)=\frac{3.35p^2+10.35p+10}{5.5p^3+17.6p^2+20.3p+10.35} \quad (3.76)$$

Tables 3.27 and 3.28 showcase the usage of the numerator and denominator coefficients of $H_{3L}(p)$ and Tables 3.29, and 3.30 uses $H_{3U}(p)$.

Table 3.27: Numerator coefficient for Lower Limit B_3^-

p^2	3.25	9.55
p^1	10	
p^0	9.55	

Table 3.28: Denominator coefficient for Lower Limit A_3^-

p^3	5.4	19.7
p^2	17.2	10
p^1	16.56	
p^0	10	

Table 3.29: Numerator coefficient for Upper Limit B_3^+

p^2	3.35	10
p^1	10.35	
p^0	10	

Table 3.30: Denominator coefficient for Upper Limit A_3^+

p^3	5.5	20.3
p^2	17.6	10.35
p^1	17.06	
p^0	10.35	

Combination of the required lower and upper limit coefficients from the above tables result in the reduced z-domain model as

$$R_2(z) = \frac{[10,10.35]z + [-0.8,0]}{[17.2,17.6]z^2 + [14.56,15.06]z + [10.14,11.39]} \tag{3.77}$$

Error in Table 3.31 substantiates towards the establishment of an algorithm.

Figures 3.15 and 3.16 validate the algorithm through the tracking of step responses. And Figures 3.17 and 3.18 depict the frequency response for stability check of lower and upper limit transfer functions respectively.

Table 3.31: Error for 2nd order reduced models for E.3.5.2

Methods	Error	
	Lower Limit	Upper Limit
Proposed Algorithm	4.1847×10^{-4}	4.4196×10^{-4}
Multipoint Pade Appr. [66]	0.0721	0.0392
Mikhailov & Factor Division [88]	0.0194	0.0889
Gamma Delta Appr. [90]	0.0011	5.4626×10^{-5}

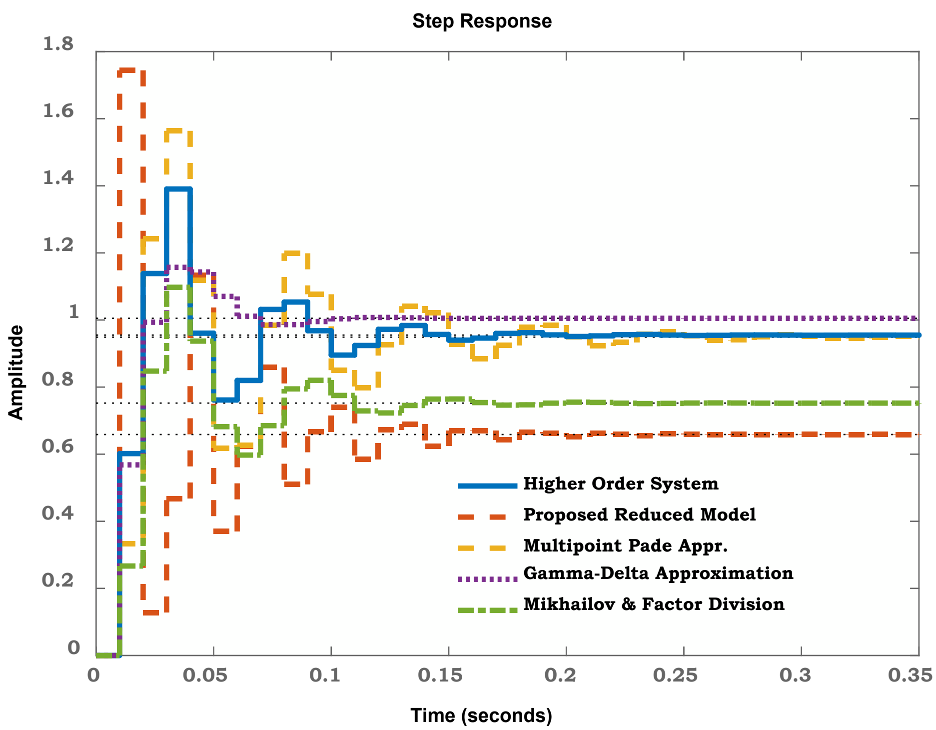

Figure 3.15: Step responses of reduced models (Lower Limit) for E.3.5.2

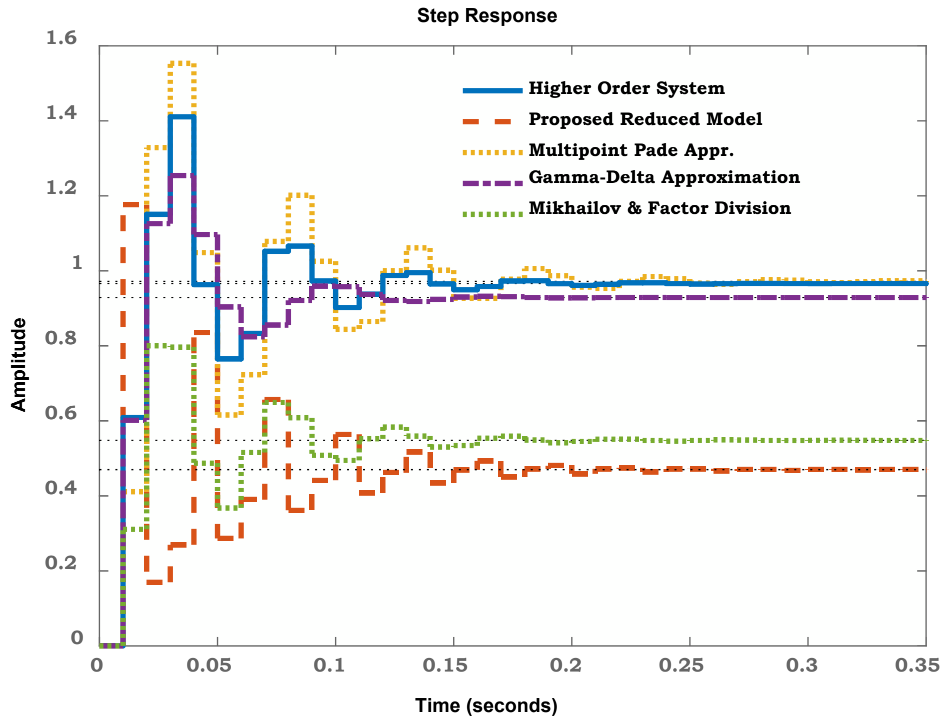

Figure 3.16: Step responses of reduced models (Upper Limit) for E.3.5.2

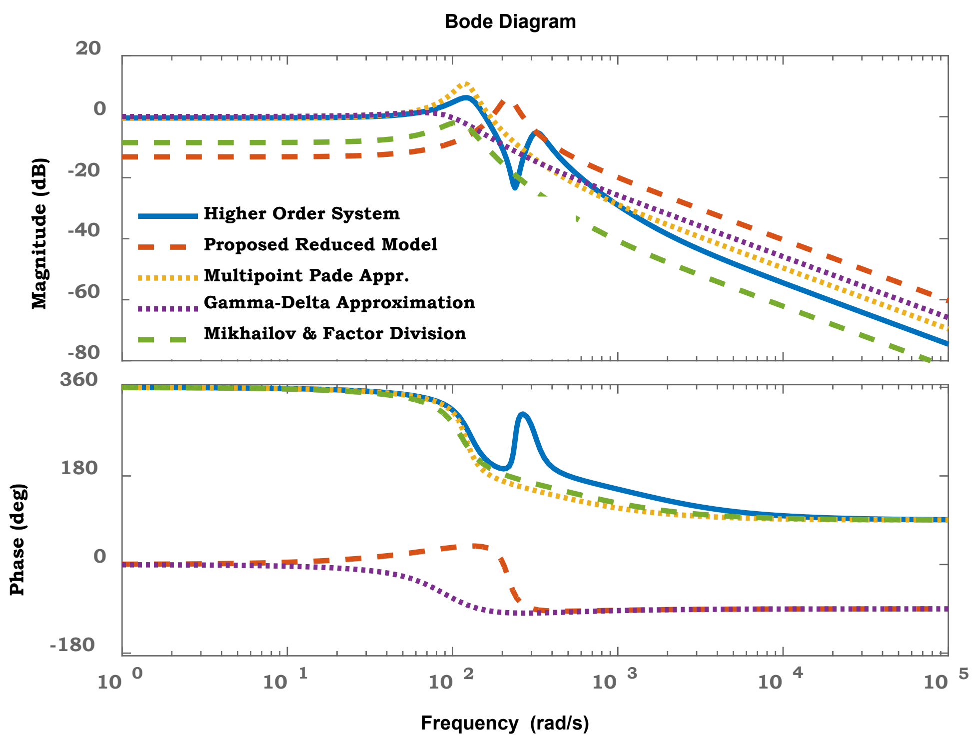

Figure 3.17: Frequency responses of reduced models (Lower Limit) for E.3.5.2

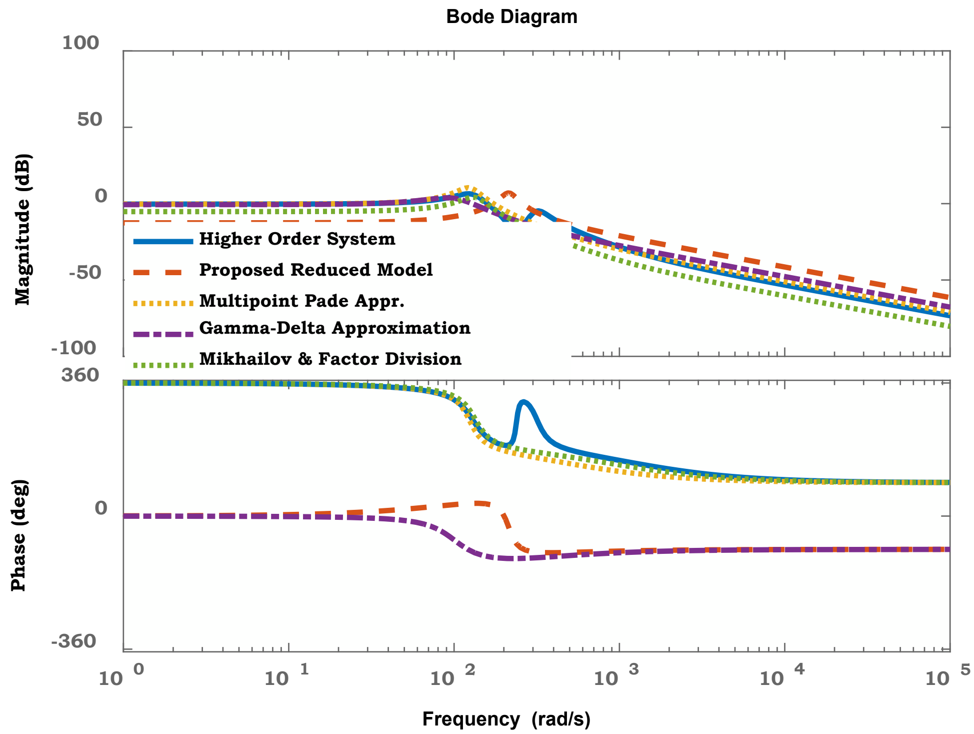

Figure 3.18: Frequency responses of reduced models (Upper Limit) for E.3.5.2

Conclusions

The novelty in the proposed algorithm is the splitting of interval system into two transfer functions. One with lower limit coefficients and other with upper limit coefficients respectively. Later on, *RA* is applied on each of the numerator and denominator polynomials.

3.6. Advanced Routh Approximation Method (A-RAM)

This algorithm unfolds the prevailing techniques based on *RA* that assures to be simple and features the advantage of retaining the model stability. This is a possible advancement of Hutton and Friedland's [24] work from fixed coefficients to discrete-time interval coefficients system, thus named *A-RAM*.

Methodology

For simplification based on *RA*, here engaged is linear transformation that result in (2.16).

The *A-RAM* uses transfer function $\hat{H}_n(p)$ related to $H_n(p)$ by the following transformation, which is obtained by reversing the order of the polynomial coefficients.

$$\hat{H}_n(p) = \frac{1}{p}H_n\left(\frac{1}{p}\right) = \frac{\left[b_0^-, b_0^+\right]p^{n-1} + \left[b_1^-, b_1^+\right]p^{n-2} + \ldots + \left[b_{n-1}^-, b_{n-1}^+\right]}{\left[a_0^-, a_0^+\right]p^n + \left[a_1^-, a_1^+\right]p^{n-1} + \ldots + \left[a_n^-, a_n^+\right]} \quad (3.78)$$

Routh table from $\hat{H}_n(p)$, for computing uncertain parameters $\hat{\alpha}_i = \left[\alpha_i^-, \alpha_i^+\right]$ and $\hat{\beta}_i = \left[\beta_i^-, \beta_i^+\right]$ with the application of denominator and numerator polynomials are shown in Table 3.32 and Table 3.33 respectively. Entries from the third row in the tables are by (3.79) and (3.80).

$$\left[a_{i,j}^-, a_{i,j}^+\right] = \left[a_{i-2,j+1}^-, a_{i-2,j+1}^+\right] - \left[\alpha_{i-1}^-, \alpha_{i-1}^+\right]\left[a_{i-1,j+1}^-, a_{i-1,j+1}^+\right] \quad i=2,3,.\ \&\ j=0,1,2.. \quad (3.79)$$

$$\left[b_{i,j}^-, b_{i,j}^+\right] = \left[b_{i-2,j+1}^-, b_{i-2,j+1}^+\right] - \left[\beta_{i-1}^-, \beta_{i-1}^+\right]\left[a_{i-1,j+1}^-, a_{i-1,j+1}^+\right] \quad i=2,3,.\ \&\ j=0,1,2... \quad (3.80)$$

Table 3.32: Denominator array for $\hat{\alpha}_i$ parameter

$\left[a_0^-, a_0^+\right] = \left[a_{0,0}^-, a_{0,0}^+\right]$	$\left[a_2^-, a_2^+\right] = \left[a_{0,1}^-, a_{0,1}^+\right]$	$\left[a_4^-, a_4^+\right] = \left[a_{0,2}^-, a_{0,2}^+\right]$	.....
$\left[a_1^-, a_1^+\right] = \left[a_{1,0}^-, a_{1,0}^+\right]$	$\left[a_3^-, a_3^+\right] = \left[a_{1,1}^-, a_{1,1}^+\right]$	$\left[a_5^-, a_5^+\right] = \left[a_{1,2}^-, a_{1,2}^+\right]$	.....
....			
$\left[a_{n-1,0}^-, a_{n-1,0}^+\right]$			
$\left[a_{n,0}^-, a_{n,0}^+\right]$			

Table 3.33: Numerator array for $\hat{\beta}_i$ parameter

$\left[b_0^-, b_0^+\right] = \left[b_{1,0}^-, b_{1,0}^+\right]$	$\left[b_2^-, b_2^+\right] = \left[b_{1,1}^-, b_{1,1}^+\right]$	$\left[b_4^-, b_4^+\right] = \left[b_{1,2}^-, b_{1,2}^+\right]$	.....
$\left[b_1^-, b_1^+\right] = \left[b_{2,0}^-, b_{2,0}^+\right]$	$\left[b_3^-, b_3^+\right] = \left[b_{2,1}^-, b_{2,1}^+\right]$	$\left[b_5^-, b_5^+\right] = \left[b_{2,2}^-, b_{2,2}^+\right]$	.....
.....			
$\left[b_{n-1,0}^-, b_{n-1,0}^+\right]$			
$\left[b_{n,0}^-, b_{n,0}^+\right]$			

where

$$\hat{\alpha}_i = \frac{\left[a_{i-1,0}^-, a_{i-1,0}^+\right]}{\left[a_{i,0}^-, a_{i,0}^+\right]} \quad \text{with } i=1,2,3.... \tag{3.81}$$

$$\hat{\beta}_i = \frac{\left[b_{i,0}^-, b_{i,0}^+\right]}{\left[a_{i,0}^-, a_{i,0}^+\right]} \quad \text{with } i=1,2,3.... \tag{3.82}$$

Thereafter, the first k, parameters of $\hat{\alpha} - \hat{\beta}$ tables are retained for obtaining the k^{th} order model.

$$\hat{R}_k(p) = \frac{\hat{B}_k(p)}{\hat{A}_k(p)} \tag{3.83}$$

where $\hat{A}_k(p) = \hat{A}_{k-2}(p) + \hat{\alpha}_k p \hat{A}_{k-1}(p)$ (3.84)

$$\hat{B}_k(p) = \hat{\beta}_k + \hat{B}_{k-2}(p) + \hat{\alpha}_k p \hat{B}_{k-1}(p) \tag{3.85}$$

with $\hat{A}_{-1}(p) = 1$, $\hat{A}_0(p) = 1$, $\hat{B}_{-1}(p) = 0$, $\hat{B}_0(p) = 0$

Using the above equations, the first and second order reduced models are drawn as

$$\hat{R}_1(p) = \frac{\left[\beta_1^-, \beta_1^+\right]}{1 + \left[\alpha_1^-, \alpha_1^+\right] p} \tag{3.86}$$

$$\hat{R}_2(p) = \frac{\left[\beta_2^-, \beta_2^+\right] + \left[\alpha_2^-, \alpha_2^+\right]\left[\beta_1^-, \beta_1^+\right] p}{1 + \left[\alpha_2^-, \alpha_2^+\right] p + \left[\alpha_2^-, \alpha_2^+\right]\left[\alpha_1^-, \alpha_1^+\right] p^2} \tag{3.87}$$

After $\hat{R}_k(p)$ computation, $R_k(p)$ is obtained by proper reciprocation and the desired z-domain model is obtained by $p = z - 1$ transformation.

The proposed algorithm is represented by flow diagram in Figure 3.19.

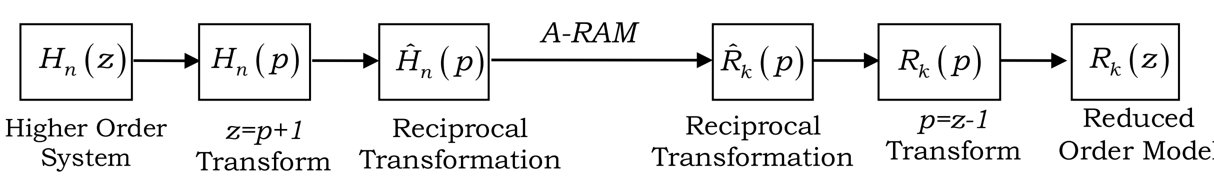

Figure 3.19: Flow diagram of the algorithmic steps for A-RAM

Example

The examples establish a comparative study of the proposed method with the prevailing techniques

E.4.6.1. Consider the third order system from [68], [83], [90], [107] with p-domain equivalent as

$$H_3(z)=\frac{[1,2]z^2+[3,4]z+[8,10]}{[6,6]z^3+[9,9.5]z^2+[4.9,5]z+[0.8,0.85]} \tag{3.88}$$

$$H_3(p)=\frac{[1,2]p^2+[5,8]p+[12,16]}{[6,6]p^3+[27,27.5]p^2+[40.9,42]p+[20.7,21.35]} \tag{3.89}$$

By algorithm, reciprocal of the above system is (3.90) and $\hat{\alpha}-\hat{\beta}$ parameters obtained are;

$$\hat{H}_3(p)=\frac{[12,16]p^2+[5,8]p+[1,2]}{[20.7,21.35]p^3+[40.9,42]p^2+[27,27.5]p+[6,6]} \tag{3.90}$$

$$\left[\alpha_1^-,\alpha_1^+\right]=[0.49,0.52],\ \left[\alpha_2^-,\alpha_2^+\right]=[1.66,1.75]$$

$$\left[\beta_1^-,\beta_1^+\right]=[0.28,0.39],\ \left[\beta_2^-,\beta_2^+\right]=[0.20,0.34]$$

These parameters lead to the reduced models as

$$R_1(z)=\frac{[0.28,0.39]}{z+[-0.51,-0.48]} \tag{3.91}$$

and

$$R_2(z)=\frac{[0.20,0.34]z+[0.14,0.48]}{z^2+[-0.34,-0.25]z+[0.07,0.25]} \tag{3.92}$$

The error for $R_1(z)$ and $R_2(z)$ along with those obtained by the prevailing techniques shown in Table 3.34, confirms the merit of the proposed method. Step response of the reduced models for lower and upper limit transfer functions are shown in Figure 3.20 and Figure 3.21 respectively for E.3.6.1. Later Figure 3.22 and Figure 3.23 depict the frequency responses for the two limit transfer functions correspondingly.

Table 3.34: Error for 1st and 2nd order reduced models for E.3.6.1

Methods	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Algorithm	0.0140	0.0030	0.0011	$4.44\text{x}10^{-05}$
Pade & Dominant Pole [68]	0.1398	0.0195	0.1810	0.0741
Dominant Pole/Direct Series [83]	0.4839	0.4134	0.3237	0.3229
Gamma-Delta Appr. [90]	0.0157	0.0035	0.1292	0.0443
Direct Truncation [107]	2.1491	2.7778	0.0278	0.0077

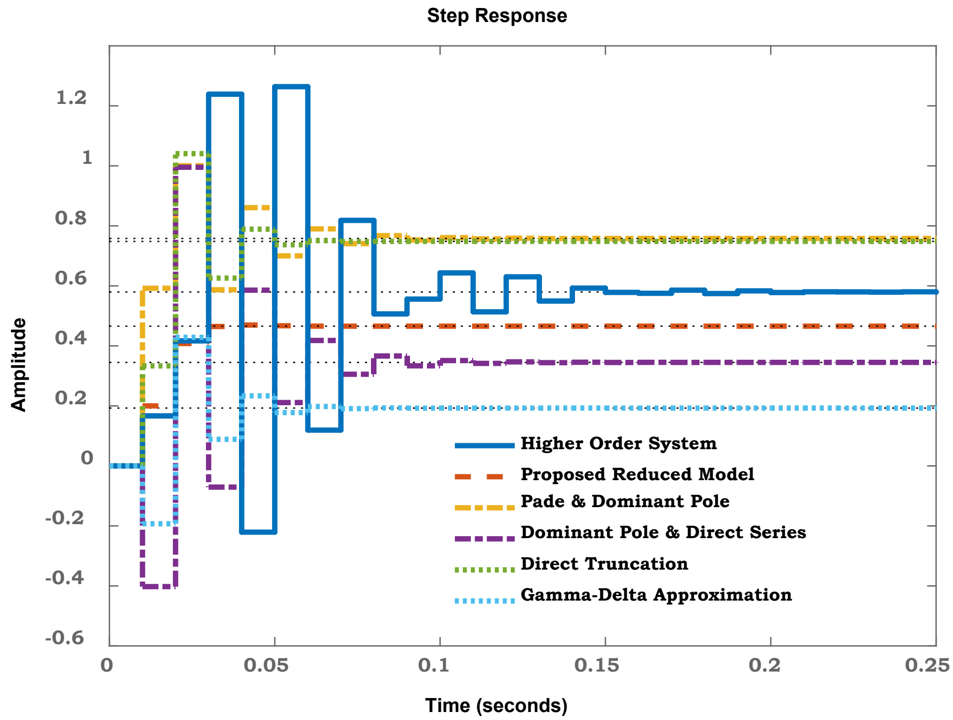

Figure 3.20: Step responses of reduced models (Lower Limit) for E.3.6.1

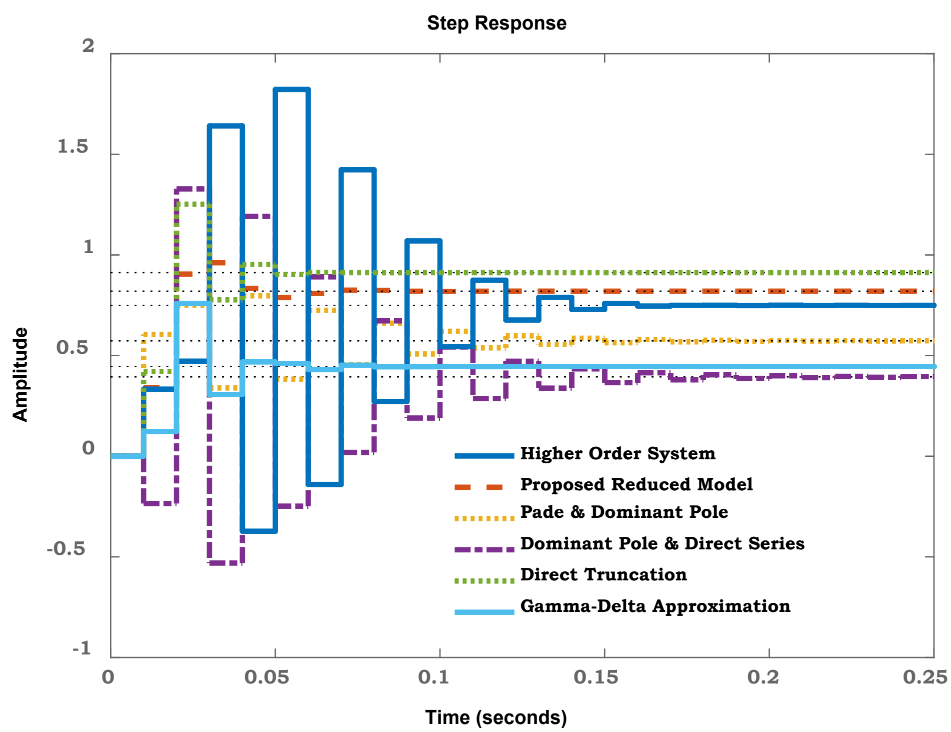

Figure 3.21: Step responses of reduced models (Upper Limit) for E.3.6.1

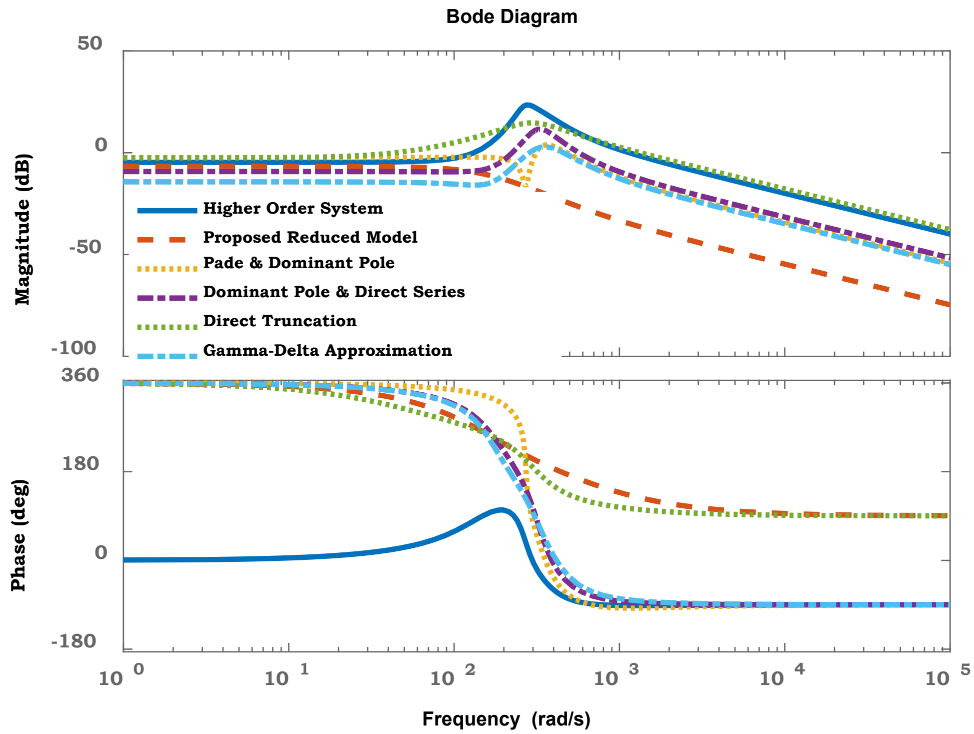

Figure 3.22: Frequency responses of reduced models (Lower Limit) for E.3.6.1

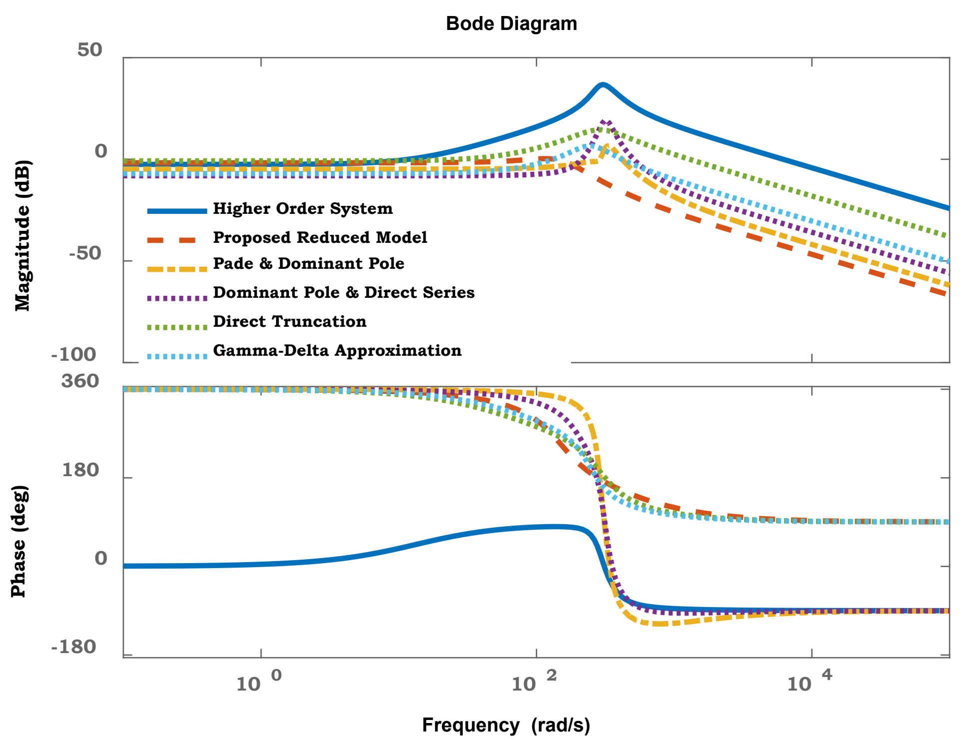

Figure 3.23: Frequency responses of reduced models (Upper Limit) for E.3.6.1

E.4.4.2. Consider a real-time digital control system with $H_8(z)$ as

$$H_8(z)=\frac{\begin{array}{l}[1.6484,1.7156]z^7+[1.0937,1.1383]z^6+[-0.2142,-0.2058]z^5\\+[0.1490,0.1550]z^4+[-0.5263,-0.5057]z^3+[-0.2672,-0.2568]z^2\\+[0.0431,0.0449]z+[-0.0061,-0.0059]\end{array}}{\begin{array}{l}[23.52,24.48]z^8+[-1.7156,-1.6484]z^7+[-1.1383,-1.0937]z^6\\+[0.2058,0.2142]z^5+[-0.1550,-0.1490]z^4+[0.5057,0.5263]z^3\\+[0.2568,0.3672]z^2+[-0.0449,-0.0431]z+[0.0059,0.0061]\end{array}} \tag{3.93}$$

Using algorithmic steps and parameters computed, the reduced models are as

$$R_1(z)=\frac{[0.01,0.01]}{z+[-0.88,-0.86]} \tag{3.94}$$

and

$$R_2(z)=\frac{[0.03,0.03]z+[-0.03,-0.02]}{z^2+[-1.64,-1.58]z+[0.63,0.70]} \tag{3.95}$$

Error for $R_1(z)$ and $R_2(z)$ are shown in Table 3.35. Their step response are in Figures 3.24 and 3.25 for lower and upper limits respectively. Figures 3.26 and 3.27 depict their frequency responses correspondingly.

Table 3.35: Error for 1st and 2nd order reduced models for E.3.6.2

Method	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Algorithm	0.0036	0.0036	0.0016	0.0016

Conclusions

The proposed algorithm is an approach for approximating a higher-order interval system by a lower order interval model based on *RA* method. Accompanied examples confirm the algorithm to be efficient that uphold the dynamic characteristics of the original system giving better and satisfactory results.

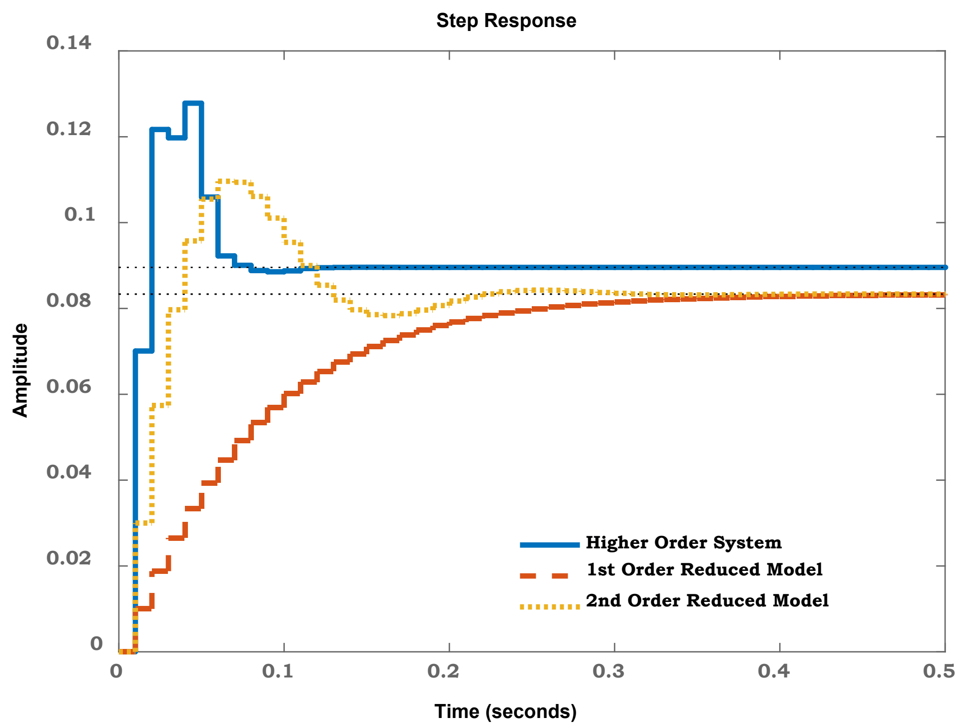

Figure 3.24: Step responses of reduced models (Lower Limit) for E.3.6.2

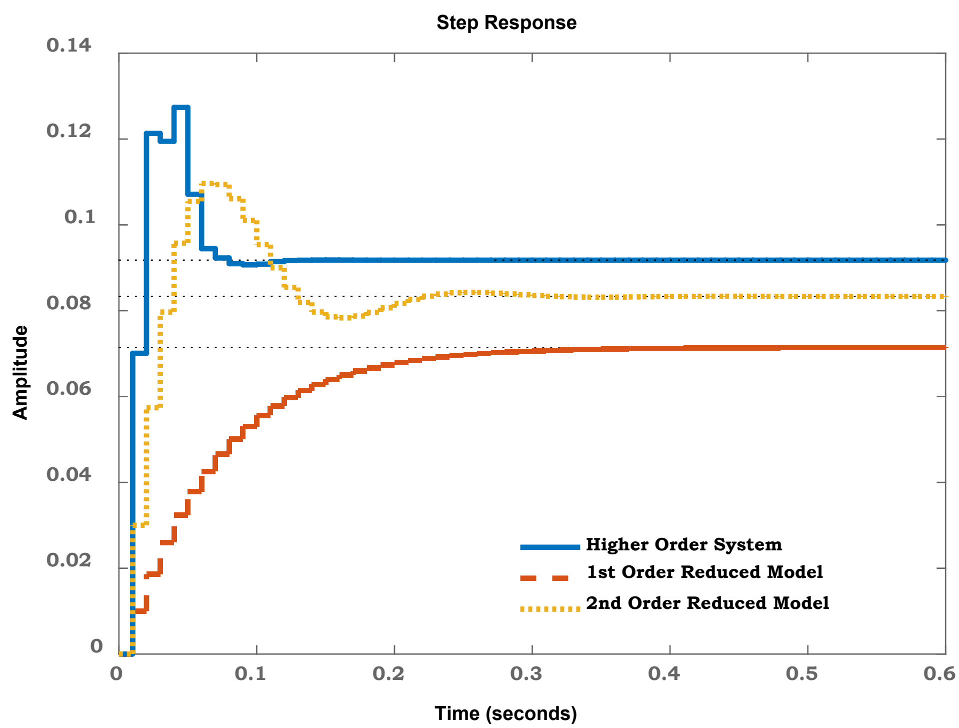

Figure 3.25: Step responses of reduced models (Upper Limit) for E.3.6.2

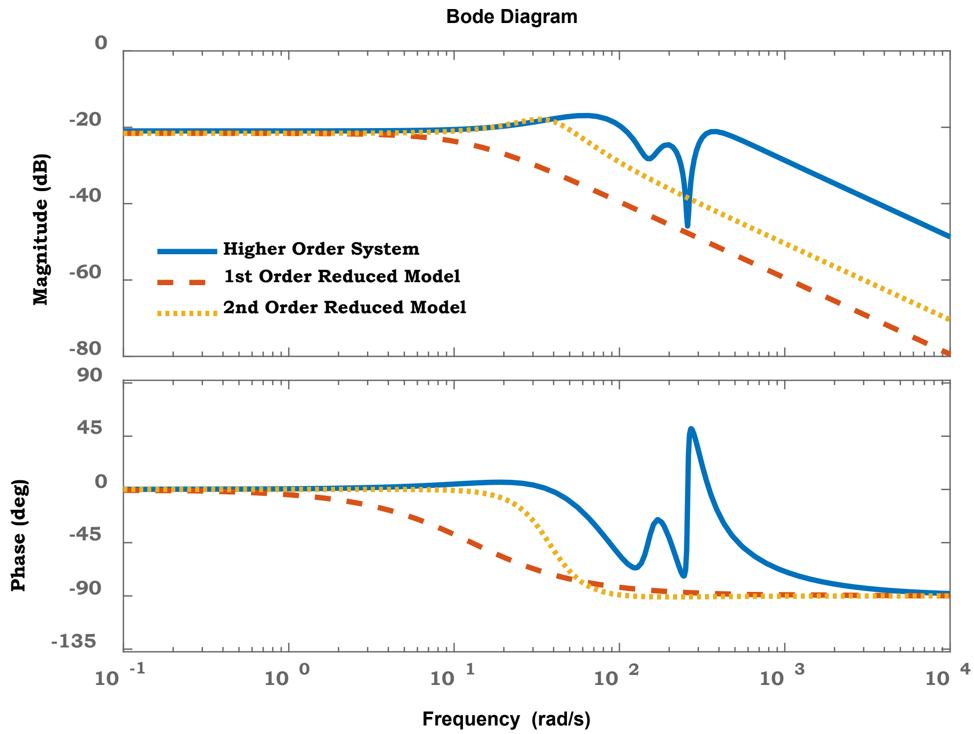

Figure 3.26: Frequency responses of reduced models (Lower Limit) for E.3.6.1

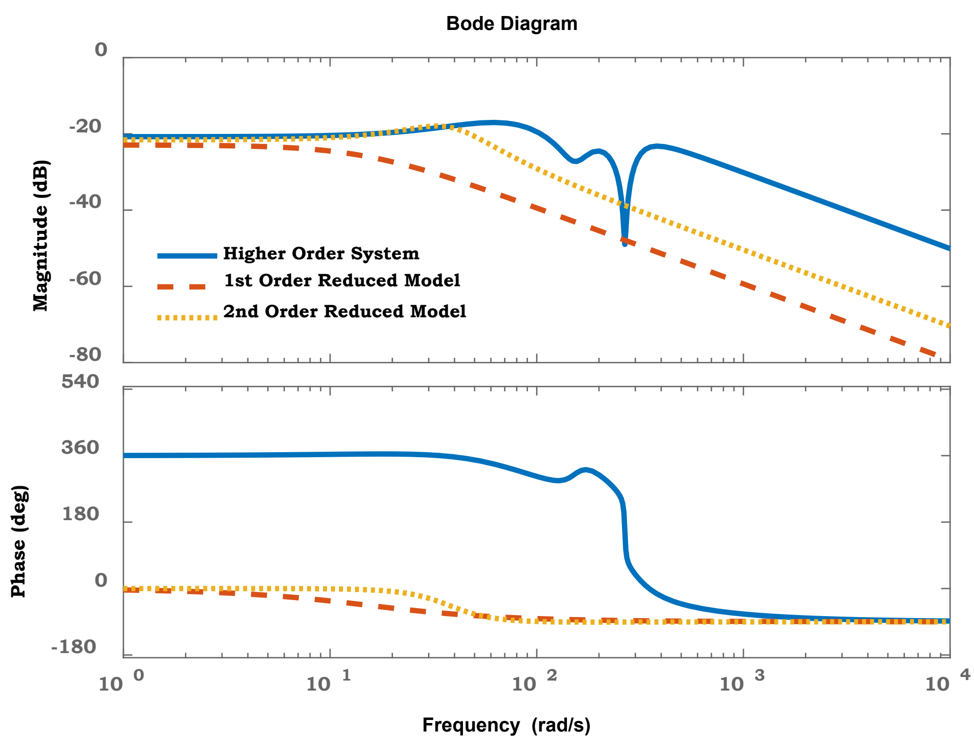

Figure 3.27: Frequency responses of reduced models (Upper Limit) for E.3.6.2

3.7. Extended Direct Routh Approximation Method (E-DRAM)

Similar to the technique in section 3.6, an extension is presented from non-interval coefficient system to interval coefficient system, based on Direct Routh Approximation Method (DRAM) in [54]. This DRAM is accepted as an amendment of Routh Approximation Method (RAM) demonstrated in [24], in a manner for being free from the reciprocal transformation. The proposed algorithm is designated *E-DRAM*.

Methodology

Routh approximation rules employ $z = 1 + p$ transformation on $H_n(z)$ resulting in (2.16).

Using above transfer function, interval $\alpha's$ is calculated using the denominator polynomial stated in Table 3.36 as

$$\left[\alpha_i^-, \alpha_i^+\right] = \frac{\left[a_{i-1,0}^-, a_{i-1,0}^+\right]}{\left[a_{i,0}^-, a_{i,0}^+\right]} \qquad \text{for } i=1,\ 2,\ 3.... \tag{3.96}$$

and

$$\left[a_{i,j}^-, a_{i,j}^+\right] = \left[a_{i-2,j+1}^-, a_{i-2,j+1}^+\right] - \left[\alpha_{i-1}^-, \alpha_{i-1}^+\right]\left[a_{i-1,j+1}^-, a_{i-1,j+1}^+\right] \tag{3.97}$$

with *i=2,3,4.......* and *j=0,1,2,.....*

Table 3.36: Denominator array for α-parameter

$\left[a_0^-, a_0^+\right] = \left[a_{0,0}^-, a_{0,0}^+\right]$	$\left[a_2^-, a_2^+\right] = \left[a_{0,1}^-, a_{0,1}^+\right]$	$\left[a_4^-, a_4^+\right] = \left[a_{0,2}^-, a_{0,2}^+\right]$	.....
$\left[a_1^-, a_1^+\right] = \left[a_{1,0}^-, a_{1,0}^+\right]$	$\left[a_3^-, a_3^+\right] = \left[a_{1,1}^-, a_{1,1}^+\right]$	$\left[a_5^-, a_5^+\right] = \left[a_{1,2}^-, a_{1,2}^+\right]$	.....
....			
$\left[a_{n-1,0}^-, a_{n-1,0}^+\right]$			
$\left[a_{n,0}^-, a_{n,0}^+\right]$			

Similarly, interval $\beta's$ is obtained from numerator polynomial as shown in Table 3.37,

$$\left[\beta_i^-, \beta_i^+\right] = \frac{\left[b_{i,0}^-, b_{i,0}^+\right]}{\left[a_{i,0}^-, a_{i,0}^+\right]} \quad \text{for } i=1,\ 2,\ 3.... \tag{3.98}$$

and

$$\left[b_{i,j}^{-},b_{i,j}^{+}\right]=\left[b_{i-1,j+1}^{-},b_{i-1,j+1}^{+}\right]-\left[\beta_{i-1}^{-},\beta_{i-1}^{+}\right]\left[a_{i-1,j+1}^{-},a_{i-1,j+1}^{+}\right] \tag{3.99}$$

for i=3,4……. and j=0,1,2,…..

Table 3.37: Numerator array for β-parameter

$\left[b_0^-,b_0^+\right]=\left[b_{1,0}^-,b_{1,0}^+\right]$	$\left[b_2^-,b_2^+\right]=\left[b_{1,1}^-,b_{1,1}^+\right]$	$\left[b_4^-,b_4^+\right]=\left[b_{1,2}^-,b_{1,2}^+\right]$	…..
$\left[b_1^-,b_1^+\right]=\left[b_{2,0}^-,b_{2,0}^+\right]$	$\left[b_3^-,b_3^+\right]=\left[b_{2,1}^-,b_{2,1}^+\right]$	$\left[b_5^-,b_5^+\right]=\left[b_{2,2}^-,b_{2,2}^+\right]$	…..
…..			
$\left[b_{n-1,0}^-,b_{n-1,0}^+\right]$			
$\left[b_{n,0}^-,b_{n,0}^+\right]$			

From the computed $\alpha-\beta$ parameters, the reduced order numerator and denominator are drawn as

$$B_1(p)=\left[\beta_1^-,\beta_1^+\right] \tag{3.100a}$$

$$A_1(p)=\left[\alpha_1^-,\alpha_1^+\right]+p \tag{3.100b}$$

$$B_2(p)=\left[\beta_2^-,\beta_2^+\right]p+\left[\alpha_2^-,\alpha_2^+\right]\left[\beta_1^-,\beta_1^+\right] \tag{3.101a}$$

$$A_2(p)=p^2+\left[\alpha_2^-,\alpha_2^+\right]p+\left[\alpha_2^-,\alpha_2^+\right]\left[\alpha_1^-,\alpha_1^+\right]\text{and so on...} \tag{3.101b}$$

In general, $B_k(p)$ and $A_k(p)$ can be specified by

$$B_k(p)=\left[\beta_k^-,\beta_k^+\right]p^{k-1}+p^2B_{k-2}(p)+\left[\alpha_k^-,\alpha_k^+\right]B_{k-1}(p) \tag{3.102a}$$

$$A_k(p)=p^2A_{k-2}(p)+\left[\alpha_k^-,\alpha_k^+\right]A_{k-1}(p) \tag{3.102b}$$

with $A_{-1}(p)=1/p$, $A_0(p)=1$, $B_{-1}(p)=0$, $B_0(p)=0$

Thereafter, inverse transformation of derived $R_k(p)$, results in the desired z-domain reduced model transfer function.

The algorithmic steps of the proposed technique is shown as a flow diagram in Figure 3.28.

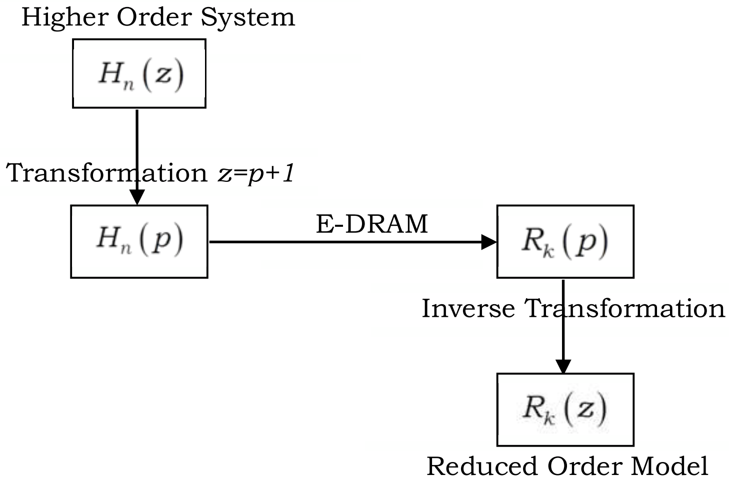

Figure 3.28: Flow diagram of the algorithmic steps for E-DRAM

Example

E.3.7.1. Consider the third order system represented by (3.103) from [68], [83], [90], [107] and (3.104) gives its *p*-domain representation

$$H_3(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{3.103}$$

$$H_3(p) = \frac{[1,2]p^2 + [5,8]p + [12,16]}{[6,6]p^3 + [27,27.5]p^2 + [40.9,42]p + [20.7,21.35]} \tag{3.104}$$

α's and β's parameters obtained from (3.104) are

$$[\alpha_1^-, \alpha_1^+] = [0.21, 0.22], \qquad [\alpha_2^-, \alpha_2^+] = [0.717, 0.759],$$

$$[\beta_1^-, \beta_1^+] = [0.036, 0.074], \quad [\beta_2^-, \beta_2^+] = [0.132, 0.220].$$

These parameters in (3.100) and (3.101) lead to the following reduced models

$$R_1(z) = \frac{[0.036, 0.074]}{z + [-0.78, -0.79]} \tag{3.105}$$

and

$$R_2(z) = \frac{[0.132, 0.220]z + [-0.195, -0.079]}{z^2 + [-1.283, -1.241]z + [0.391, 0.449]} \tag{3.106}$$

The error for $R_1(z)$ and $R_2(z)$ along with those obtained by other prevailing methods shown in Table 3.38, confirms the fact that a satisfactory approximation is achieved.

Figures 3.29 and 3.30 depict the step response and Figures 3.31 and 3.32 present the frequency response for the lower and upper limit reduced models respectively.

Table 3.38: Error for 1st and 2nd order reduced models for E.3.7.1

Methods	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Algorithm	0.0171	0.0673	0.0012	0.0128
Pade & Dominant Pole [68]	0.1398	0.0195	0.1810	0.0741
Dominant Pole/Direct Series [83]	0.4839	0.4134	0.3237	0.3229
Gamma-Delta Appr. [90]	0.0157	0.0035	0.1292	0.0443
Direct Truncation [107]	2.1491	2.7778	0.0278	0.0077

E.3.7.2. Let transfer function of seventh order be (3.107) with its p-domain representation as (3.108)

$$H_7(z)=\frac{\begin{array}{l}[0.0077,0.0080]z^6+[-0.0092,-0.0088]z^5+[-0.0124,-0.0119]z^4\\+[0.0259,0.0270]z^3+[-0.0177,-0.0170]z^2+[0.0047,0.0049]z\\+\left[-4.8073\,\mathrm{e}^{-4},-4.6187\,\mathrm{e}^{-4}\right]\end{array}}{\begin{array}{l}[0.98,1.02]z^7+[-5.5417,-5.3243]z^6+[12.5538,13.0662]z^5\\+[-17.3502,-16.6698]z^4+[13.4456,13.9944]z^3+[-6.8626,-6.5934]z^2\\+[1.8208,1.8952]z+[-0.2276,-0.2186]\end{array}} \tag{3.107}$$

$$H_7(p)=\frac{\begin{array}{l}[0.0077,0.0080]p^6+[0.037,0.0392]p^5+[0.0571,0.0641]p^4\\+[0.0383,0.0514]p^3+[0.0091,0.0246]p^2+[-0.0024,0.0083]p\\+[-0.0890,-0.0823]\end{array}}{\begin{array}{l}[0.98,1.02]p^7+[1.3183,1.8157]p^6+[-0.1164,2.5464]p^5\\+[-3.4067,4.4967]p^4+[-6.9512,7.1912]p^3+[-7.6345,7.5885]p^2\\+[-4.5896,4.5376]p+[-1.1819,1.1697]\end{array}} \tag{3.108}$$

Parameters from (3.108) are obtained as

$$\left[\alpha_1^-,\alpha_1^+\right]=[-0.5397,0.7737],\left[\alpha_2^-,\alpha_2^+\right]=[-0.5050,0.3508]$$

$$\left[\beta_1^-,\beta_1^+\right]=[0.0042,0.0061],\left[\beta_2^-,\beta_2^+\right]=[-0.0109,0.0076]$$

These parameters result in the following reduced models

$$R_1(z)=\frac{[0.0042,0.0061]}{z+[-0.4603,-0.2263]} \tag{3.109}$$

and

$$R_2(z) = \frac{[-0.0109, 0.0076]z + [-0.0107, 0.0130]}{z^2 + [-2.5050, -1.6492]z + [0.2585, 1.7764]} \quad (3.110)$$

The error for reduced models of order 1 and 2 are shown in Table 3.39. Figures 3.33 and 3.34 present the frequency response for the lower and upper limit reduced models respectively for E.3.7.2.

Table 3.39: Error for 1st and 2nd order reduced models for E.3.7.2.

Method	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Algorithm	1.3375×10^{-5}	3.0385×10^{-6}	3.5183×10^{-4}	5.9116×10^{-8}

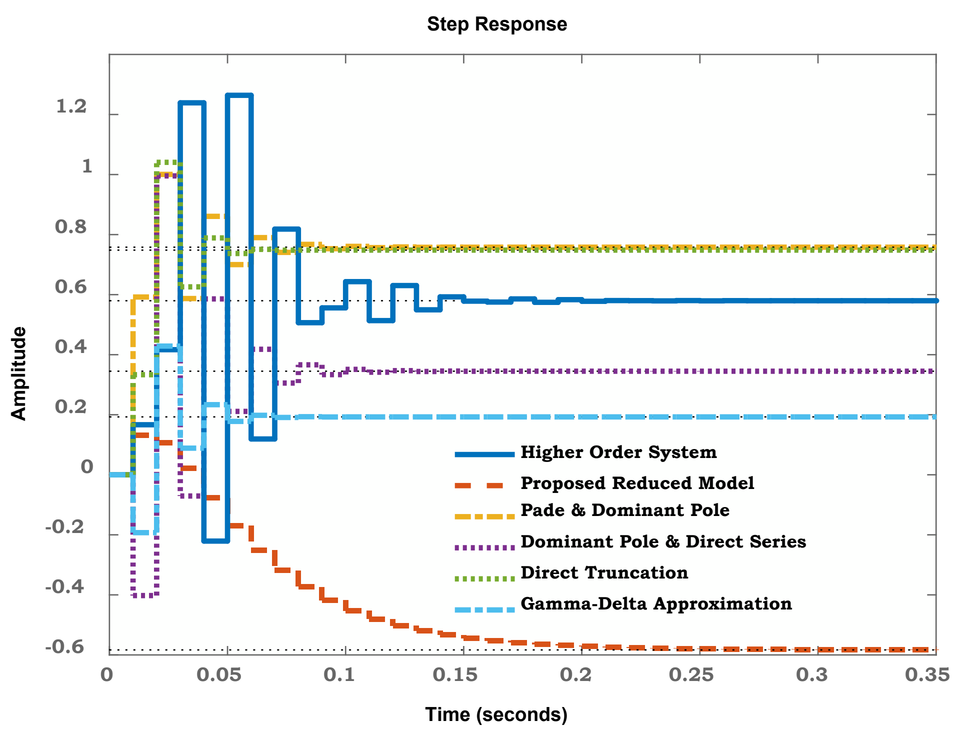

Figure 3.29: Step responses of reduced models (Lower Limit) for E.3.7.1

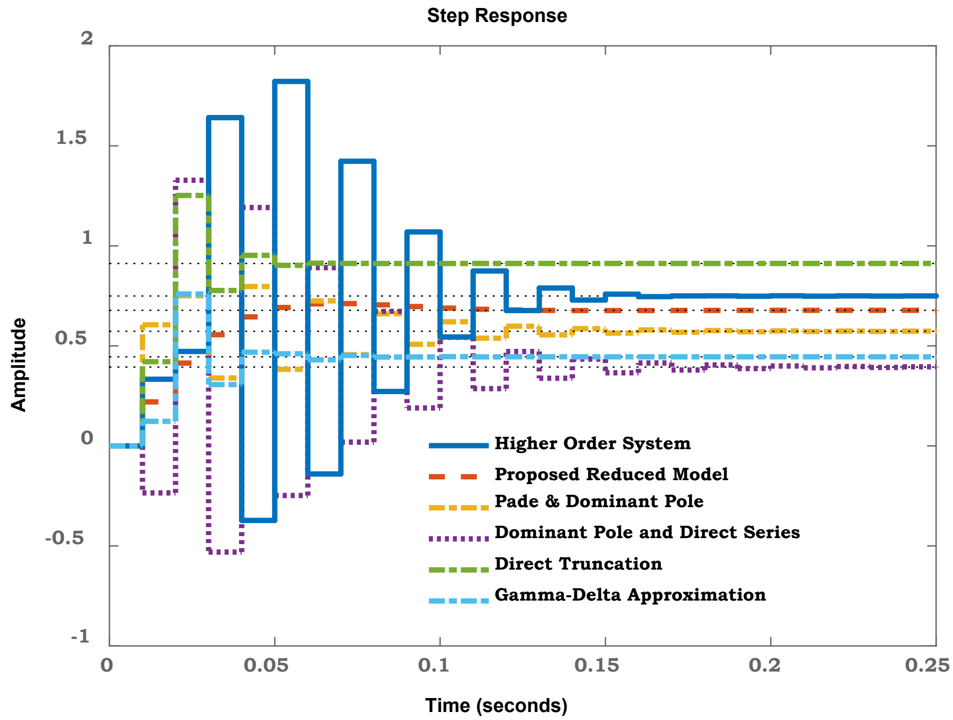

Figure 3.30: Step responses of reduced models (Upper Limit) for E.3.7.1

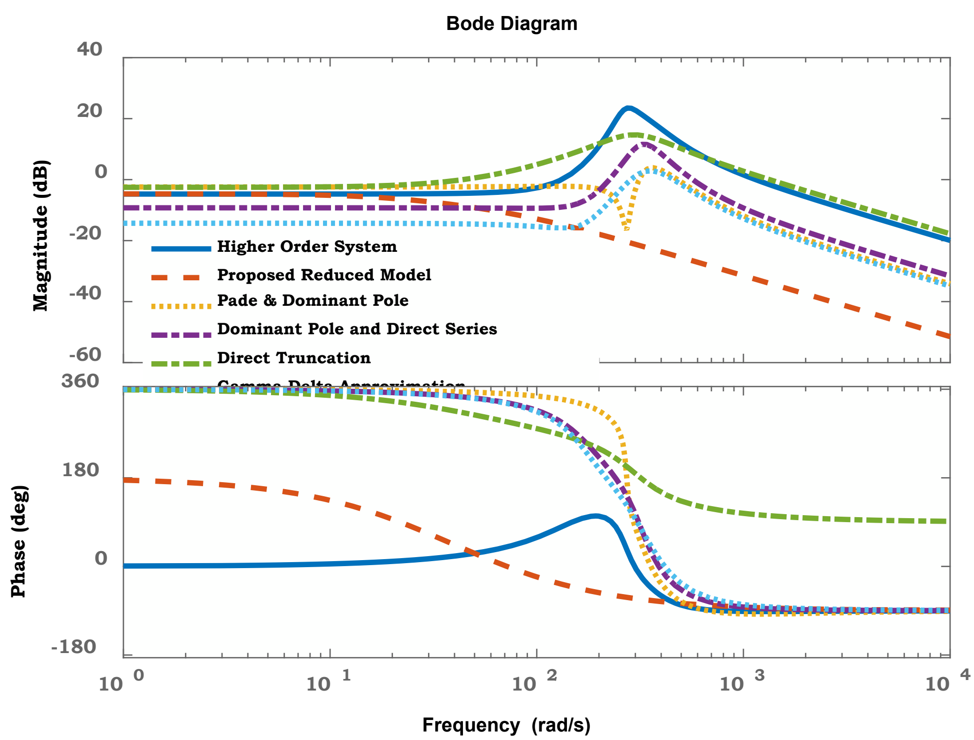

Figure 3.31: Frequency responses of reduced models (Lower Limit) for E.3.7.1

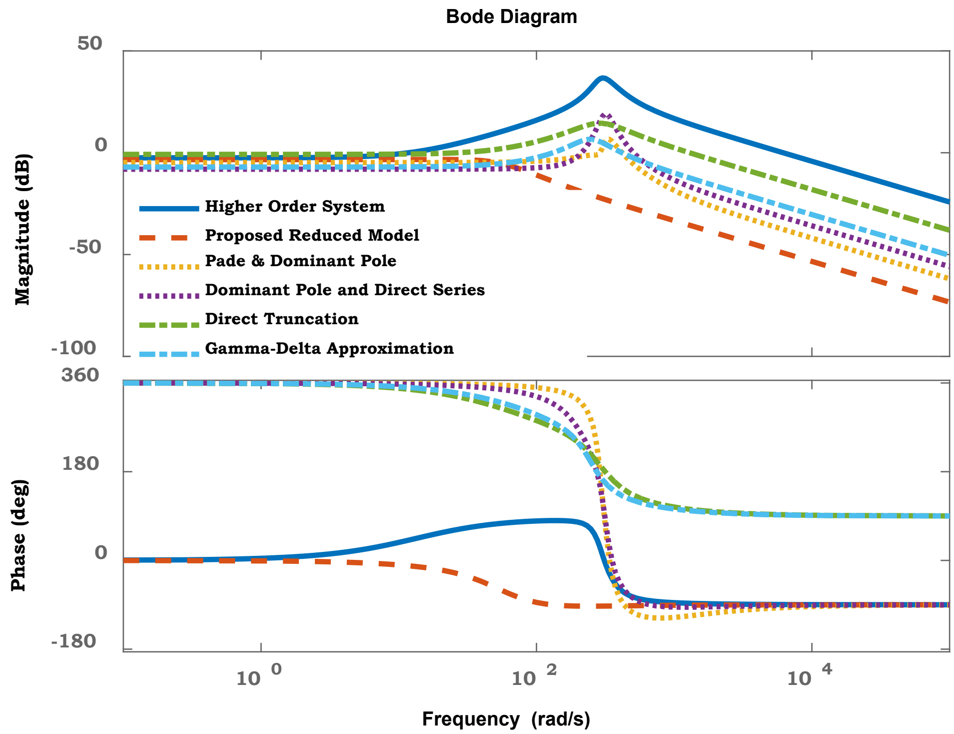

Figure 3.32: Frequency responses of reduced models (Upper Limit) for E.3.7.1

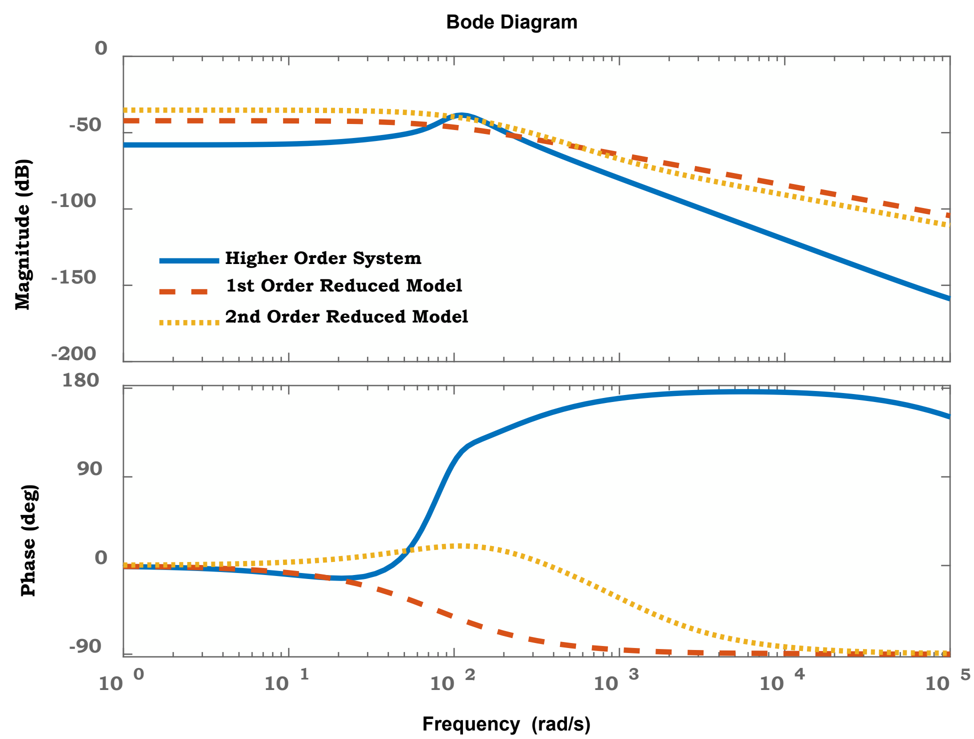

Figure 3.33: Frequency responses of reduced models (Lower Limit) for E.3.7.2

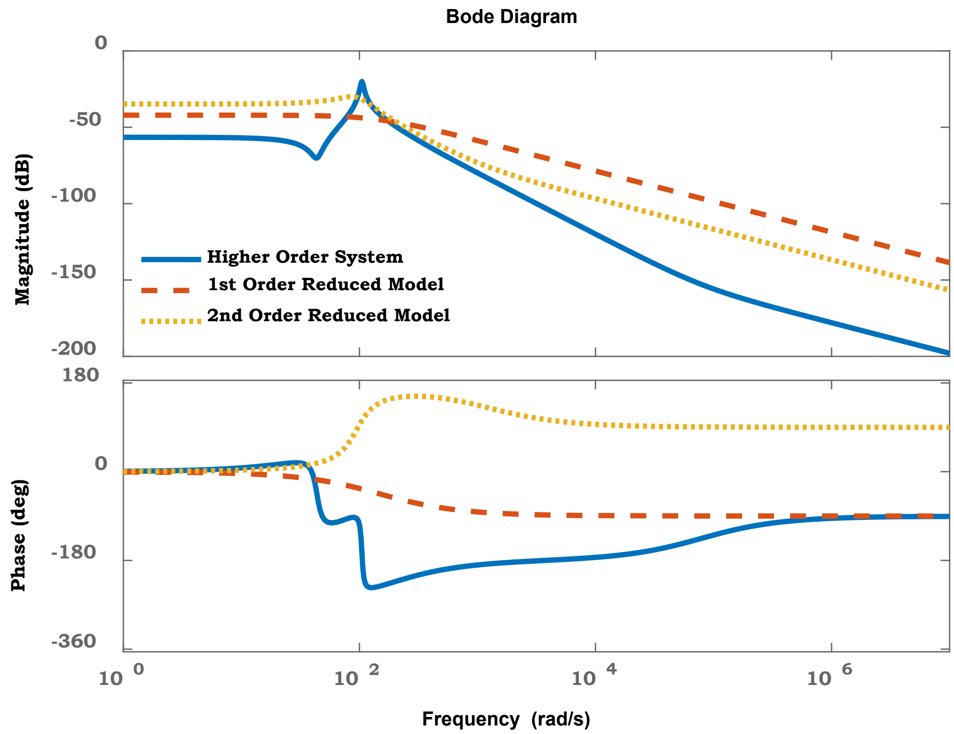

Figure 3.34: Frequency responses of reduced models (Upper Limit) for E.3.7.2

Conclusions

The proposed algorithm is an acceptable extension of model order reduction for discrete-time interval system based on *RA* yielding better and satisfactory results.

3.8. Routh Approximant

Adding to the proposed algorithms, here presented is yet another extension of a prevailing technique in continuous-time domain [32] to discrete-time domain. The existence and the prolongation, both belong to interval coefficient committee.

Methodology

Here considered is bilinear transformation for Routh algorithm implication that results in (2.15).

$A_n(w)$ from the obtained $H_n(w)$ (2.15) is employed to draft the entries of the first two rows of Routh array drafted in Table 3.40. σ_k where $k = 1, 2,\ldots$ in Table 3.40 is of interval nature $\left[\sigma_k^-, \sigma_k^+\right]$ as required for the computation of the entries down the table and the reduced models.

Table 3.40: Routh Table for $\left[\sigma_k^-,\sigma_k^+\right]$

$\sigma_1=\dfrac{a_0}{a_1}$	$\left[a_0^-,a_0^+\right]$	$\left[a_1^-,a_1^+\right]$	$\left[a_2^-,a_2^+\right]$	$\left[a_3^-,a_3^+\right]$
	$\left[a_1^-,a_1^+\right]$	$\left[a_2^-,a_2^+\right]$	$\left[a_3^-,a_3^+\right]$	………
$\sigma_2=\dfrac{c_1}{c_2}$	$\left[c_1^-,c_1^+\right]$	$\left[c_2^-,c_2^+\right]$	$\left[c_3^-,c_3^+\right]$	………
	$\left[c_2^-,c_2^+\right]$	$\left[c_3^-,c_3^+\right]$	………	………
$\sigma_3=\dfrac{d_1}{d_2}$	$\left[d_1^-,d_1^+\right]$	$\left[d_2^-,d_2^+\right]$	………	………
	$\left[d_2^-,d_2^+\right]$	………		

The entries down the third row is figured as

For i = *odd*

$$\left[c_1^-,c_1^+\right]=\left[a_i^-,a_i^+\right] \qquad i=1,\ 3,\ 5,\ \ldots. \tag{3.111a}$$

$$\left[d_1^-,d_1^+\right]=\left[c_2^-,c_2^+\right] \tag{3.111b}$$

$$\left[d_i^-,d_i^+\right]=\left[c_{i+1}^-,c_{i+1}^+\right] \qquad i=3,\ 5,\ \ldots. \tag{3.111c}$$

For i = *even*

$$\left[c_i^-,c_i^+\right]=\left[a_i^-,a_i^+\right]-\left\{\left[a_{i+1}^-,a_{i+1}^+\right]\left[a_0^-,a_0^+\right]\right\}\Big/\left[a_1^-,a_1^+\right] \tag{3.112a}$$

$$\left[d_i^-,d_i^+\right]=\left[c_{i+1}^-,c_{i+1}^+\right]-\left\{\left[c_1^-,c_1^+\right]\left[c_{i+2}^-,c_{i+2}^+\right]\right\}\Big/\left[c_2^-,c_2^+\right] \tag{3.112b}$$

Preferred order reduced models are confronted through the numerator and denominator polynomials in Table 3.41. The set combination is used to derive the reduced order model in w-domain which later is transformed back to z-domain by inverse transformation.

Example

E.3.8.1. Consider transfer function available from [66] as

$$H_3(z)=\frac{[3.25,3.35]z^2+[3.5,3.65]z+[2.8,3]}{[5.4,5.5]z^3+[1,1.1]z^2+[1.5,1.6]z+[2.1,2.15]} \tag{3.113}$$

Steps from the algorithm, transforms the transfer function in w-domain to (3.114)

$$H_3(w)=\frac{[-2.85,2.4]w^3+[1.3,2.35]w^2+[-9.05,-8.9]w+[9.55,10]}{[3.65,4]w^3+[19.8,20.45]w^2+[9.15,9.8]w+[10,10.35]} \tag{3.114}$$

Table 3.41: Reduced polynomials for various order

Order	Element	Reduced Polynomials
$k=1$	Den.	$A_1(w)=[1,1]w+\dfrac{[\sigma_1^-,\sigma_1^+]}{[a_0^-,a_0^+]}[a_0^-,a_0^+]$
	Num.	$B_1(w)=\dfrac{[\sigma_1^-,\sigma_1^+]}{[a_0^-,a_0^+]}[b_0^-,b_0^+]$
$k=2$	Den.	$A_2(w)=[1,1]w^2+\dfrac{[\sigma_1^-,\sigma_1^+][\sigma_2^-,\sigma_2^+]}{[a_0^-,a_0^+]}\{[a_0^-,a_0^+]+[a_1^-,a_1^+]w\}$
	Num.	$B_2(w)=\dfrac{[\sigma_1^-,\sigma_1^+][\sigma_2^-,\sigma_2^+]}{[a_0^-,a_0^+]}\{[b_0^-,b_0^+]+[b_1^-,b_1^+]w\}$
In general	Den.	$A_k(w)=[1,1]w^k+\dfrac{[\sigma_k^-,\sigma_k^+]!}{[a_0^-,a_0^+]}\left\{\begin{matrix}[a_0^-,a_0^+]+[a_1^-,a_1^+]w\\+\ldots+[a_{k-1}^-,a_{k-1}^+]w^{k-1}\end{matrix}\right\}$
	Num.	$B_k(w)=\dfrac{[\sigma_1^-,\sigma_1^+]!}{[a_0^-,a_0^+]}\left\{\begin{matrix}[b_0^-,b_0^+]+[b_1^-,b_1^+]w\\+\ldots+[b_{k-1}^-,b_{k-1}^+]w^{k-1}\end{matrix}\right\}$

Routh table for obtaining the $[\sigma_k^-,\sigma_k^+]$ coefficients required for computation of the reduced denominator is constructed as in Table 3.42.

Table 3.42: Denominator array for $[\sigma_k^-,\sigma_k^+]$

w^3	$[10,10.35]$	$[9.15,9.8]$	$[19.8,20.45]$	$[3.65,4]$
w^2	$[9.15,9.8]$	$[19.8,20.45]$	$[3.65,4]$	
w^1	$[9.15,9.8]$	$[15.28,16.73]$		
w^0	$[15.28,16.73]$			

The parameters calculated from the above table are $[\sigma_1^-,\sigma_1^+]=[1.02,1.13]$ and $[\sigma_2^-,\sigma_2^+]=[0.54,0.64]$. Using these parameters in the numerator and denominator polynomial of order $k = 2$ extracted from Table 3.41 result the reduced model in z-domain as

$$R_2(z)=\frac{[-0.14,0.24]z^2+[1.01,1.44]z+[0.98,1.37]}{[2.03,2.42]z^2+[-0.9,-0.56]z+[0.84,1.23]} \tag{3.115}$$

The error computed by the proposed algorithm is made known in Table 3.43 and is minimum making it acceptable for its proficiency when compared with the prevailing technique. Justification to the proposed technique is also supported by the tracking of step responses of the reduced and higher order systems shown in Figures 3.35 and 3.36 for lower and upper limit transfer functions respectively. Figures 3.37 and 3.38 demonstrate the frequency responses for the derived reduced models correspondingly.

Table 3.43: Error for 2nd order reduced models for E.3.8.1

Methods	Error	
	Lower Limit	Upper Limit
Proposed Algorithm	0.0463	0.0215
Multipoint Pade [66]	0.0721	0.0409

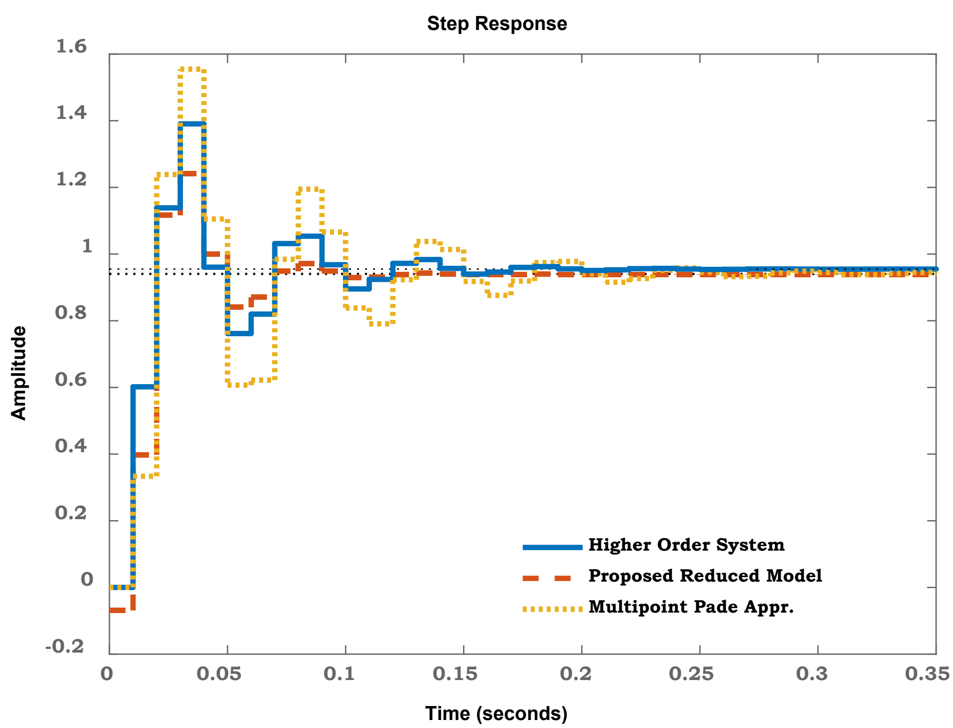

Figure 3.35: Step responses of reduced models (Lower Limit) for E.3.8.1

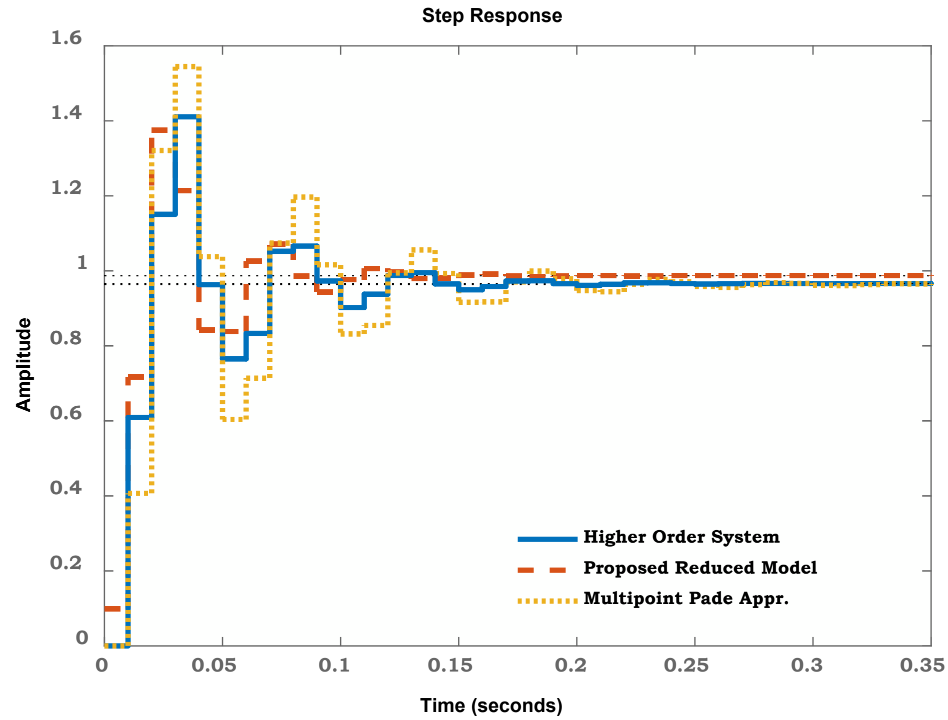

Figure 3.36: Step responses of reduced models (Upper Limit) for E.3.8.1

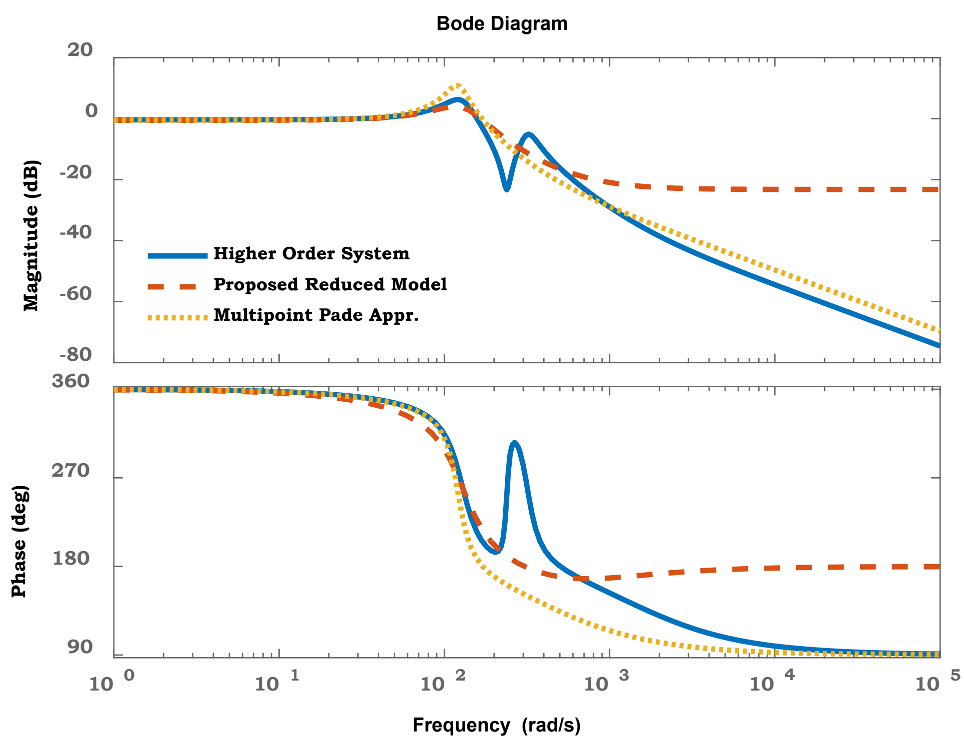

Figure 3.37: Frequency responses of reduced models (Lower Limit) for E.3.8.1

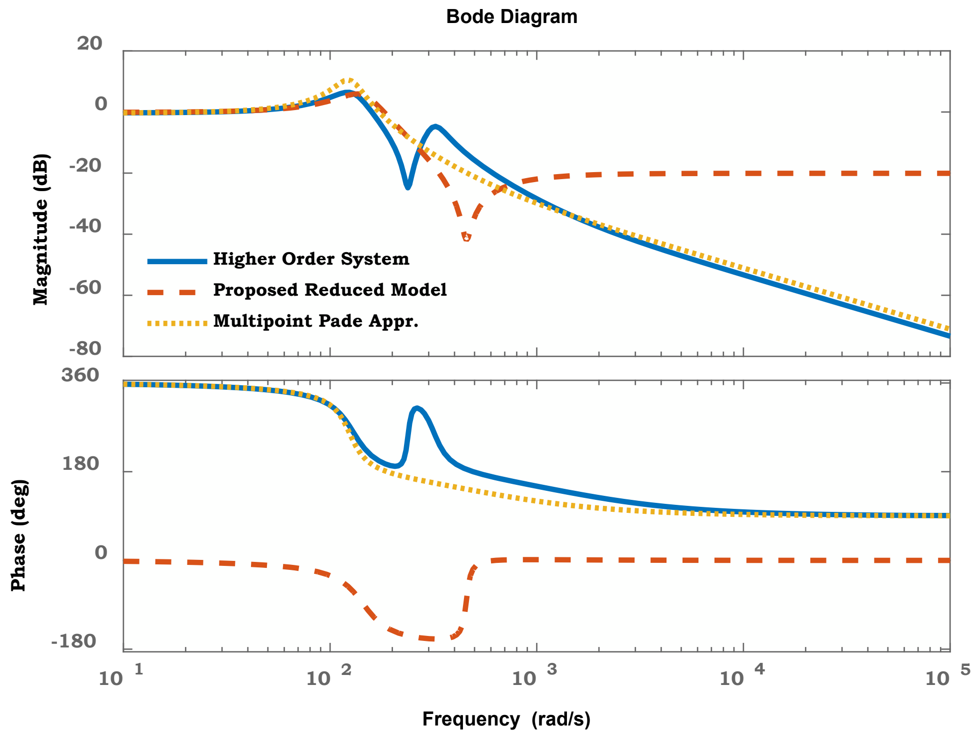

Figure 3.38: Frequency responses of reduced models (Upper Limit) for E.3.8.1

E.3.8.2. Consider another higher order transfer function from [68], [83], [90], [107] be

$$H_3(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{3.116}$$

The parameters calculated from the Table 3.40 are $\left[\sigma_1^-, \sigma_1^+\right] = [1.02, 1.09]$ and $\left[\sigma_2^-, \sigma_2^+\right] = [3.19, 4.40]$.

Reduced models with the polynomials extracted from Table 3.41 for *k=1, 2* generates

$$R_1(z) = \frac{[0.57,0.84]z + [0.57,0.84]}{[2.02,2.09]z + [0.02,0.09]} \tag{3.117}$$

$$R_2(z) = \frac{[-6.10,0.05]z^2 + [3.61,7.47]z + [5.51,11.67]}{[7.25,10.55]z^2 + [4.45,7.66]z + [-0.44,2.85]} \tag{3.118}$$

Table 3.44 confer the error computed for the first and second order models and is observed to be minimal demonstrating its support for the acceptance of the proposed algorithm.

Figures 3.39 and 3.40 represent the step responses and Figures 3.41 and 3.42 depict the frequency responses for the reduced models for lower and upper limits respectively.

Table 3.44: Error for 1st and 2nd order reduced models for E.3.8.2

Methods	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Algorithm	0.2356	0.0069	0.7080	0.1414
Pade & Dominant Poles [68]	0.1398	0.0195	0.1810	0.0741
Dominant Pole/Direct Series [83]	0.4839	0.4134	0.3237	0.3229
Gamma-Delta Appr. [90]	0.0157	0.0035	0.1292	0.0443
Direct Truncation [107]	2.1491	2.7780	0.0278	0.0077

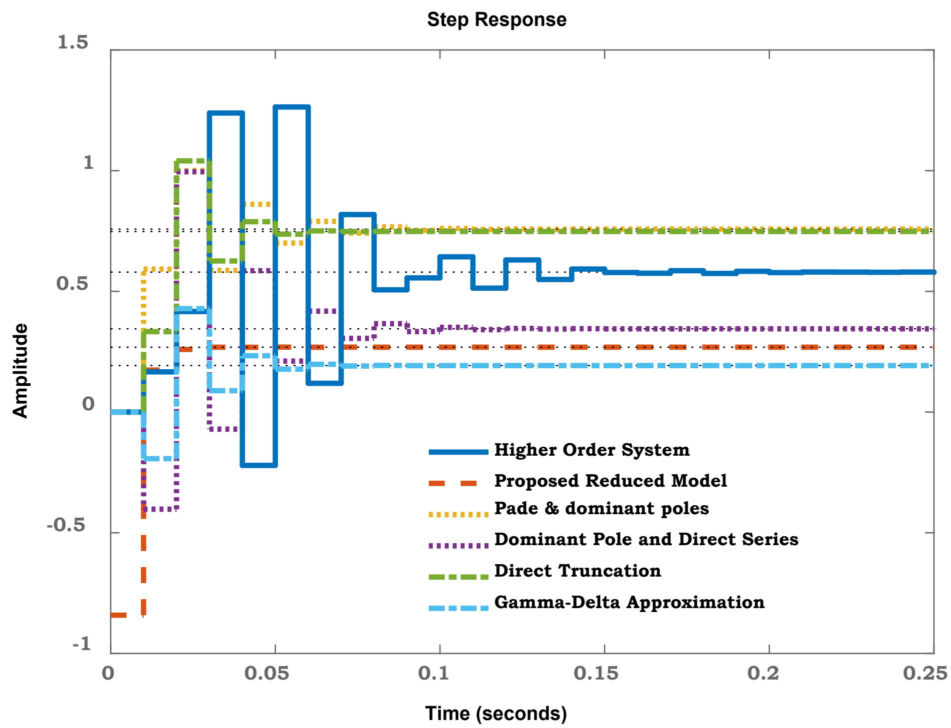

Figure 3.39: Step responses of reduced models (Lower Limit) for E.3.8.2

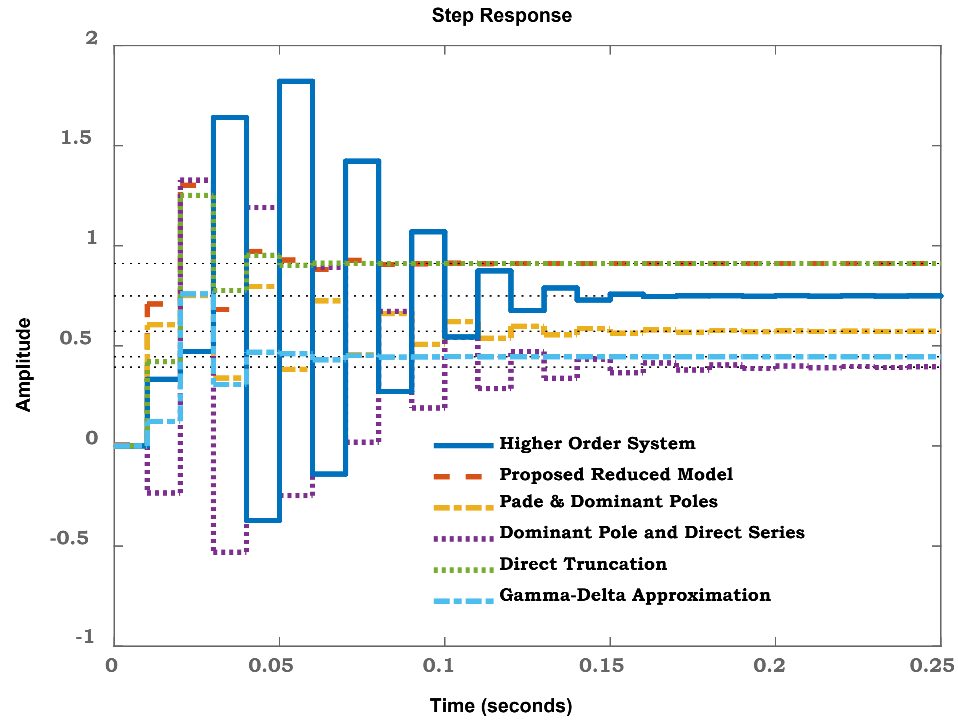

Figure 3.40: Step responses of reduced models (Upper Limit) for E.3.8.2

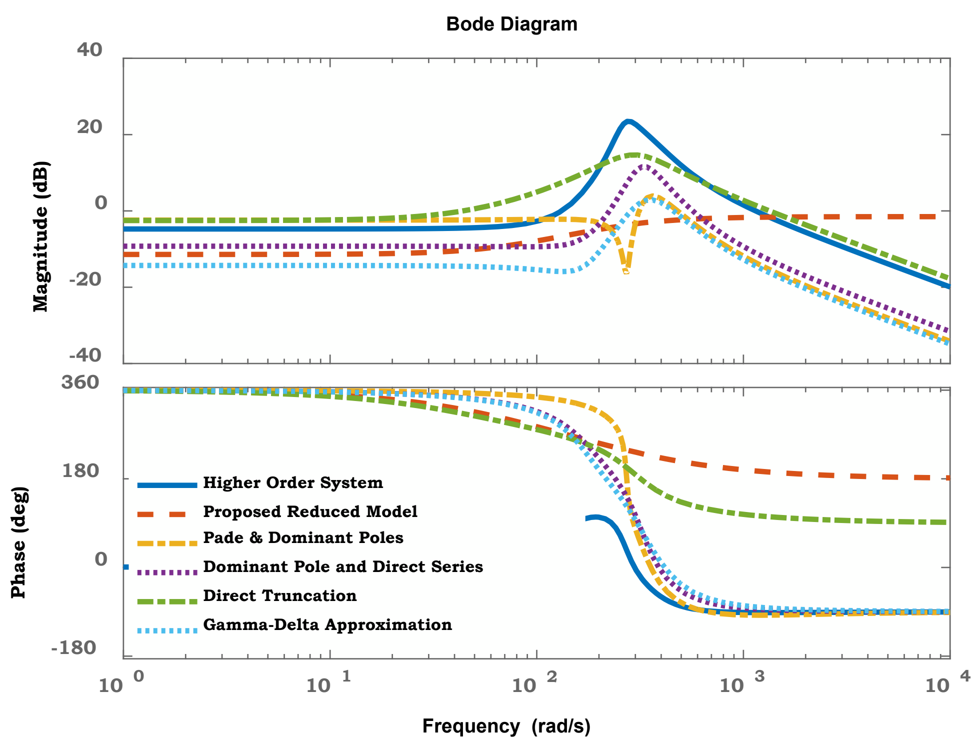

Figure 3.41: Frequency responses of reduced models (Lower Limit) for E.3.8.2

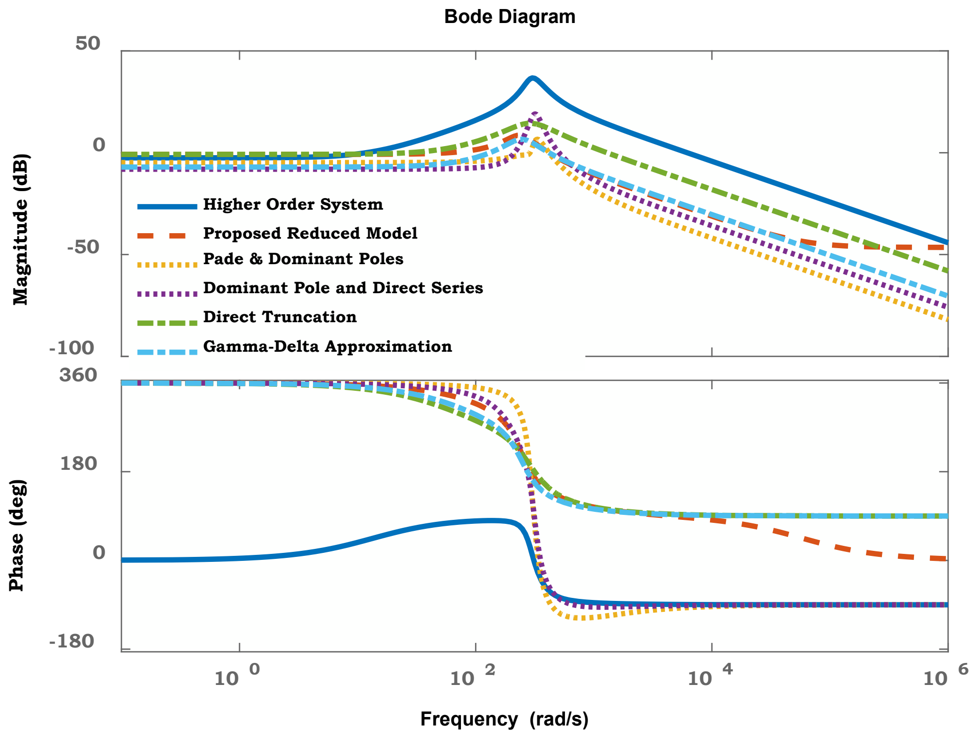

Figure 3.42: Frequency responses of reduced models (Upper Limit) for E.3.8.2

Conclusions

The attempt to present a promising propagation of continuous-time domain approximation technique to discrete-time interval systems is achieved here.

3.9. Routh-Pade Approximation

Algorithm discussed under this heading is a mingled form of the prevailing techniques that employ their individual algorithmic steps. Enlisted algorithms are *Pade Approximation* and *Routh Algorithm* for deriving the reduced numerator and denominator polynomials respectively. In this approach, the preservation of model stability is considered the prime focus. The algorithm is as follows;

Methodology

Euler forward method *i.e.* $z=1+p$ results (2.13) to (2.16) and its corresponding Routh array as drafted in Table 3.45, where

$$\left[a_{i,j}^{-}, a_{i,j}^{+}\right] = \left[a_{i-2,j+1}^{-}, a_{i-2,j+1}^{+}\right] - \left[\alpha_{i-2}^{-}, \alpha_{i-2}^{+}\right]\left[a_{i-1,j+1}^{-}, a_{i-1,j+1}^{+}\right] \quad (3.119)$$

with $i=3,4,\ldots\ldots,n.$ and $j=1,2,\ldots\ldots$

and $\left[\alpha_i^-,\alpha_i^+\right]=\dfrac{\left[\alpha_{i,1}^-,\alpha_{i,1}^+\right]}{\left[\alpha_{i+1,1}^-,\alpha_{i+1,1}^+\right]}$ where *i=1,2,.....n* (3.120)

Table 3.45: Routh array for denominator

$\left[a_0^-,a_0^+\right]=\left[a_{1,1}^-,a_{1,1}^+\right]$	$\left[a_2^-,a_2^+\right]=\left[a_{1,2}^-,a_{1,2}^+\right]$	$\left[a_4^-,a_4^+\right]=\left[a_{1,3}^-,a_{1,3}^+\right]$	…
$\left[a_1^-,a_1^+\right]=\left[a_{2,1}^-,a_{2,1}^+\right]$	$\left[a_3^-,a_3^+\right]=\left[a_{2,2}^-,a_{2,2}^+\right]$	$\left[a_5^-,a_5^+\right]=\left[a_{2,3}^-,a_{2,3}^+\right]$	…
$\left[a_{3,1}^-,a_{3,1}^+\right]$	$\left[a_{3,2}^-,a_{3,2}^+\right]$		
$\left[a_{4,1}^-,a_{4,1}^+\right]$	….		
•			
$\left[a_{n,1}^-,a_{n,1}^+\right]$			

Denominator polynomial for transfer function with order *k(<n)* is constructed with $(n-k)th$ and $(n+1-k)th$ rows of Table 3.45 as

$$A_k(p)=a_{(n-k),1}p^k+a_{(n+1-k),1}p^{k-1}+a_{(n-k),2}p^{k-2}+\dots \tag{3.121}$$

The numerator $B_k(p)$ is obtained by Pade approximation. Consider the required reduced model of order k be

$$\frac{B_k(p)}{A_k(p)}=\frac{\left[U_0^-,U_0^+\right]+\left[U_1^-,U_1^+\right]p+\dots+\left[U_{k-1}^-,U_{k-1}^+\right]p^{k-1}}{\left[V_0^-,V_0^+\right]+\left[V_1^-,V_1^+\right]p+\dots+\left[V_k^-,V_k^+\right]p^k} \tag{3.122}$$

Equate (3.122) and (2.16), as in (3.123), cross multiply and compare left and right hand side for coefficients of similar power. The comparison of the two sides offer the required coefficients.

$$\frac{B_k(p)}{A_k(p)}=\frac{B_n(p)}{A_n(p)} \tag{3.123a}$$

$$\left\{\left[U_0^-,U_0^+\right]+\dots+\left[U_{k-1}^-,U_{k-1}^+\right]p^{k-1}\right\}A_n(p)=B_n(p)\left\{\left[V_0^-,V_0^+\right]+\dots+\left[V_k^-,V_k^+\right]p^k\right\} \tag{3.123b}$$

Substitution of the obtained parameters in (3.122) gives the desired order reduced model which on appropriate inverse transformation gives $R_k(z)$.

Example

E.3.9.1. Consider transfer function available from [66], [86]

$$H_3(z)=\frac{[3.25,3.35]z^2+[3.5,3.65]z+[2.8,3]}{[5.4,5.5]z^3+[1,1.1]z^2+[1.5,1.6]z+[2.1,2.15]} \tag{3.124}$$

Steps from the algorithm transforms $H_3(z)$ as follows

$$H_3(p)=\frac{[3.25,3.35]p^2+[10,10.35]p+[9.55,10]}{[5.4,5.5]p^3+[17.2,17.6]p^2+[19.7,20.3]p+[10,10.35]} \tag{3.125}$$

Its Routh array for computing the denominator is shown in Table 3.46 that results $A_2(p)$ as

$$A_2(p)=[17.2,17.6]p^2+[16.4,17.24]p+[10,10.35] \tag{3.126}$$

For numerator the coefficients of second order reduced model are $[U_0^-,U_0^+]=[9.55,10]$ and $[U_1^-,U_1^+]=[5.18,9.13]$ which results, $R_2(p)$ and its equivalent z-domain model $R_2(z)$ as

$$R_2(p)=\frac{[5.18,9.13]p+[9.55,10]}{[17.2,17.6]p^2+[16.4,17.24]p+[10,10.35]} \tag{3.127}$$

$$R_2(z)=\frac{[5.18,9.13]z+[0.42,4.82]}{[17.2,17.6]z^2+[-18.8,-17.16]z+[9.96,11.55]} \tag{3.128}$$

Table 3.46: Denominator array for E.3.9.1

p^3	[5.4,5.5]	[19.7,20.3]
p^2	[17.2,17.6]	[10,10.35]
p^1	[16.4,17.24]	
p^0	[10,10.35]	

Table 3.47, shows the error of the reduced model obtained by the proposed algorithm and by the prevailing techniques. Figures 3.43 and 3.44 shows the step response of the reduced models for lower and upper limits respectively. Figures 3.45 and 3.46 present the frequency responses of the reduced models correspondingly for lower and upper limits.

Table 3.47: Error for 2nd order reduced models for E.3.9.1

Methods	Error	
	Lower Limit	Upper Limit
Proposed Algorithm	0.0904	0.0082
Multipoint Pade [66]	0.0721	0.0409
Least Squares Methods [86]	$1.6246\text{x}10^{-06}$	$7.7560\ \text{x}10^{-04}$

E.3.9.2. Let transfer function from [68], [90] and its *p*-domain equivalent as

$$H_3(z)=\frac{[1,2]z^2+[3,4]z+[8,10]}{[6,6]z^3+[9,9.5]z^2+[4.9,5]z+[0.8,0.85]} \tag{3.129}$$

$$H_3(p)=\frac{[1,2]p^2+[5,8]p+[12,16]}{[6,6]p^3+[27,27.5]p^2+[40.9,42]p+[20.7,21.35]} \tag{3.130}$$

Table 3.48: Denominator array for E.3.9.2

p^3	$[6,6]$	$[40.9,42]$
p^2	$[27,27.5]$	$[20.7,21.35]$
p^1	$[36.16,37.49]$	
p^0	$[20.7,21.35]$	

By proposed algorithm, the reduced order model obtained with coefficients $\left[U_0^-,U_0^+\right]=[11.63,16.50]$ and $\left[U_1^-,U_1^+\right]=[-7.51,14.25]$is as

$$R_2(z)=\frac{[-7.51,14.25]z+[10.63,15.5]}{[27,27.5]z^2+[-18.84,-16.51]z+[10.21,12.69]} \tag{3.131}$$

Error for validation of the algorithm with the existing techniques is shown in Table 3.49. Figures 3.47 and 3.48 present the step response and Figures 3.49 and 3.50 depict the frequency responses of the reduced models for lower and upper limits transfer functions respectively.

Table 3.49: Error for 2nd order reduced models for E.3.9.2

Methods	Error	
	Lower Limit	Upper Limit
Proposed Algorithm	0.1079	0.0342
Pade and Dominant Pole [68]	0.1810	0.0741
Gamma-Delta Appr. [90]	0.1292	0.0443

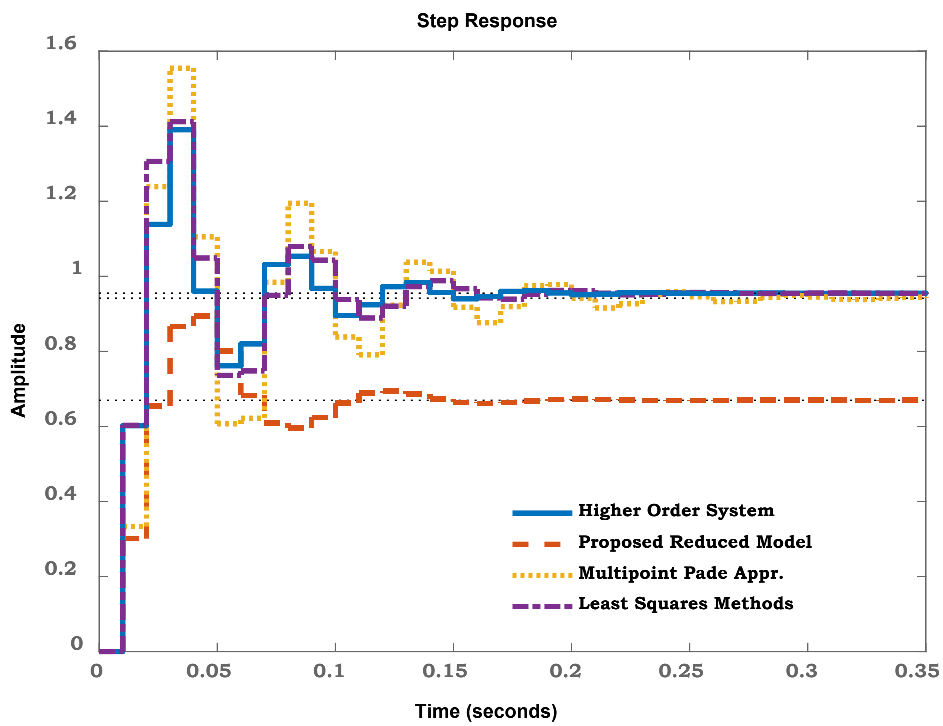

Figure 3.43: Step responses of reduced models (Lower Limit) for E.3.9.1

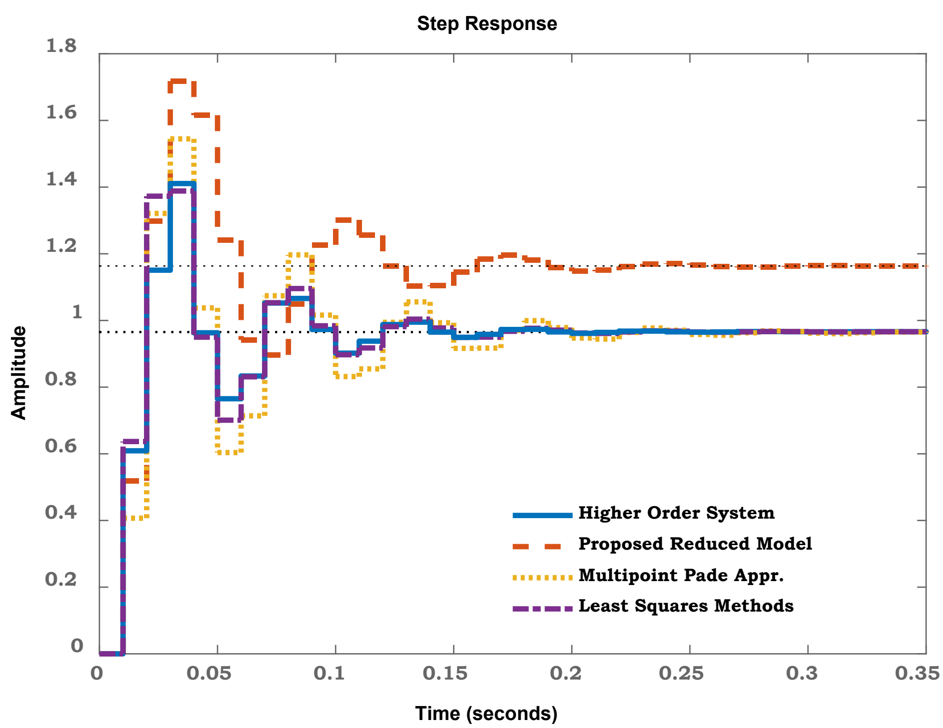

Figure 3.44: Step responses of reduced models (Upper Limit) for E.3.9.1

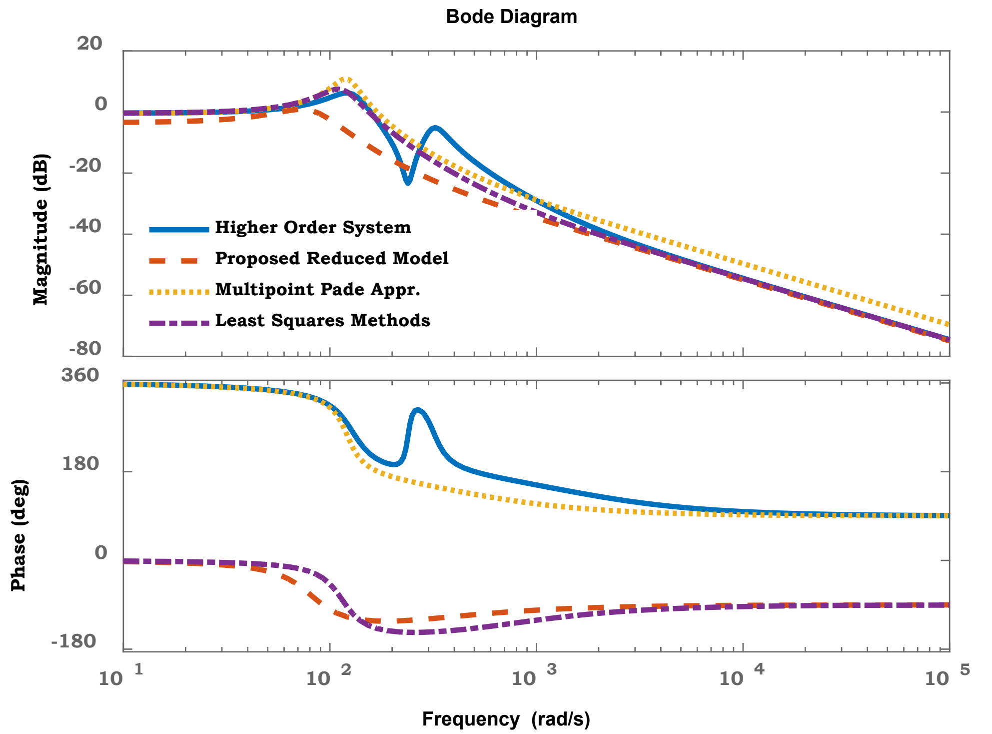

Figure 3.45: Frequency responses of reduced models (Lower Limit) for E.3.9.1

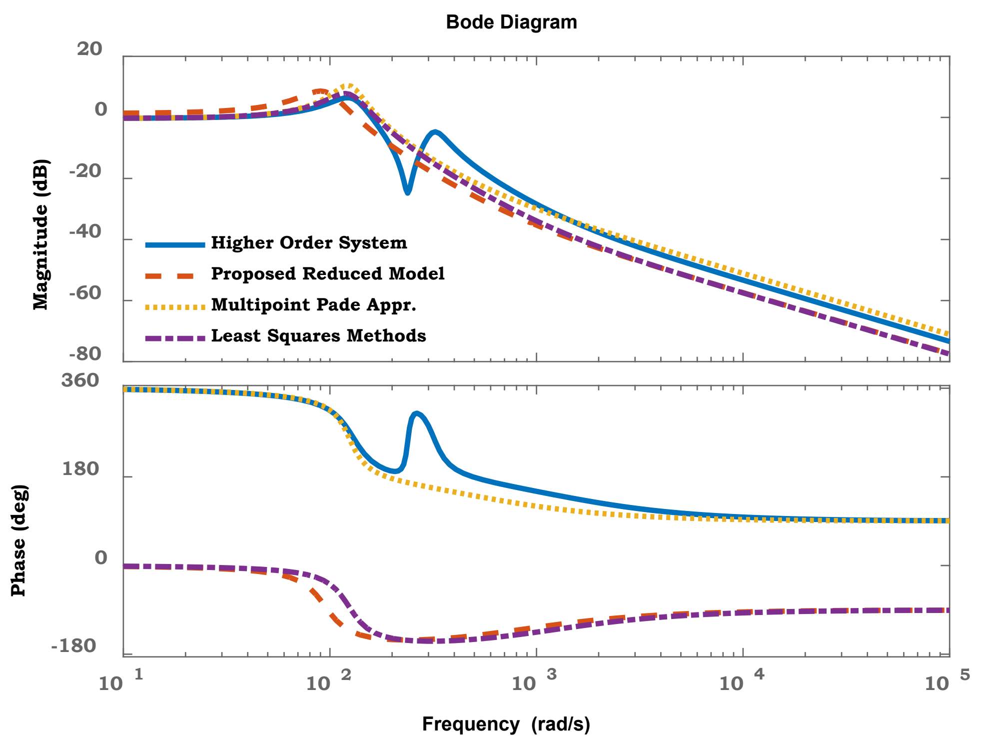

Figure 3.46: Frequency responses of reduced models (Upper Limit) for E.3.9.1

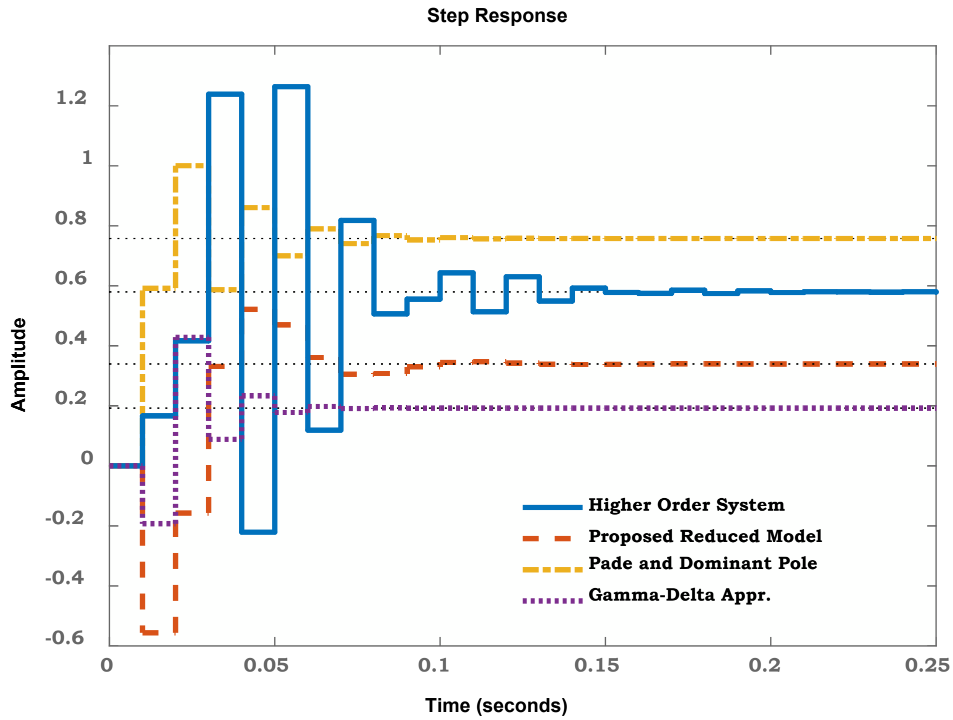

Figure 3.47: Step responses of reduced models (Lower Limit) for E.3.9.2

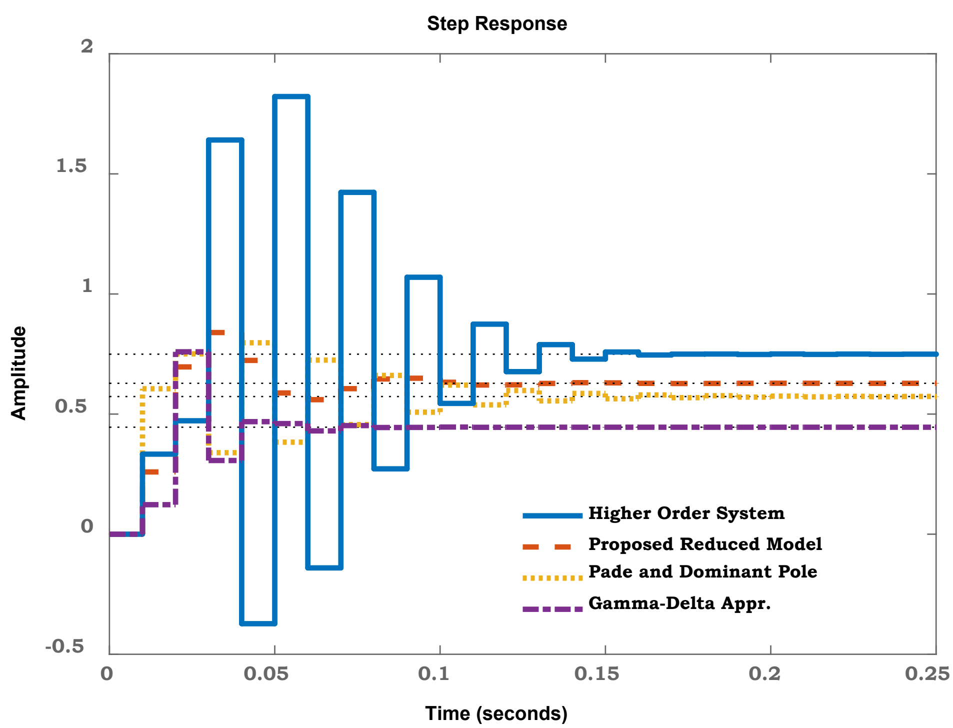

Figure 3.48: Step responses of reduced models (Upper Limit) for E.3.9.2

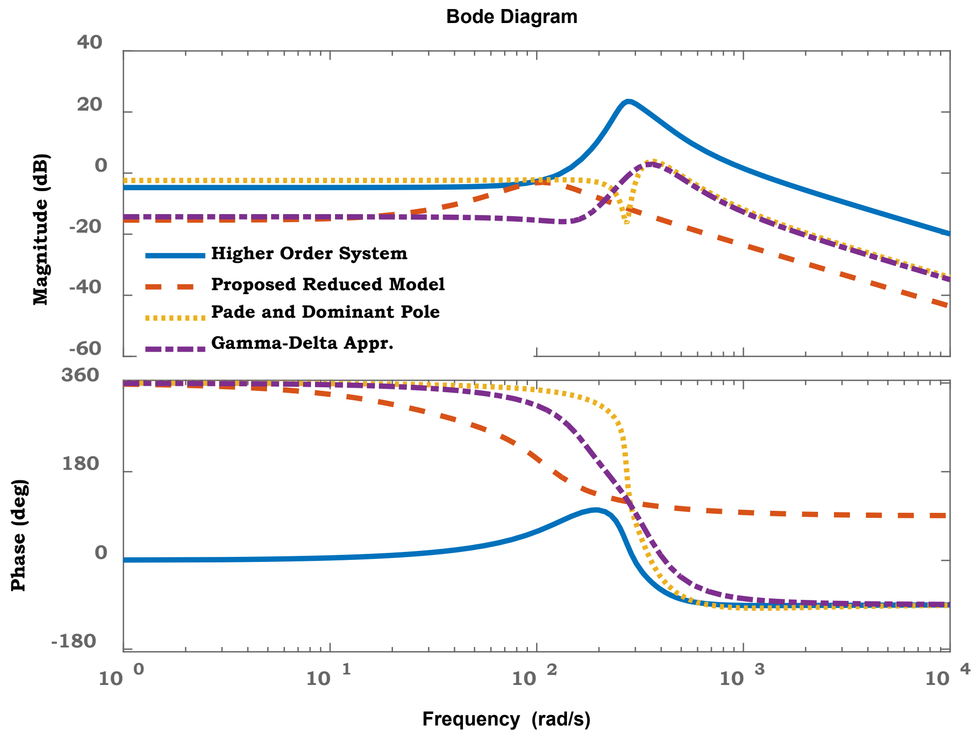

Figure 3.49: Frequency responses of reduced models (Lower Limit) for E.3.9.2

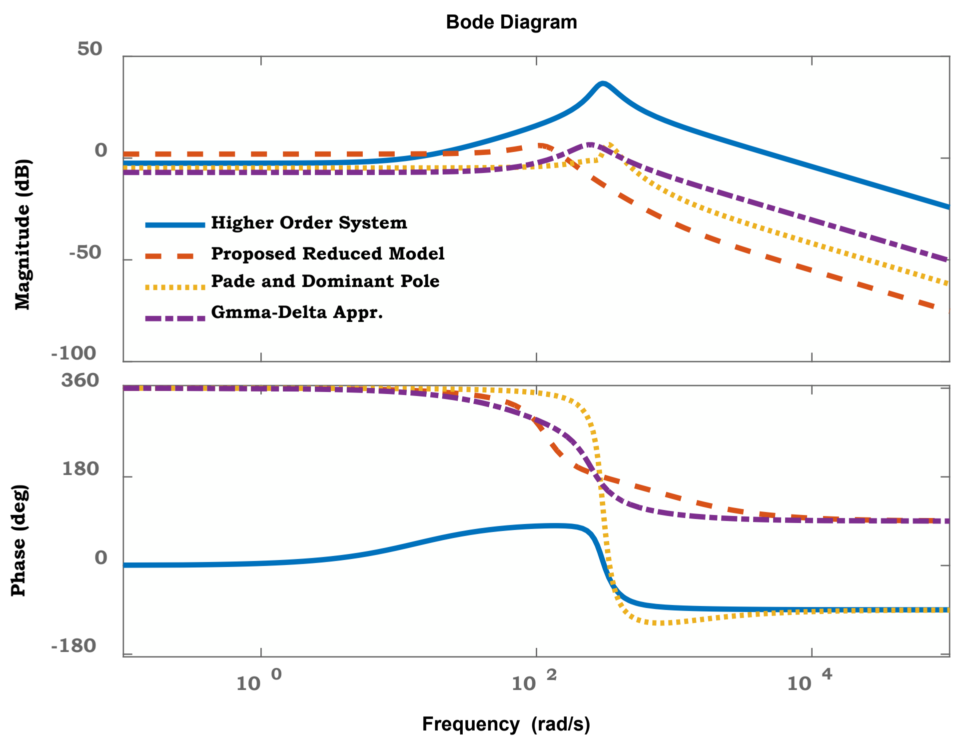

Figure 3.50: Frequency responses of reduced models (Upper Limit) for E.3.9.2

Conclusions

An existing method for order reduction of continuous-time interval systems is extended to discrete-time interval systems. Mixed approach where denominator is obtained by direct truncation of Routh table and numerator by Pade approximation is exemplified here.

3.10. Amalgamated Approximation

Under this heading, three techniques are revisited which when interlaced among themselves result two varied approximation techniques. The prevailing techniques utilized here are *Routh Approximation*, *Direct Truncation* and *Pade Approximation*. The first approximation is used to derive the reduced denominator and the latter two are used for numerator derivation.

Methodology

Bilinear transformation considered here, results (2.13) as (2.15). Consider the reciprocal form of the above denominator polynomial $A_n(w)$ to obtain the reduced denominator polynomial represented as $\hat{A}_n(w)$;

$$\hat{A}_n(w) = \frac{1}{w} A_n\left(\frac{1}{w}\right) = \left[a_0^-, a_0^+\right] w^n + \left[a_1^-, a_1^+\right] w^{n-1} + \ldots + \left[a_n^-, a_n^+\right] \tag{3.132}$$

Use $\hat{A}_n(w)$ to draft the first two rows of the Routh array shown in Table 3.50.

Table 3.50: Routh array for denominator

$\left[a_0^-, a_0^+\right] = \left[a_{1,1}^-, a_{1,1}^+\right]$	$\left[a_2^-, a_2^+\right] = \left[a_{1,2}^-, a_{1,2}^+\right]$	$\left[a_4^-, a_4^+\right] = \left[a_{1,3}^-, a_{1,3}^+\right]$	..
$\left[a_1^-, a_1^+\right] = \left[a_{2,1}^-, a_{2,1}^+\right]$	$\left[a_3^-, a_3^+\right] = \left[a_{2,2}^-, a_{2,2}^+\right]$	$\left[a_5^-, a_5^+\right] = \left[a_{2,3}^-, a_{2,3}^+\right]$	..
$\left[a_{3,1}^-, a_{3,1}^+\right]$	$\left[a_{3,2}^-, a_{3,2}^+\right]$		
....			
$\left[a_{n,1}^-, a_{n,1}^+\right]$			

Entries down the third row in the table is computed by

$$\left[a_{i,j}^-, a_{i,j}^+\right] = \left[a_{i-2,j+1}^-, a_{i-2,j+1}^+\right] - \left[\alpha_{i-2}^-, \alpha_{i-2}^+\right]\left[a_{i-1,j+1}^-, a_{i-1,j+1}^+\right] \tag{3.133}$$

where $i=3,4,\ldots\ldots,n$ and $j=1,2,\ldots\ldots$

with $\left[\alpha_i^-,\alpha_i^+\right]=\dfrac{\left[\alpha_{i,1}^-,\alpha_{i,1}^+\right]}{\left[\alpha_{i+1,1}^-,\alpha_{i+1,1}^+\right]}$ *i=1,2,...k,.....n* (3.134)

provided $\left[\alpha_{i+1,1}^-,\alpha_{i+1,1}^+\right]\notin[0]$

The reduced denominator, $\hat{A}_k(w)$ is obtained according to (3.135) as stated for non-interval system [24]

$$\hat{A}_k(w)=\left[\alpha_k^-,\alpha_k^+\right]w\hat{A}_{k-1}(w)+\hat{A}_{k-2}(w) \tag{3.135}$$

with $\hat{A}_{-1}(w)=1$, $\hat{A}_0(w)=1$

For instance, if *k=1, 2* then denominator polynomial is

$$\hat{A}_1(w)=\left[\alpha_1^-,\alpha_1^+\right]w+[1,1] \tag{3.136}$$

and

$$\hat{A}_2(w)=\left[\alpha_1^-,\alpha_1^+\right]\left[\alpha_2^-,\alpha_2^+\right]w^2+\left[\alpha_2^-,\alpha_2^+\right]w+[1,1] \tag{3.137}$$

The resulting $\hat{A}_k(w)$ is reciprocated back to $A_k(w)$ which on inverse transformation give the required $A_k(z)$.

The numerator $B_k(w)$ is computed by implicating two algorithms discussed below;

***Algorithm 1:* Direct Truncation**

Direct Truncation [107] is hired for obtaining the reduced numerator polynomial declared as

$$N_k(z)=\left[n_{k-1}^-,n_{k-1}^+\right]z^{k-1}+\left[n_{k-2}^-,n_{k-2}^+\right]z^{k-2}+\ldots+\left[n_0^-,n_0^+\right] \tag{3.138}$$

***Algorithm 2:* Pade Approximation**

Another prevailing technique; Pade approximation used for obtaining the numerator polynomial is illustrated here. Once the denominator $A_k(w)$ exist, numerator $B_k(w)$ is obtained by matching first t time moments and l Markov parameters, such that $t+l=k$.

Assume the reduced model of order k be

$$\frac{B_k(w)}{A_k(w)}=\frac{\left[U_0^-,U_0^+\right]+\left[U_1^-,U_1^+\right]w+\ldots+\left[U_{k-1}^-,U_{k-1}^+\right]w^{k-1}}{\left[V_0^-,V_0^+\right]+\left[V_1^-,V_1^+\right]w+\ldots+\left[V_k^-,V_k^+\right]w^k} \tag{3.139}$$

Equate (3.139) and (2.15), cross multiply and compare left & right hand side for similar coefficients and compute the desired coefficients.

$$\frac{B_k(w)}{A_k(w)} = \frac{B_n(w)}{A_n(w)} \tag{3.140a}$$

$$\left\{\left[U_0^-, U_0^+\right] + \ldots + \left[U_{k-1}^-, U_{k-1}^+\right] w^{k-1}\right\} A_n(w) = B_n(w) \left\{\left[V_0^-, V_0^+\right] + \ldots + \left[V_k^-, V_k^+\right] w^k\right\}$$

(3.140b)

Place the obtained coefficient in (3.139) and apply inverse transformation to obtain $R_k(z)$.

Example

E.3.10.1. Consider the higher order system available from [68], [83], [90], [107], [108] be

$$H_3(z) = \frac{[1,2]z^2 + [3,4]z + [8,10]}{[6,6]z^3 + [9,9.5]z^2 + [4.9,5]z + [0.8,0.85]} \tag{3.141}$$

By the proposed algorithm, its w-domain representation is

$$H_3(w) = \frac{[-9,-5]w^3 + [17,27]w^2 + [-34,-24]w + [12,16]}{[0.55,1.2]w^3 + [5.9,6.65]w^2 + [19.45,20.2]w + [20.7,21.35]} \tag{3.142}$$

The denominator polynomial for drafting the Routh array is (3.143)

$$\hat{A}_3(w) = \frac{1}{w} A_3\left(\frac{1}{w}\right) = [20.7,21.35]w^3 + [19.45,20.2]w^2 + [5.9,6.65]w + [0.55,1.2]$$

(3.143)

From $\hat{A}_3(w)$, the Routh array is outlined in Table 3.51;

Table 3.51: Denominator array for E.3.10.1

w^3	$[20.70,21.35]$	$[5.90,6.65]$
w^2	$[19.45,20.20]$	$[0.55,1.20]$
w^1	$[4.58,6.08]$	
w^0	$[0.55,1.20]$	

Required parameters procured from the above table are

$$\left[\alpha_1^-, \alpha_1^+\right] = [1.02,1.09], \ \left[\alpha_2^-, \alpha^+\right] = [3.19,4.40]$$

The second order reduced denominator polynomial by (3.137) result in

$$\hat{A}_2(w) = [3.27,4.83]w^2 + [3.19,4.40]w + [1,1] \tag{3.144}$$

On appropriate reciprocal and inverse transformation gives the reduced denominator as

$$A_2(z) = [7.46, 10.24]z^2 + [4.54, 7.66]z + [-0.13, 2.63] \tag{3.145}$$

Numerators by the two varied algorithms resulting to the overall reduced model are;

Algorithm 1

Direct truncation, result the reduced model as

$$R_2(z) = \frac{[3,4]z + [8,10]}{[7.46, 10.23]z^2 + [4.54, 7.66]z + [-0.12, 2.63]} \tag{3.146}$$

Algorithm 2

Pade approximation through (3.140) provide $\left[U_0^-, U_0^+\right] = [1.83, 3.73]$ and $\left[U_1^-, U_1^+\right] = [-9.73, -2.05]$ which result, $B_2(w)$ as

$$B_2(w) = [-9.7351, -2.0504]w + [1.8389, 3.7368] \tag{3.147}$$

The overall reduced model after inverse transformation as

$$R_2(z) = \frac{[-7.89, 1.68]z^2 + [3.67, 7.47]z + [3.88, 13.47]}{[7.46, 10.23]z^2 + [4.54, 7.66]z + [-0.12, 2.63]} \tag{3.148}$$

Table 3.52 displays the results obtained by the proposed algorithms and the existing ones. Figures 3.51 and 3.52 demonstrate the step response and Figures 3.53 and 3.54 give the frequency responses of reduced models for lower and upper limits respectively.

Table 3.52: Error for 1st and 2nd order reduced models for E.3.10.1

Methods	Error	
	Lower Limit	Upper Limit
Proposed Algorithm 1	*0.0553*	*0.0033*
Proposed Algorithm 2	*1.1265*	*0.2183*
Pade and Dominant Pole [68]	0.1810	0.0741
Dominant Pole and Direct Series [83]	0.3237	0.3229
Gamma-Delta Approximation [90]	0.0278	0.0077
Direct-Truncation [107]	0.1292	0.0443
Routh-Pade Approximation [108]	0.1079	0.0342

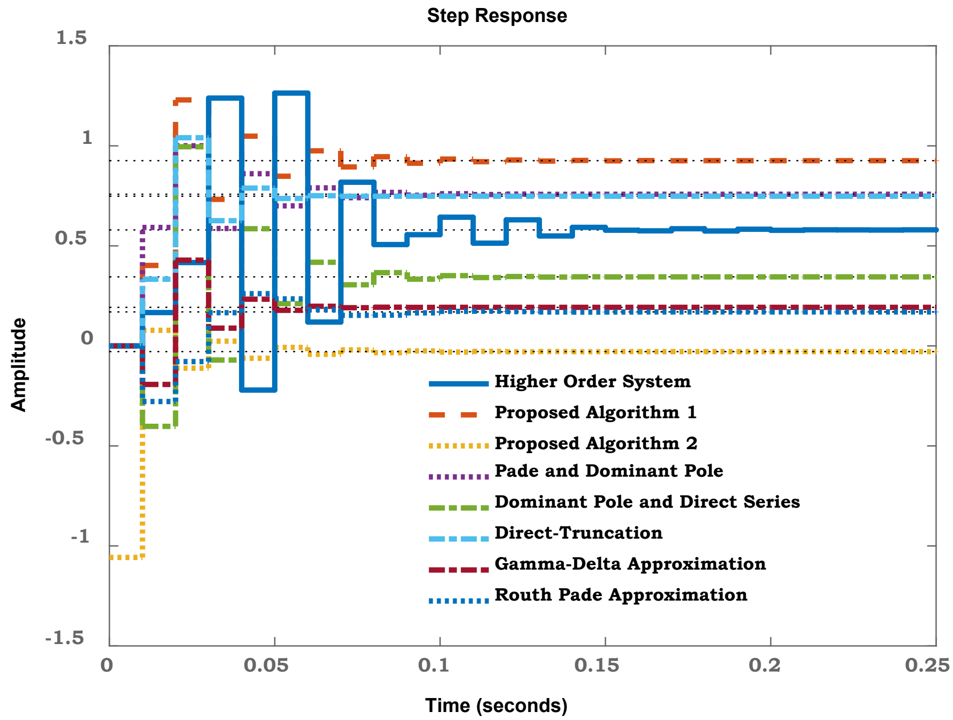

Figure 3.51: Step responses of reduced models (Lower Limit) for E.3.10.1

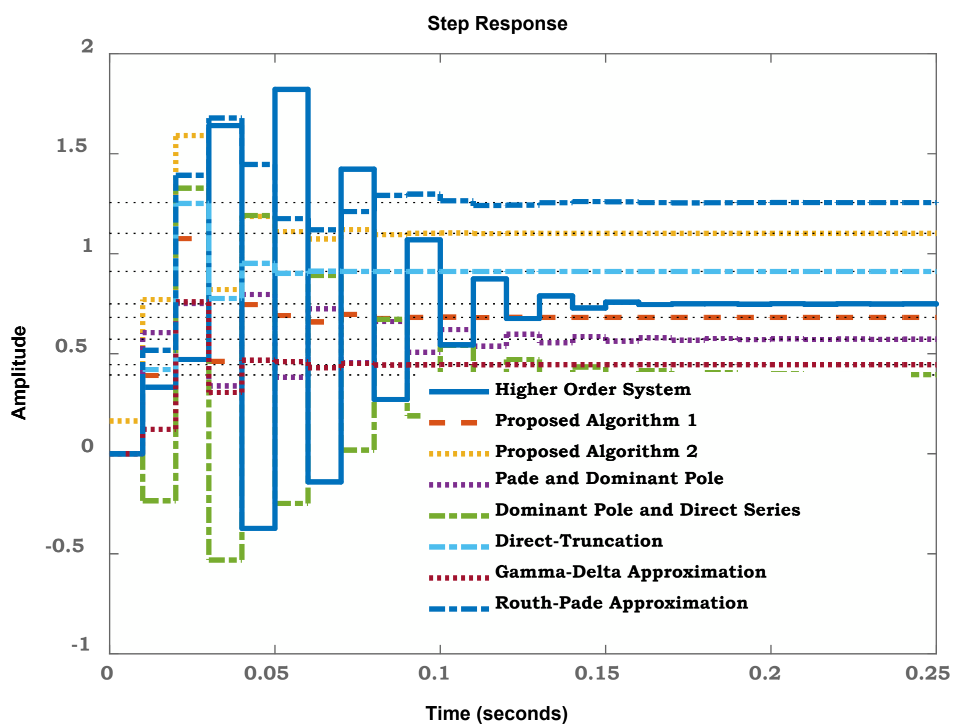

Figure 3.52: Step responses of reduced models (Upper Limit) for E.3.10.1

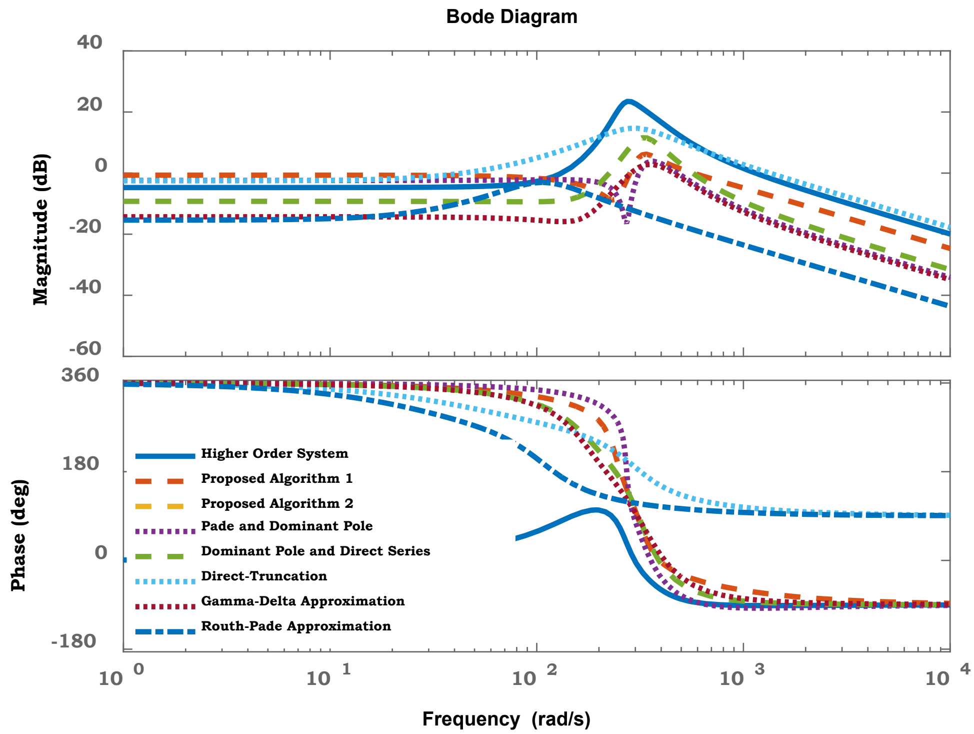

Figure 3.53: Frequency responses of reduced models (Lower Limit) for E.3.10.1

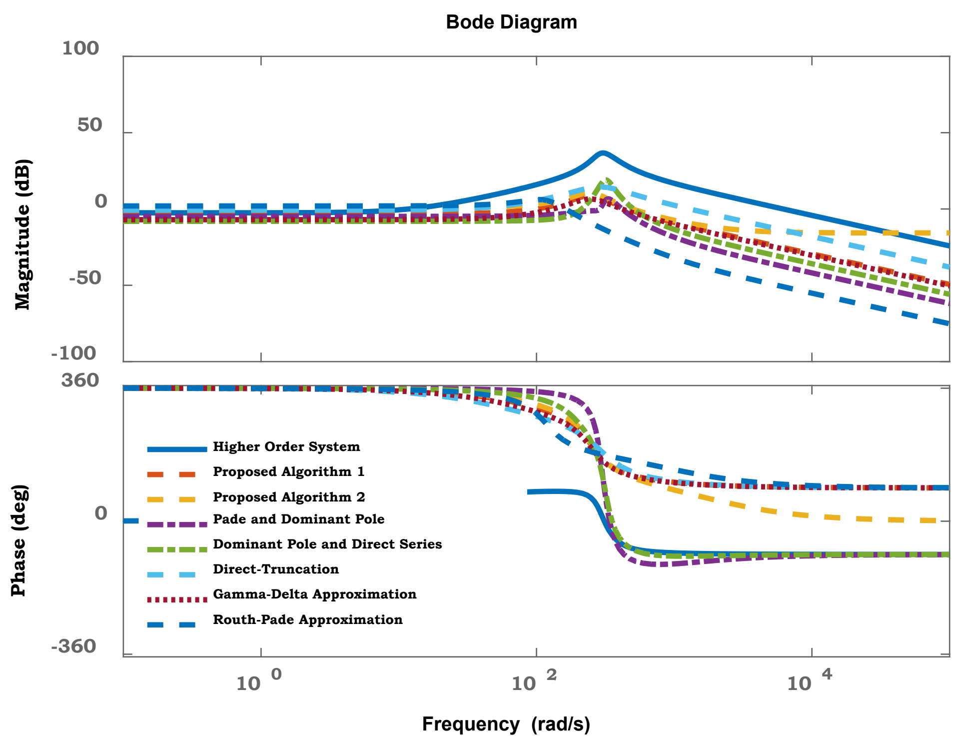

Figure 3.54: Frequency responses of reduced models (Upper Limit) for E.3.10.1

E.3.10.2. Consider a real-time digital control system

$$H_8(z)=\frac{\begin{array}{l}[1.6484,1.7156]z^7+[1.0937,1.1383]z^6+[-0.2142,-0.2058]z^5\\+[0.1490,0.1550]z^4+[-0.5263,-0.5057]z^3+[-0.2672,-0.2568]z^2\\+[0.0431,0.0449]z+[-0.0061,-0.0059]\end{array}}{\begin{array}{l}[23.52,24.48]z^8+[-1.7156,-1.6484]z^7+[-1.1383,-1.0937]z^6\\+[0.2058,0.2142]z^5+[-0.1550,-0.1490]z^4+[0.5057,0.5263]z^3\\+[0.2568,0.3672]z^2+[-0.0449,-0.0431]z+[0.0059,0.0061]\end{array}} \tag{3.149}$$

By the algorithms, the reduced models are obtained as

Algorithm 1

$$R_1(z)=\frac{[-0.006,-0.005]}{[1.11,1.13]z+[-0.88,-0.86]} \tag{3.150}$$

$$R_2(z)=\frac{[0.04,0.04]z+[-0.006,-0.005]}{[1.38,1.45]z^2+[-1.91,-1.89]z+[0.64,0.71]} \tag{3.151}$$

Algorithm 2

$$R_1(z)=\frac{[0.01,0.01]z+[0.01,0.01]}{[1.11,1.13]z+[-0.88,-0.86]} \tag{3.152}$$

$$R_2(z)=\frac{[0.02,0.05]z^2+[0.007,0.01]z+[-0.04,-0.01]}{[1.38,1.45]z^2+[-1.91,-1.89]z+[0.64,0.71]} \tag{3.153}$$

Table 3.53, present the error computed for the obtained reduced models. Figures 3.55 and 3.56 present the step responses and Figures 3.57 and 3.58 depict the frequency responses of reduced models for lower and upper limits respectively.

Table 3.53: Error for 1st and 2nd order reduced models for E.3.10.2

Methods	Error			
	1st Order		2nd Order	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
Proposed Algorithm 1	0.0057	0.0056	0.0017	0.0018
Proposed Algorithm 2	0.0037	0.0022	0.0011	0.0015

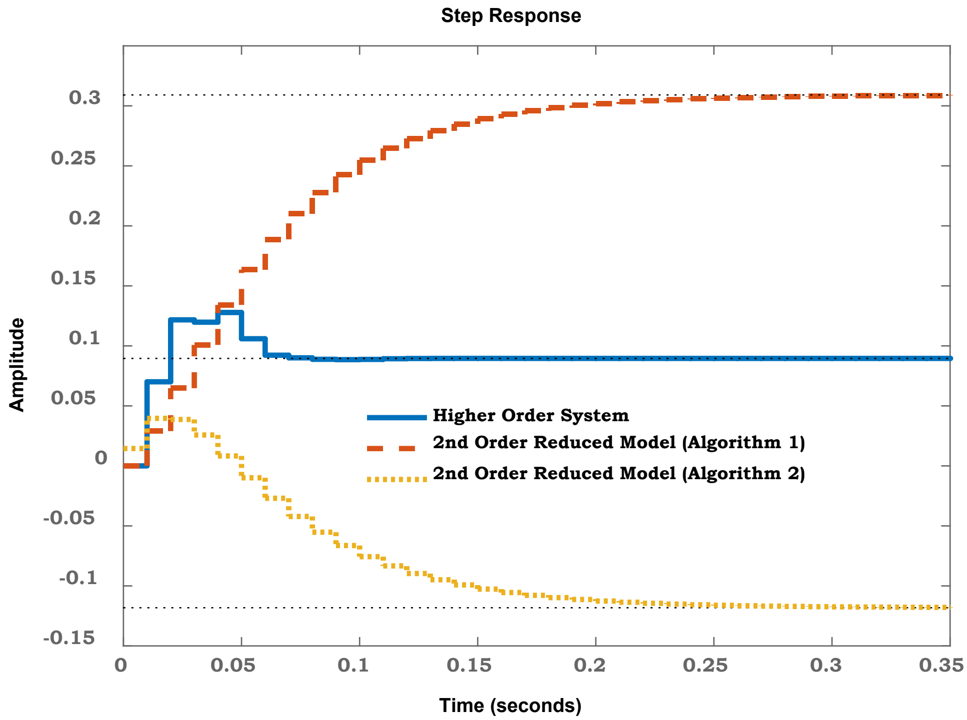

Figure 3.55: Step responses of reduced models (Lower Limit) for E.3.10.2

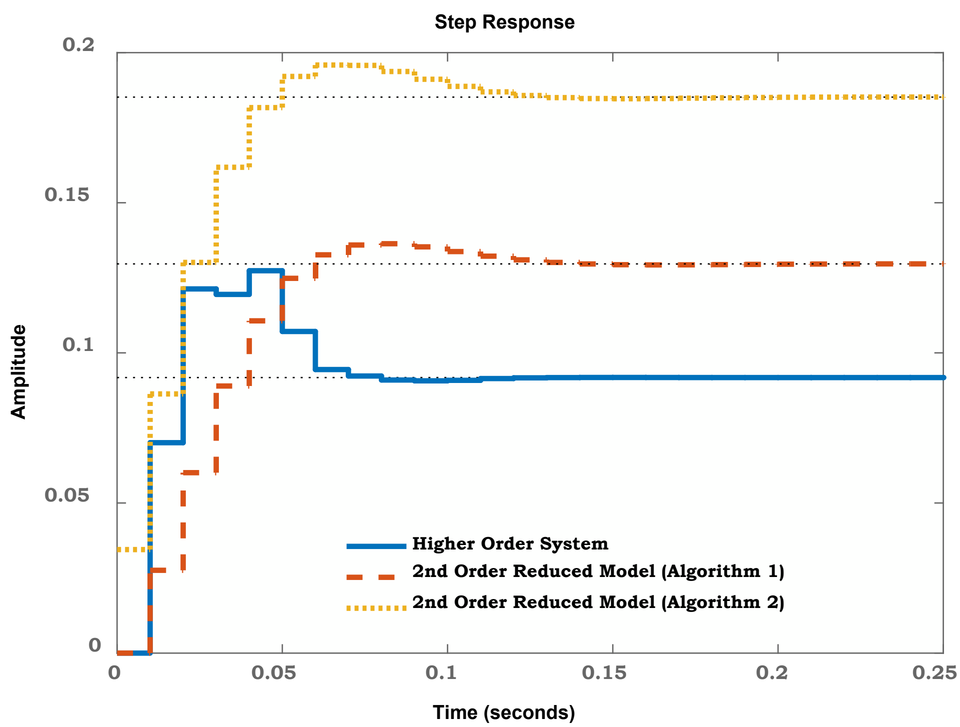

Figure 3.56: Step responses of reduced models (Upper Limit) for E.3.10.2

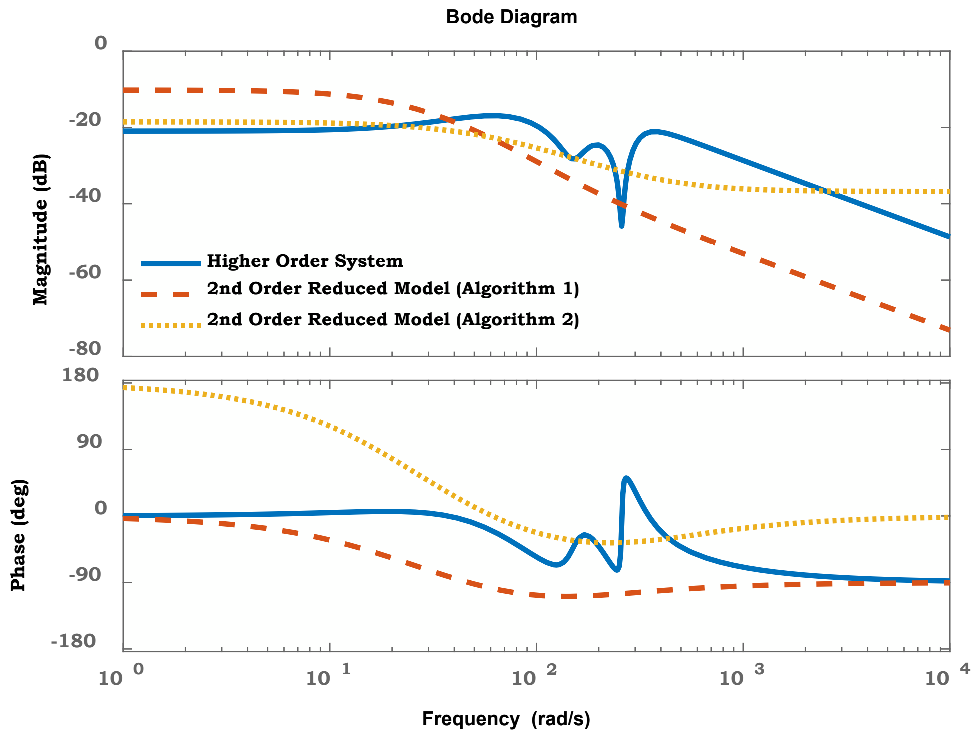

Figure 3.57: Frequency responses of reduced models (Lower Limit) for E.3.10.2

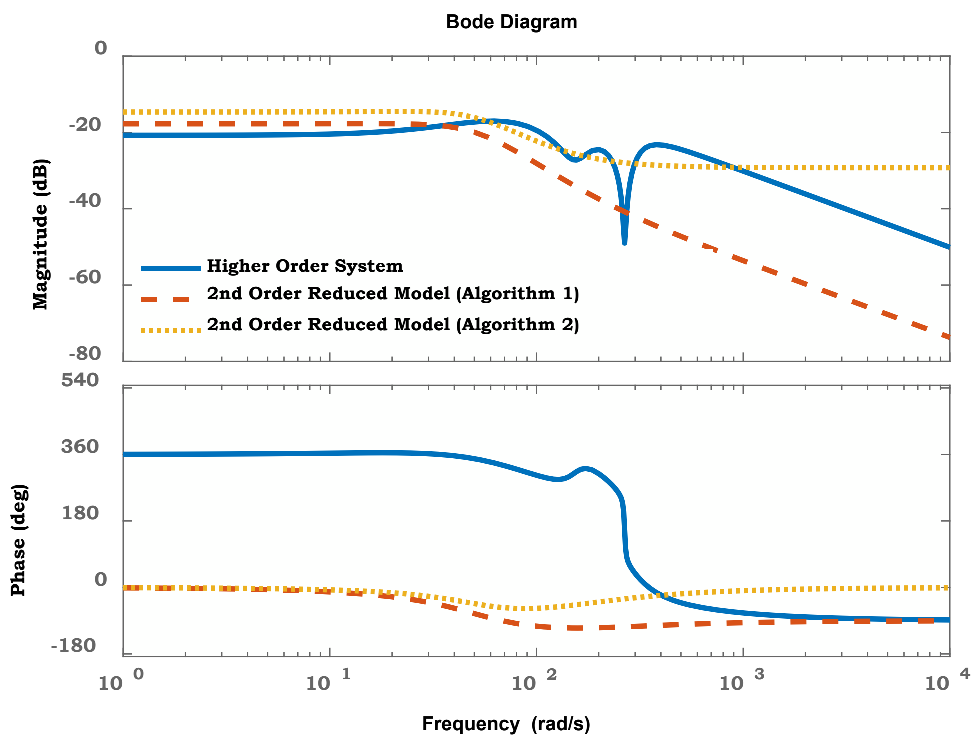

Figure 3.58: Frequency responses of reduced models (Upper Limit) for E.3.10.2

Discussion

From the above table of errors, limitation of getting a relatively higher error sum is observed (*For example: In Table 3.52; lower limit of Proposed Algorithm 1 is higher than* [90]*; Upper limit of Proposed Algorithm 2 is higher than* [68], [90], [107]). Similar limitation is observed for E.3.10.2. This limitation is considered with a confrontation that these error differences are very minute and the algorithms proposed are computationally simple and easy relative to the prevailing ones. Negligence of this limitation is also strengthened, when these proposed algorithms are applied to the real-time systems and error sum obtained is minimal as desired.

Conclusions

Two new techniques for order reduction of discrete-time interval system are explained successfully. Though the considered methodologies exist, yet, proofs themselves to be new as per the elaboration here. The method to find the reduced denominator polynomial is fresh. From the two algorithms for numerator polynomial derivation, *Algorithm 1* uses Direct Truncation which earlier exists for discrete-time interval system but here it's used in mixed form. During the course of computing the reduced model a limitation derived is also discussed.

3.4. Summary

This chapter conclude with the establishment of reduction methodologies based on *Routh Approximation Approach* for discrete-time interval systems. Altogether, *ten* algorithms are elaborated in this chapter.

Next chapter will deal with the algorithms based on *Assorted Approach* grounded on various procedural steps for computation of reduced models.