Chapter 1
Introduction

1.1 Preamble

Daily evolution in today's world is experiencing a tremendous growth on every aspect ranging from a small module to large system. Their study calls for an appropriate mathematical representation formulated by system identification. The overall derivation results in a vigorous illustration involving a large number of differential and difference equations. These derived systems are often too massive and troublesome for analysis making them computationally inconvenient for design, simulation, and implementation. Significantly, these problems demand an algorithms or methodologies to cut down the order of the structure to a user-friendly approximate model for their exhaustive investigation. As a solution to this issue emerged the practice of deriving an approximate model representing the higher order system. The outgrowth aims to preserve fewer of the critical characteristics of the system such as stability, transient and steady state response, etc. The exercise is known as Model Order Reduction (*MOR*) which till date is under thorough research. The requirement primarily considers the computational simplicity, accuracy, and storage capabilities. Nevertheless, the reduced models are assumed to be a replica of higher order systems, which when substituted in place of the original complex systems makes them simple for either study or analysis or simulation or control. *MOR* algorithms showcased advancement from their birth to continuous/discrete time non-interval systems to the present day's available continuous/discrete time interval systems. In continuation, the discussion ahead in the thesis is about the *Model Order Reduction of Discrete-Time Interval Systems*.

1.2. Motivation

Mathematical representations of dynamical systems incurred via system identification and modeling result in the complexity of their study and analysis. Additionally, the excessive demand for accurate analysis of the system tends to derive a higher order structure making their investigation even more troublesome. The claim inclines the motivation towards *Model Order Reduction*, with an aim to produce a small dimension approximate with the same response or dynamic characteristics as of the original system. Small size models tend to have less storage and minimum computation time. Few of the aspirations for the *MOR* are

i) Minimum error estimate.

ii) Retaining dynamic characteristics as stability and transient response.

iii) Being computationally straightforward and efficient.

These days the demand of thorough and rigorous sketch of a dynamic system contributes to an outburst of the uncertainties available within the system. These changes resulting from the un-modeled dynamics, sensor noises, disturbances, standard errors, parameter variations, actuator constraints and many others are vital to gather for better understanding about the system. Their encounter results in the precise structure of higher order system of uncertain nature making their study troublesome. Since the uncertainty in the system cannot be overlooked, the research directed towards the order reduction of such establishments, which is stated to be the key objective of the thesis.

The mathematical representation of dynamic systems matured from non-interval systems to present date interval systems. Thus, the arena of *MOR* enhanced from non-interval to interval systems. Literature in the next section reports the available reduction methodologies for interval systems. Here, a researcher can examine, the various algorithms accessible for continuous-time interval systems and very few for discrete-time interval systems. The availability of fewer algorithms for discrete-time interval systems seized to be a possible domain about the prime motivation for the origin of this thesis work through the development of algorithms for *Model Order Reduction of Discrete-Time Interval Systems*.

1.3. A Short Account of Order Reduction Techniques

This section reports a summary of the available reduction methodologies from its initial work for the non-interval system to interval systems till date. As it's hard to list the full available literature, here mentioned are only a few articles that pose significant role in the advancement of the methodologies developed and relevant to present work. The literature below showcases the algorithms under continuous-time and discrete-time interval domains.

The outburst of *MOR* technique in the late sixties through the critical discussion in [1] fulfilled the demand for simplified model of a complex system. It illustrated the importance of study and analysis through an example of the practical available chemical reactor. The eruption guided other researchers to work on different formulations of *MOR* algorithms. The arena for *MOR* techniques initiated from non-interval system both in continuous-time and discrete-time domain advanced to present day interval system. Commencing with the algorithms in non-interval systems include remarkable methods like Pade Approximation, Aggregation Method, Routh Approximation (*RA*) and many others. Fewer of the review-cum-survey composition for *MOR* algorithms include Pade approximation and continued fractions [2], aggregation method [3], balanced truncation [4], [5], singular value decomposition [6], proper generalized decomposition [7]. Other surveys include approximation of the typical characteristics of the high-order systems as impulse or step response, time moments, transfer functions and others [8], [9]. Literature also present the technique that considers the asymptotic waveform evaluation based on Galerkin and multipoint Galerkin asymptotic waveform evaluation and matrix-Pade via Lanczos [10]. A technique based on frequency domain identification method using the nonlinear least square method and subspace-based identification method [11] is also available. Among the reduction methodologies, over the years of growing interest for the order reduction, *RA* grabbed a significant attention from the researchers. The reason for the noteworthy acceptability is its computational simplicity, ease of access and the assurance of retaining the reduced model stability [12]. A book [13] presenting a critical review of reduction algorithms is also available. Few other survey reports are [14], [15]. Above all, comparison among various methods for obtaining reduced-order models for large-scale systems grounded on their applicability to diverse practical available systems is available in [16]–[20]. The above offered exhaustive survey report for the non-interval system is not of much concern regarding the work in the thesis.

The focused topic of interest is the order reduction of interval system. Methodologies available in the literature for such systems are classified under continuous-time and discrete-time domains. Again, the former falls out of the realm of the thesis, but a glimpse of major and significant acceptable algorithms is reported here. Later sub-section submits the topic of interest, the discrete-time interval system, elaborated at par, since its dawn. The discussion authorizes an arena for the development of order reduction methodologies for *Discrete-time Interval Systems*, illustrated in the thesis.

1.3.1. Continuous-time Interval Systems

Execution of higher order interval systems to their lower approximate pioneered with the work of *Bandyopadhyay et. al.* in [21]. They proposed the direct extension of Routh-Pade from the non-interval system [22] to interval systems. Many other reduction methodologies witnessed their direct extensions from non-interval systems to interval systems. One such extension is noticed in [23] that use the computation of α-β parameters as employed for non-interval systems in [24]. Similarly, in [25], the calculation of γ-δ parameters is prolonged from [26]. The only difference between the two methodologies is the nature of coefficients dealt with *i.e.* non-interval and interval form. Another expansion is of balanced truncation approximation method in [27].

Algorithms in [21], [25] guaranteed to retain the stability of the reduced models if the higher order interval system is asymptotically stable. After a tenure of half decade, these algorithms are questioned for the retention of model stability in [28]. Here authors commented, unlike the non-interval *RA*, the interval *RA* do not preserve the intervals of the first time moments or Markov parameters. The above debate sprang up to critic, with some severe comments as;

1) interval Routh approximants depend on the interval arithmetic implementations of the Routh expansion and inversion algorithms;

2) interval Routh expansion algorithms cannot guarantee the success in constructing a full interval Routh array because they possibly generate an interval entry in the first column of the array that includes zero; and

3) an unstable interval *RA* may be obtained for a stable higher order interval system even if an optimization approach implements the interval Routh expansion and inversion algorithms.

Finally, based on robust frequency responses plot, the conclusion is made that an interval Routh approximant is not appropriate for the robust controller design because the interval arithmetic operations are irreversible. In continuation to the comments aforementioned, authors in [29] also mentioned the method proposed in [21] as erroneous *i.e.* a stable family of interval polynomials may yield an unstable family of reduced order polynomials. Additional to the comment, the authors proposed a modified method of direct truncation of Routh table for interval polynomials that guarantee a stable family of reduced order interval polynomials if the original family of higher order polynomials is stable. The justification for the amendment is via examples that offer the step response of reduced order models *w.r.t.* higher order systems. The revision stated in [30] is not to generate a stable reduced interval model, as its already being acknowledged in [28]. Here, the reason for the loss of stability preservation through the method is the fact that the reduced-order interval polynomials have member polynomials not generated from the member polynomials of the higher-order interval polynomials. This comment is replied in [31], where it affirmed the initial statement of [29] to be correct. It presents the main idea of shrinking the uncertainty of the elements of the last existing line of the table, in the procedure of building the new line. For instance, according to the algorithm in [29], for $i \geq 3,\ 1 \leq j \leq \left[(n-i+3)/2\right]$,

$$C_{i,j} = C_{i-2,j+1} - \frac{\tilde{C}_{i-2,1}}{\tilde{C}_{i-1,1}} \cdot C_{i-1,j+1} \tag{1.1}$$

where $\tilde{C}_{i,j}$ is the midpoint considered from the interval $C_{i,j} \hat{=} \left[C_{i,j}^{-}, C_{i,j}^{+}\right]$.

Precisely, to obtain "self-contained" interval, Routh tables one should replace the intervals in the first column by their centers, according to the choice made for $\tilde{C}_{i,j}$ in (1.1) when constructing all the rows as observed in the stated examples. Additionally, it provides two conditions for computing stable Routh table for reduced interval model.

In the process of execution, the proposal in [32], challenge for computational simplicity by avoiding the derivation of the time moments of the higher order system in advance. It only formulates the γ-table.

Ismail and *Bandyopadhyay* in [33] propose the implementation of Pade approximation. Later on, *Ismail* presented the reduction methodology for linear structured uncertain systems over the desired frequency interval in [34], [35]. It discusses the involvement of stability equation method to preserve the stability of the sixteen Kharitonov's polynomials and later compute the denominator coefficients of the reduced models. In [34], determination of the numerator coefficients of the reduced models is by matching the first (*k-1*) terms of Chebyshev polynomial series expansions of non-interval systems with that of the corresponding Chebyshev polynomial series expansions of non-interval models. The reduced order coefficients obtained by minimizing errors between the unit step responses of the sixteen fixed Kharitonov's systems and the corresponding sixteen set Kharitonov's lower models along with steady state constraints is in [35].

Beck et. al. proposed the model reduction of uncertain systems of varied forms as for multidimensional in [36] and for unstable in [37]. The former methodology involves a complete generalization of balanced realizations, Gramians, and balanced truncation as solutions to a pair of linear matrix inequalities which generalize Lyapunov equations. The latter methodology is the generalization of coprime factors reduction method introduced by Meyer [38] to uncertain system models. The generalization along with a balanced truncation algorithm is made known in [39] and applies to a class of systems containing linear parameter variation and uncertain models that do not satisfy the structured $\ell 2$-induced stability constraint required in the standard non-factored case. Another available methodology that guarantees the induced $\ell 2$-norm error and its use for polytopic uncertain linear systems are in [40].

Authors in [41] developed an algorithm for fixed-parameter models based on balancing transformations to parametric uncertain models. The combination of reduced sub-models obtained using serial and parallel decompositions in a symbolic environment provides the reduced model.

A Gramian-based approach to model reduction of uncertain systems is in [42]. It introduces the controllability and observability Gramians regarding certain parameterized algebraic Riccati inequalities that enable a balanced truncation model reduction procedure for desired uncertain model. Additionally, it also presents an investigation for model reduction through H_∞ which later on considers time-delay in [43]. Linear matrix inequalities provide the solutions to these problems regarding particular and a coupling nonconvex rank constraint. Also, the development of reduced-order model with specialized structures, such as zeroth-order model, delay-free model, no parameter uncertainties model is studied. The linear matrix inequalities engage as a general criterion for deriving reduced model in [44]. This approach uses different uncertain systems, such as linear time varying systems, Markovian jump systems, and hybrid jump systems both in continuous- and discrete-time domains.

In recent times, evolutionary techniques like Genetic Algorithm and Particle Swarm Optimization grabbed attention from the researchers for providing an optimized result affecting the derivation of reduced models. Algorithms that employ such techniques are presented in [45], [46]. These techniques mixed with pole clustering algorithm and improved generalize least-squares method are in [47] and [48] respectively. Participation of these evolutionary techniques guaranteed the stability of reduced order model if the high order system is stable. Another novel algorithm for interval system conferred in [49] offers approximation based on generalized time moments matching and Luus-Jaakola optimization practice that minimizes the performance indices.

Analogous to the amalgamated methods for order reduction of the non-interval system, algorithms for interval systems that include *RA* method for deriving stable denominator polynomial combined with different algorithms for obtaining numerators are also reported. Few of them are the direct series expansion locating the time moments [50], Kharitonov's polynomials [51], factor division method [52], [53]. The algorithm applicable to a non-interval system in [54] is extended directly to the interval systems in [55]. Reduction of a particular class of interval system using the conventional approach of Routh criterion is covered in [56] that deal with only a linear system instead of interval system while applying interval arithmetic rules to obtain the reduced order model. Authors in [57] determine the reduced-order numerator and denominator polynomials by Kharitonov's approach and the *RA* method respectively. The method is similar to the work reported in [51], with the only difference in the process of validation of the obtained reduced interval models *i.e.* the former employs fundamental square error and other uses impulse energy response. Another available mixed methodology for order reduction in [58] engages the Eigenvalues obtained using Eigenspectrum preserving dynamic characteristics as centroid and stiffness of the higher order interval system. Here the numerator polynomial is obtained using Pade approximation through the time moments and Markov parameters retention.

Apart from formulating algorithms for order reduction of interval systems, the prime concern is also to preserve stability. In recent times, this limitation is said to be removed in [59] proposing a revised Routh algorithm.

Keeping the prime focus as stability preservation, recently *Kumar et. al.* proposed algorithms classified as mixed algorithms. Their algorithms cited in [60]–[65] are a mixture of two or more well-known prevailing techniques namely Mikhailov criterion, Cauer second form, differentiation method, factor division, moment matching, *RA*, and direct truncation method.

Above discourse accounts the significant amount of literature for continuous-time interval systems. Next heading is about the topic of interest *i.e.* discrete-time interval systems.

1.3.2. Discrete-time Interval Systems

This section confers the available reduction methodologies for discrete-time interval systems from their dawn.

As discussed earlier, discrete domain is acknowledged to be readily available and computationally simple triggering researchers to formulate reduction techniques for interval systems, similar to the non-interval systems. Here, the topic being relevant to the work in the thesis converses at par underneath;

Exploitation of reduction technique for discrete-time interval systems initiated with the works of *Ismail* in [66]. They rectified the earlier observed drawback of not giving consideration to the initial transient response [67]. A possible cause for this drawback is the existence of Pade approximant for about $z=1$ for deriving reduced model. As a solution, they provided multipoint Pade approximation, where the higher-order system derives the Pade approximant of the lower model about $2k$ points, offering the numerator and denominator coefficients of the reduced models.

Authors in [68], reported a combined approach for system reduction using Pade approximation for numerator coefficients and retention of dominant poles for denominator coefficients. The literature explains the advantages of Pade approximation including computational simplicity and the fitting of time moments. It also features the disadvantage of generating unstable reduced order model even though the original system is stable. Seeking the advantage of Routh approximation, together with Pade approximation, authors of [69] proposed a Routh-Pade approximation for discrete-time interval systems without any assurance for deriving stable reduced models.

Presented under the above subheading is the optimization problem formulated regarding coupled (nonconvex) linear matrix inequalities [40], employed for model reduction of interval continuous-time systems in polytopic domains. This linear matrix inequalities are engaged here for discrete-time interval systems [70]. *Assuncao* and *Peres* have attempted the reduction methodology in a way that the H_2 and the H_∞ norm of the error between the original interval system and the reduced one are guaranteed minimum. It also presents an effort with the mixed H_2/H_∞ for order reduction.

Dolgin and *Zeheb* researched the order reduction of interval discrete-time *SISO* systems in [71], [72]. The investigation directed towards the approximation of high-order system by fixed-coefficients and unpredictable low-order system. The estimate considered the idea of the infinity norm of “absolute error” (*i.e.* minimizing the maximum of distances between the original and the reduced systems over all frequencies), which later on modified to the notion of distance liable to interval systems *i.e.* “signed distance.” For computation of the reduced models, they employed manipulated linear semi-infinite programming. The method is nevertheless, an extension of the Complex Chebyshev Approximation problem enabling the treatment of interval systems.

The above authors introduced a novel approach by directly minimizing the maximal “distance” (error) between the higher-order interval system and the reduced-order fixed models in [73]. The formulation performs the special treatment as linear semi-infinite programming problem with linear constraints. An example of degree 18 illustrates the approach by approximating it to a fixed-coefficients polynomial of degree 5 pointing reliable results.

The outburst of evolutionary approaches in the 21st century impressed discrete-time interval systems also as *Hsu, Lu* and *Wang* proposed derivation of reduced-order models by Genetic Algorithm [74] based on the resemblance of discrete sequence energy between the higher-order and reduced models. Later on in 2012, again *GA* is used for deriving the reduced model in [75] with an improvisation of computing minimum error.

Choo formulated the derivation of reduced denominator polynomial by applying interval arithmetic to dominant poles of the higher order system in [76]. It remarked two limitations; *a*) obtained poles may be of larger ranges than desired ones and *b*) unstable polynomial from the stable polynomial may be derived. However, the author attempted for a simple technique to partially overcome the stability problem as stated in the earlier literature.

A direct extension of the earlier available methodology for non-interval systems is made known in [77] by *Pappa* and *Babu*, using differentiation technique. The algorithm retains the initial Markov parameters and time moments of the higher order system. The proposed algorithm is an extension of [78] where modified differentiation combines with Pade’s approximation. The proposal excludes the additional computation of Routh type arrays, reciprocal transformations and the time moments of the n^{th} order higher order system. The proposal was an altered version of the differentiation reported in [79] for non-interval systems.

In early 2009, *Zhang, Boukas,* and *Shi* investigated a class of polytopic discrete-time uncertain switched linear systems with average dwell time switching [80]. The proposal discusses the stability criterion for general discrete-time switched systems and a μ-dependent approach for model order reduction posting the conditions for existence derived via linear matrix inequalities formulation. The obtained models are stated to be robustly exponentially stable and achieve an exponential H_∞ performance. Authors suggested the future scope extending the ideas and methods in the article to systems with time delays.

The description by *Li* in [81] addresses the coprime factorization and model reduction problems for discrete-time uncertain systems which are possibly robustly unstable. It is a possible extension of the work investigated in [82] for continuous-time uncertain systems. Here studied coprime factor model reduction problem is revisited from [39] which extends the coprime factor approach [38] for linear time invariant systems to the underlying uncertain systems. The study of this issue grounds on the results in [39] and the author's previous work on interval systems [42], [82] that reviews two classes of continuous uncertain systems. Here, they investigated the generalized controllability, observability Gramians, and balanced truncation model reduction approaches. The proposal in [81], emphasize for not being trivial as there is a significant difference between continuous and discrete systems especially in the presence of uncertainty. A sufficient condition is also presented to guarantee the closed-loop stability when the reduced model replaces the original model.

Singh and *Chandra* presented a method for model reduction using dominant poles retention and direct series expansion method in [83] for obtaining denominator and numerator polynomials respectively. The direct series expansion method used for calculating time moments is submitted to be a novel. The same authors proposed a mixed algorithm in [84] where the reduced model is derived using pole-clustering method along with Padé approximation technique by minimizing the errors between first k-time moments of the higher order system and the reduced order patterns. In [85], the method employs pole clustering for the denominator, and the numerator is derived by retaining first k time moments/Markov parameters of high-order discrete interval system that minimizes the error between the higher order system and the reduced order models. The minimization of the objective function is the Luus Jaakola algorithm as the weighted squared sum of errors.

Mixed approach for order reduction is readily available in [86] using least squares and time moments matching techniques. In [87] Cauer second form, α-table, β-table, factor division algorithms in an amalgamated form are accessible. Another combined procedure is in [88] that uses Mikhailov criterion and factor division method.

In recent past, *Sharma* and *Kumar* provided a modified γ–δ Routh approximation method [89]. Here, the denominator is constructed based on the changed γ-table concept conferred by *Dolgin* and *Zehab* [29]. However, the presented methodology is an extension of the work in [90].

The above-cited literature showcases the existence of fewer order reduction methods for discrete-time interval systems. Thus, the thesis directs towards the development of techniques in this particular domain.

1.4. Thesis Contributions

The primary objective of the thesis is to develop algorithms for order reduction of discrete-time interval systems. Most of the algorithms discussed confer to retain the dynamic characteristics of the higher order system and others not. A brief discussion of the limitations discovered during the discourse of the proposed algorithm is also available.

The thesis formulates the algorithms based on *a*) Routh Approximation and *b*) Assorted Techniques.

The proposed algorithms aim to produce reduced-order models of their higher-order representations efficiently. They are verified and validated over the examples available from the literature. Lastly, the thesis also addresses the possible future works.

1.5. Road Map of the Thesis

The illustration of work is spread over six chapters and organized as outlined below commencing with

Chapter 1 that gives an insight into the introduction and motivation to the work performed. It reports the essential literature survey for the algorithms available from the day of origin for the non-interval system to present day's interval system. Briefly, it states the thesis contribution and road map to the thesis write-up.

Chapter 2 sketches the preliminaries essential for understanding the algorithms with admittance to the need of *MOR* for interval system. Desired assets include interval arithmetic, stability check algorithm, performance analysis tools, problem statement and the discrete transformation.

Chapter 3 reports the algorithms based on *Routh Approximation* approach. These algorithms desire an appropriate discrete transformation for application of Routh algorithm. Linear or bilinear transformations is used to fulfill the expectation followed by the specific procedural steps to meet up with the discrete-time interval systems.

Chapter 4 presents the other set of developed algorithms for the order reduction of discrete-time interval systems based on *Assorted* approach. These algorithms exhibit themselves to be unique in their preparation.

Chapter 5 demonstrates two separate assessments of the results in the thesis. One is an evaluation among the two frequently used discrete transformation techniques namely linear and bilinear transformation used for order reduction of discrete-time systems. Another is the overall analysis of proposed algorithms and their results observed throughout the thesis including the limitations if discovered any.

Last ***Chapter 6***, concludes the write-up with an inevitable development of successful algorithms of varied forms for discrete-time interval systems. The chapter also suggests the future scope of work to be carried out.

1.6. Summary

MOR aim at approximating a complex system by a simpler model, while preserving most of the dynamic characteristics to the possible extent. The proposal in the chapters ahead for order reduction of discrete-time interval system focuses on the mentioned aim. Throughout the thesis, the label of *System*, and *Model* are used for the higher and lower order mathematical representations respectively.

Next chapter discusses the desired preliminaries for *MOR*.