# Chapter 2
# Preliminaries

## 2.1. Preamble

This chapter is an insight to the desired supplements for the acquaintance towards the development of proposed algorithms. The chapter fragments into seven sections demonstrating their significance for the proposals ahead in the write-up. Commencing with a brief discussion on *MOR* techniques is followed by the definition of uncertain or interval systems. A short discussion about the involved arithmetic rules for mathematical computation is also available. Performance analysis engaged to validate the algorithms is then made known proceeded by the stability check methodology applied over interval systems. Towards the end of the chapter, illustrated are the essential materials like the problem statement and the desired transformation.

## 2.2. Model Order Reduction

Da Vinci mentioned, "*Simplicity is the Ultimate Sophistication.*" Later Einstein conceived "*Everything must be made as simple as Possible, But Not Simpler.*" The statements are worth stated but practically not commendable. Every individual believed the importance of being simple, but when it comes to mathematical modeling, the problem becomes troublesome. The primary reason for this is the decision of the level of desired details to be incorporated or neglected in the numerical representation of a given scenario that does not pose much of the effect on the capability of the system. Dealing with such question is tough!!! The answer to this issue is the development of *MOR* algorithms that simplifies the complexity of the systems to an acceptable limit. At present, *MOR* is well known and established area of research posturing significant algorithms. Few of the basic algorithms among them are Pade and Routh approximation, moment matching technique, aggregation method and many others.

*MOR* is a division of control systems theory, in which the complexity of large-scale dynamical systems is reduced, without disturbing the system's physical meaning. There is always an attempt to preserve the prime characteristics of the system or their input-output behavior. Reduced models are the mirror image of the large-scale dynamical systems which after derivation are efficiently engaged for efficient computational simulations, designing and other purposes. Models of dynamical systems are useful primarily for two reasons: *(i)* simulation and *(ii)* control. Fewer of the desirable characteristics of *MOR* techniques are listed below;

*a*) *Accuracy*: The algorithm should be able to extract the minimal error computation between the higher order system and the reduced models.

*b*) *User independence*: The algorithm is desired to offer minimal user intervention for developing the reduced models.

*c*) P*roperties Preservation*: It should retain, not much, but fewer of the characteristics of the higher order system such as transient response, stability.

*d*) *Computational efficiency*: It should be computationally efficient *i.e.* the cost of implementation of the reduced model should justify the higher order system realization.

## 2.3. Uncertain or Interval System

Control system theory and their analysis ground on the concept of linear and non-interval systems. Recently, the research community encountered an unusual category of problem, *i.e.* uncertainty in the system. The behavior of dynamical systems is understood and recognized through the knowledge and technology that primarily rely on the link between data and mathematical models. But data are frequently subjected to uncertainty and when taken into account result in the system parameters and even their structure to be of ambiguous nature. Below is a brief discussion of such structure and their importance;

Uncertainty in a broad sense is the lack of exact knowledge, regardless of the cause for its existence. Precisely, a thorough study of any environmental, management or technical system discovers the various types of uncertainties lying within. Uncertainty classifies into two categories based on their fundamental nature; *(a)* Aleatoric uncertainties and *(b)* Epistemic uncertainties. Former is the representative of unknowns that vary each time when the same experiment is performed *i.e.* inherent randomness and natural variability. And latter is due to things known in principle but not in practice. It is the result of imperfect knowledge, and scientific uncertainty, arising from language issues. The first is usually irreducible, whereas the latter can be quantified and reduced.

These uncertainties when considered for deriving mathematical representation offer the system in their best approximation. Uncertainty is not only due to the lack of the system knowledge but often being the subject to perturbations that change those dynamics over time. In physical systems, these uncertainties are the consequence of un-modeled dynamics, sensor noises, disturbances, standard errors, parameter variations, actuator constraints and many others. The presence of these changes alters the coefficient of the transfer function from deterministic to uncertain parameters. Thus, systems having coefficients of unknown nature are known as uncertain systems. And when these systems are bounded by a finite range or boundary is designated as *Interval Systems*. In literature, both these interval and uncertain systems are used as a choice by the researcher to assign a name. Here, in particular, they cited as *Interval Systems*. The presence of uncertainties in the systems results in foster inconvenience for the systems simulation, design and implementation. Few examples of practical system incurring changes in their mathematical representations are;

Cold rolling mill [91] with transfer functions of the steering type and the displacement type guides are

$$H(s)=\frac{[4.2,21]s^2+[3,16]s+[0.5,2.6]}{s^4+[3,8]s^3+[1,2.5]s^2+[0.05,0.15]s} \tag{2.1}$$

$$H(s)=\frac{[4.5,24]s^3+[5,29]s^2+[0.9,5]s+[0.05,0.3]}{s^5+[4,9]s^4+[4,9]s^3+[0.7,2]s^2+[0.03,0.08]s} \tag{2.2}$$

Vehicle Suspension Systems [92]

$$H(s)=\frac{[1361111.11,3062500]s+[18055555.56,40625000]}{\begin{array}{l}s^4+[441,661.5]s^3+[47516.66,71275]s^2\\+[1361111.11,3062500]s+[18055555.56,40625000]\end{array}} \tag{2.3}$$

Oblique Wing Aircraft [93]

$$H(s)=\frac{[54,74]s+[90,166]}{s^4+[2.8,4.6]s^3+[50.4,80.8]s^2+[30.1,33.9]s+[-0.1,0.1]} \tag{2.4}$$

Electric Motors [94]

$$H(s)=\frac{50*10^{-3}}{[0.0000096,0.0000336]s^3+[0.0012,0.0028]s^2+[0.002025,0.002475]s} \tag{2.5}$$

Additionally, these uncertainties significantly affect the stability and performance of the system. Since the uncertainty in the system cannot be ignored or neglected, and also no competent method broke for cutting down the uncertainty descriptions, the investigation proceeded towards the order reduction of such systems which is the prime objective of this thesis.

Literature in the earlier chapter reported about the availability of less number of techniques for order reduction of such regimes. And among them, only a few ensure the stability of the models. Literature also showcases the applicability of most order reduction techniques for non-interval systems to interval systems. Additionally, for an exception when the interval arithmetic operation results in unexpected results, a little modification in the algorithms call for a better result. From the literature, it is also evident that as compared to the algorithms available for continuous-time domain systems, the algorithms for discrete-time domain systems are very less, directing the work presented here. The contributions in this work intend towards model order reduction of discrete-time interval systems. The proposed algorithms guarantee the preservation of fewer of the dynamic characteristics of the higher order system to lower order models.

## 2.4. Interval Arithmetic

Mathematics is involved in each and every part of the existence of nature to mathematical modeling calling for a set of arithmetic rules. Similar is the case with interval systems which desire for an interval arithmetic to deal with their computation. Interval Arithmetic came into existence in the late fifties but got its first appearance in mid-sixties. A brief history about this summons below;

The idea of Interval Computation pioneered in the Ph.D. Dissertation by *R. E. Moore* at Stanford University, the USA in 1962 [95]. He presented the first application in 1959, but its first monograph appeared in 1966 [96]. Interval computation did not survive much in the US and moved to Europe, mainly to Germany. Reason gathered for this mobility is their less cost-conscious and less worried about the inaccuracy of the sensors. They confronted "if a sensor is not sufficient, spend some more money and buy a better one." Scientists being the primary users of this technique did not accept this, as they were working at the cutting edge of accuracy and were engaged with the best possible sensors to measure micro-quantities. Moving to Germany, Interval Computation got its importance as it is a part of their standard qualifying exam for different areas of Numerical Mathematics and it also offered the first specialized journal for it. Germany also hosts regular conferences for Interval Computations till date.

In 1991, USA came forward with an outburst of activity related to *Interval Computations* as they launched a new international journal *Interval Computation* which is issued under the new title *Reliable Computing* from 1995. In continuation to this, they also hosted various international conferences and workshops. In recent activities, *Interval Computations* are welcomed worldwide through peer-reviewed journals, conferences, workshops, short-term courses and many more activities. Some of the good literature on interval arithmetic are made available at http://www.cs.utep.edu/interval-comp/. Recently, Society for Industrial and Applied Mathematics published a book on Interval Arithmetic [97].

Since, the proposed methodologies in the thesis deal with interval systems, their mathematical computation demand interval arithmetic which is very similar to the basic arithmetic rules in mathematics. Only difference stated is the parameters in the interval arithmetic are of uncertain nature as illustrated below;

Let [*a, b*] and [*c, d*] be two intervals parameters where *a, c* are minimum and *b, d* are maximum entries in the specific intervals. Analogous, to the arithmetic rules of definite numerical, interval arithmetic exist as,

Addition: $$[a,b]+[c,d]=[a+c,b+d] \tag{2.6}$$

Subtraction: $$[a,b]-[c,d]=[a-d,b-c] \tag{2.7}$$

Multiplication: $$[a,b]\times[c,d]=\left[Min(ac,ad,bc,bd),Max(ac,ad,bc,bd)\right] \tag{2.8}$$

Division: $$\frac{[a,b]}{[c,d]}=[a,b]\times\left[\frac{1}{d},\frac{1}{c}\right] \tag{2.9}$$

provided $[c,d]\notin[0]$

## 2.5. Performance Analysis

The implication of the different proposed reduction methodologies to obtain the reduced order models is incomplete until the models are validated. The analysis of the model's performance is compared with the higher order system. When an appreciable amount of validation is performed, the methodology followed by the obtained models is significantly accepted, theoretically and practically. Below discussed are the two principal tools employed for the process of validation.

### 2.5.1. Summation of Error Square (SES)

A performance index of the system is a quantitative measure of its enactment subjected to parameter variations. It is dependent on the design objective of the scheme, which in most of the cases is related to systems' time response. In conventional control systems, the output regulates in a way that its response follows a constant reference input. For instance, consider the reference input and the output response be denoted by *r(t)* and *c(t)* respectively. Then, the measure of regulation is stated as the error signal *e(t) = r(t) - c(t)*. An appropriate performance index is the cumulative effect of the error function *e(t)*, often represented by the integral of the square of the error signal (*ISE*).

Since the thesis deal with discrete-time systems, the performance index is altered from integral to summation of the error squares defined as the weighted error sum over a fixed interval of time. It determines the error between the transient responses of the higher order system, and the lower order model, expressed as;

$$J = \sum_{k=0}^{\infty}\left[y(k) - y_k(k)\right]^2 \tag{2.10}$$

where $y(k)$ and $y_k(k)$ are the unit step responses of higher order system $H_n(z)$ and reduced order model $R_k(z)$ respectively.

The minimum '*J*' guarantees an approximate model of the higher order system. Computation of '*J*' for interval systems is performed by computing the transfer function with *i)* only lower limits and *ii)* only upper limits. The assessment of interval system is uneasy in its raw form; this sets the motive for considering it into two for their better analysis providing an in-depth evaluation via their boundaries. Thus, the individual '*J*' for the two transfer functions are used for comparison with the general techniques under the error columns referred as a *Lower Limit* and *Upper Limit*, shown in error tables for examples in the chapters ahead.

### 2.5.2. Step Response

Another very familiar and straightforward tool for validation of the obtained reduced models is Step Response. It is used to track the response of the reduced order model comparative to their higher order systems.

Throughout the discussion of the methodologies in the thesis, either of these two tools is employed to perform the performance analysis.

## 2.6. Stability

Earlier section quantified the outburst of interval systems in the sixties, but their analysis grabbed recognition after their theoretical existence in the nineties. Around the 1980s, researchers around the world knew interval system but did not conceive the stability analysis. The only reason for the lack of interest in such systems is the unavailability of the theories for the purpose of analyzing and designing of control systems with uncertain parameters. Grabbing rigorous interest and attention from around the world, intended *Kharitonov* to introduce Kharitonov's theorem [98] in 1979 for analysis of such systems. This publication was unknown to a major part of the researchers for many years as it is only available in Russian literature. Later, its proof and highly acceptable literature were available in a much-simplified version by Barmish [99]. Below discussed is a brief discussion about Kharitonov theorem to perform the stability test of interval systems.

It is an extension of the standard Routh Stability Criterion studied in linear control systems to interval polynomials. It states that an interval polynomial family, having an infinite number of members, is Hurwitz stable if and only if a finite small subset of four polynomials known as the Kharitonov polynomials are Hurwitz stable. Below is its illustration;

Consider the set of real polynomials of degree $n$ of the form

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 + \delta_4 s^4 + \ldots + \delta_n s^n \tag{2.11}$$

where the coefficients lie within given ranges

$$\delta_0 \in [x_0, y_0],\ \delta_1 \in [x_1, y_1],\ \ldots\ldots\ \delta_n \in [x_n, y_n].$$

Then, the polynomial is Hurwitz if and only if the following four extreme polynomials are Hurwitz *i.e.*

$$\begin{aligned}
K^1(s) &= x_0 + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + x_5 s^5 + y_6 s^6 + \ldots \\
K^2(s) &= x_0 + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + y_5 s^5 + y_6 s^6 + \ldots \\
K^3(s) &= y_0 + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + x_5 s^5 + x_6 s^6 + \ldots \\
K^4(s) &= y_0 + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + y_5 s^5 + x_6 s^6 + \ldots
\end{aligned} \tag{2.12}$$

Once the above four polynomials formulate, their stability check via the conventional Routh Stability algorithm is performed. Thus, an interval family of polynomials $\delta(s)$ is robustly stable if, and only if, the Kharitonov polynomials are stable. This Kharitonov polynomial and the stability theorem holds true for discrete-time systems also [100], [101].

In recent times, a tremendous work on the Kharitonov theorem for their simplification is reported. For the same, a complete concept and developments of the generalized Kharitonov's theorem are elaborated in the book by *Bhattacharya et. al.* [102].

Frequency response plot depicts the pictorial representation of stability check for the examples in the chapters ahead.

## 2.7. Problem Statement

The discourse of the problem statement states the mathematical resemblance of the higher order system and the reduced order model.

Let the transfer function of a higher order interval systems of order $n$ be:

$$H_n(z)=\frac{N_n(z)}{D_n(z)}=\frac{\left[N_{n-1}^-,N_{n-1}^+\right]z^{n-1}+\left[N_{n-2}^-,N_{n-2}^+\right]z^{n-2}+....+\left[N_0^-,N_0^+\right]}{\left[D_n^-,D_n^+\right]z^n+\left[D_{n-1}^-,D_{n-1}^+\right]z^{n-1}+.....+\left[D_0^-,D_0^+\right]} \tag{2.13}$$

After the engagement of the proposed methodologies, the transfer function of the reduced model of order $k<n$ is

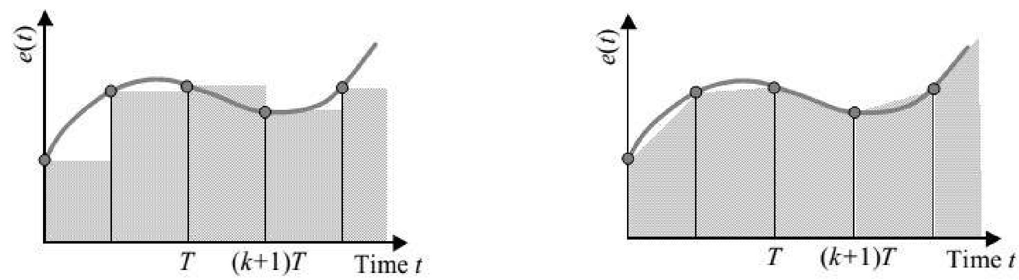
$$R_k(z)=\frac{N_k(z)}{D_k(z)}=\frac{\left[n_{k-1}^-,n_{k-1}^+\right]z^{k-1}+\left[n_{k-2}^-,n_{k-2}^+\right]z^{k-2}+.....+\left[n_0^-,n_0^+\right]}{\left[d_k^-,d_k^+\right]z^k+\left[d_{k-1}^-,d_{k-1}^+\right]z^{k-1}+.....+\left[d_0^-,d_0^+\right]} \tag{2.14}$$

## 2.8. Desired Discrete-Transformation

The thesis is devoted to the discrete-time domain which is gaining popularity in today's world. This trend is the outcome of the emergence of low-cost digital computer and microprocessor at user end presenting the simplicity for controlling and designing of the physical systems. They pose various advantages over analog control regarding reliability, flexibility, cost, performance, etc. and their handling at ease makes them highly acceptable to study and analyze systems. The majority of the physical systems are available in the continuous-time domain, and fewer are in the discrete-time domain. Study of such systems demands a proper discretization making the system convenient and easily accessible.

Discretization is an important data processing task and includes many advantages as; it is less prone to variance in estimation from small fragmented data; the amount of data under consideration reduces as redundant data can be recognized and neglected; provides better performance for the rule extraction. There are numerous ways of transformation available from [103], [104]. But well-known and easily accessible are *i*) Euler's Forward differentiation method, *ii*) Euler's Backward differentiation method, *iii*) Zero Order Hold method, *iv*) Tustin's method with frequency pre-wrapping or Bilinear transformation, and *v*) Matched Pole-Zero mapping. All these conversions fall under frequency-domain. Time-domain transformations are impulse-invariance and step-invariance methods. Each of the transformations owes their practical and theoretical importance with differences among each other, which when studied would be lengthy and exhaustive. At par, their individual study is out of scope for this thesis.

From the literature available, transformation techniques used widely for either discrete-time non-interval or interval coefficient systems are namely Tustin or bilinear or trapezoidal method and Euler's Forward differentiation method. These two approaches are elaborated below for their better understanding [105]. Let Figure 2.1 depict their integral approximations.




(a) Forward Difference (b) Trapezoidal Approximation

Figure 2.1: Different types of integral approximations

### 2.8.1 Tustin or Bilinear Transformation ($w$-domain)

In $z$-plane, the frequency appears as $z = e^{j\omega t}$, and its response loses the simplicity of logarithmic plots. It is to be noted that the $z$-transformation maps the original and complementary strips of the left of the $s$-plane into the unit circle in the $z$-plane. Thus conventional frequency response methods, do not apply to the $z$-plane. To overcome this difficulty, the pulse transfer function in the $z$-plane transforms to $w$-plane. The $w$-transformation states $z = \left(1 + \frac{T}{2} w \Big/ 1 - \frac{T}{2} w\right)$ where $T$ is the sampling period. The inverse transformation is $w = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$.

Through the $z$-transformation and the $w$-transformation, the original strip of the left half of the $s$-plane is first mapped inside of the unit circle in the $z$-plane and then allocated to the entire left half of the $w$-plane. The origin in the $z$-plane maps to the point $w = -2/T$ in the $w$-plane.

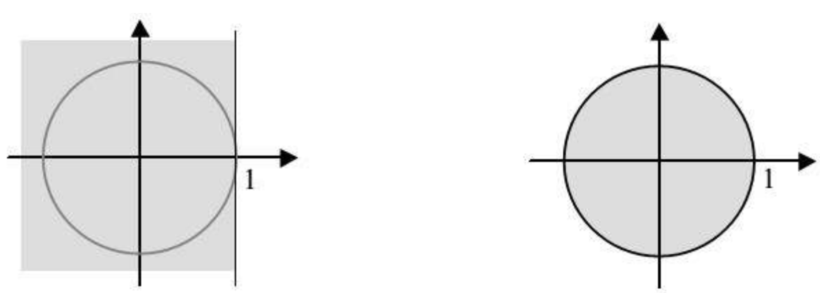
As $s$ varies from $0 \rightarrow j\frac{w_s}{2}$ along $j\omega$ axis, $z$ ranges from 1 to -1 along the unit circle in $z$-plane, and $w$ ranges from 0 to $\infty$ along the imaginary axis in the $w$-plane. The difference between the $s$-plane and $w$-plane is that the frequency range $-\frac{1}{2}\omega_s \leq \omega \leq \frac{1}{2}\omega_s$ in the $s$-plane maps to the field $-\infty < \upsilon < \infty$ in the $w$-plane, where $\upsilon$ is the fictitious frequency on the $w$-plane. Thus, there is a compression of the frequency scale. Although the $w$-plane resembles the $s$-plane geometrically, the frequency axis in the $w$-plane distorts.

### 2.8.2 Forward Difference Transformation ($p$-domain)

Tustin transformation poses fewer difficulties in its application to filter design, thus calling for matched $z$-transformation where $z = e^{pt}$. The Forward Difference conversion is successfully implemented and used in digital control systems. It's a simpler version where $z = 1 + p$ and prevail by employing forward Euler rule to the matched $z$-transform equality and retaining the first two terms in the resultant expansion. Such transformations are of particular significance in the design of audio and telephone networks.

Regarding stability transformation from $z$-to-$w$ or $p$-domain can be better understood by the figure below. By using forward difference approximation, the stability region left half plane is mapped to the half-plane to the left of 1 on the complex $z$-plane. Thus, with forward difference approximation, it is possible that an unstable continuous-time controller will approximate a stable discrete-time controller.

Figure 2.2, maps stability region $\mathrm{Re}(s) < 0$ for the two approximation methods between $s$-plane and $z$-plane.




(a) Forward Difference (b) Trapezoidal Approximation

Figure 2.2: Mapping of the stability region between the s-plane and the z-plane

The bilinear transformation (trapezoidal or Tustin's approximation) maps the left half $s$-plane into the unit disc. Hence, stable discrete (continuous) controllers approximated by stable continuous (discrete) controllers, and unstable continuous (discrete) controllers map to unstable discrete (continuous) controllers. In practice, the Tustin's approximation (bilinear transformation) is the approximation of choice for converting continuous-time (discrete) controllers to discrete-time (continuous) controllers.

The two different transformation of the higher order system results in the below depicted transfer function which later on is used throughout the thesis for the development of algorithms.

In $w$-domain

$$H_n(w) = \frac{B_n(w)}{A_n(w)} = \frac{\left[b_n^-, b_n^+\right] w^n + \left[b_{n-1}^-, b_{n-1}^+\right] w^{n-1} + \ldots. + \left[b_0^-, b_0^+\right]}{\left[a_n^-, a_n^+\right] w^n + \left[a_{n-1}^-, a_{n-1}^+\right] w^{n-1} + \ldots.. + \left[a_0^-, a_0^+\right]} \tag{2.15}$$

In $p$-domain

$$H_n(p) = \frac{B_n(p)}{A_n(p)} = \frac{\left[b_{n-1}^-, b_{n-1}^+\right] p^{n-1} + \left[b_{n-2}^-, b_{n-2}^+\right] p^{n-2} + \ldots. + \left[b_0^-, b_0^+\right]}{\left[a_n^-, a_n^+\right] p^n + \left[a_{n-1}^-, a_{n-1}^+\right] p^{n-1} + \ldots.. + \left[a_0^-, a_0^+\right]} \tag{2.16}$$

In spite of the success of the extension of continuous-time system reduction methods to discrete-time systems using bilinear or similar transformation, there remain two disadvantages. First, due to the nature of bilinear transformation, the initial value of a step response of the reduced model may not be zero, in spite of the zero initial condition of the original step response. Second, most of the above-mentioned reduction techniques are either in the frequency domain or in the time domain, and they are designed to secure a proper fitting in their respective domain. Thus, a reduced model may be satisfactory in one domain, but unsatisfactory in another domain. The second disadvantage circumvents by an attempt to assess between two transformation techniques for their implication to order reduction in the later chapter of the thesis.

## 2.9. Summary

The above accumulated preliminaries direct towards the *Model Order Reduction of Discrete-Time Interval Systems*. The chapters ahead elaborate the different algorithms under diverse categories. The flow chart in Figure 2.3 illustrates the roadmap for the development of reduction methodologies with the various decisions at few instances.

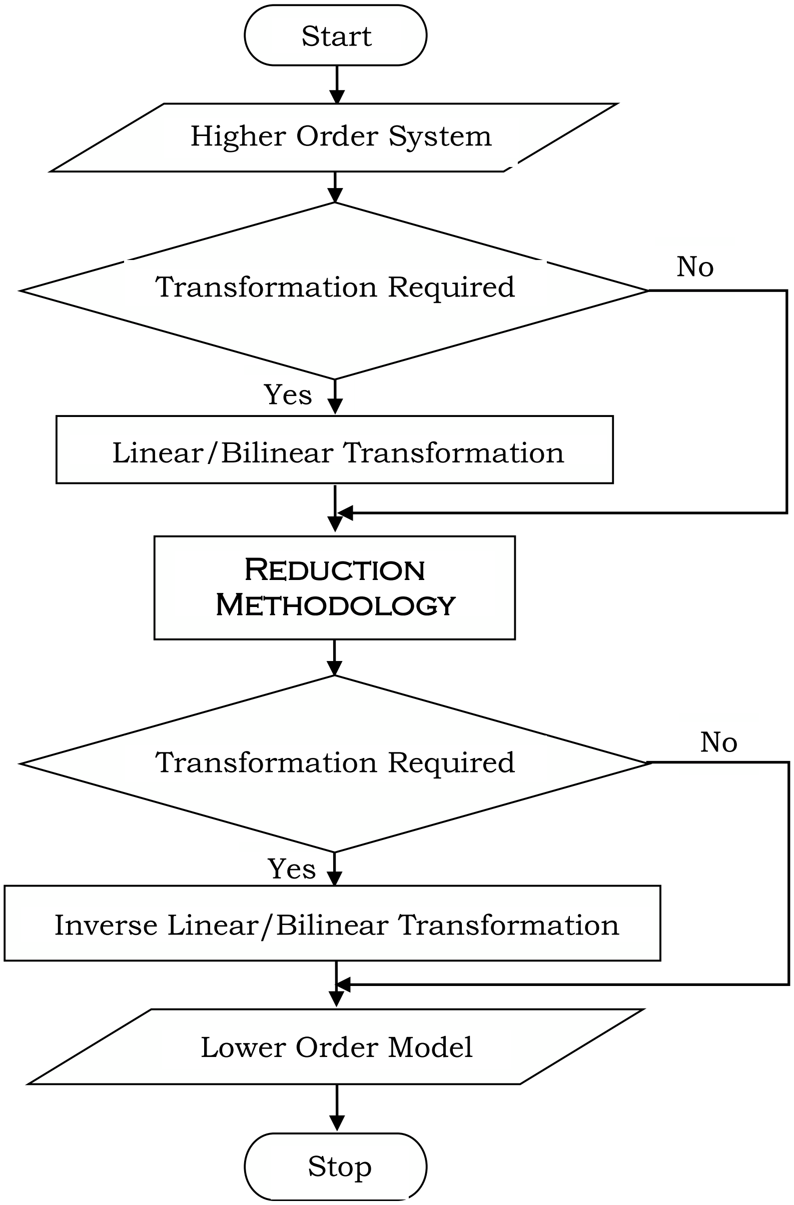



Figure 2.3: Flow chart for the development of reduction methodologies