6

An improved robust domain decomposition method for singularly perturbed parabolic reaction-diffusion systems

In this chapter, we consider the following problem

$$\begin{cases} \boldsymbol{\mathcal{L}}\boldsymbol{u} := \partial_t \boldsymbol{u} - \boldsymbol{E}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, (x,t) \in Q := \Omega \times (0,T] = (0,1) \times (0,T], \\ \boldsymbol{u}(0,t) = \boldsymbol{g}_0(t), \boldsymbol{u}(1,t) = \boldsymbol{g}_1(t), t \in (0,T], \boldsymbol{u}(x,0) = \boldsymbol{\varphi}(x), x \in \overline{\Omega}, \end{cases} \tag{6.0.1}$$

where $\boldsymbol{u} = (u_1, u_2)^T$, $\mathbf{E} = \text{diag}(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ and $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. The entries of the coupling matrix $\boldsymbol{A} = (\, a_{ij}(x,t)\,)$ are assumed to satisfy

$$a_{ij}(x,t) \leq 0, \quad i \neq j, \quad a_{ii}(x,t) > 0, \quad \sum_{j=1}^{2} a_{ij}(x,t) \geq \alpha > 0, \quad i = 1, 2.$$

Moreover, we assume that data $\boldsymbol{f} = (f_1, f_2)^T$, $\boldsymbol{g}_0 = (g_{01}, g_{02})^T, \boldsymbol{g}_1 = (g_{11}, g_{12})^T, \boldsymbol{\varphi} = (\varphi_1, \varphi_2)^T$ and the coupling matrix $\boldsymbol{A}$ are sufficiently regular and that compatibility conditions holds so that problem (6.0.1) exhibits a unique solution $\boldsymbol{u} \in C^{4,2}(\overline{Q})^2$, cf. [29, 72].

This problem is considered in Chapter 4, where we have designed a robust domain decomposition method of SWR type based on decomposing the original computational domain into five overlapping subdomains. In this chapter, we propose an improved method which splits the problem domain into three overlapping subdomains instead of splitting into five overlapping subdomains. Introducing a non uniform mesh on boundary layer subdomains and uniform mesh on a regular subdomain, we consider the central difference approximation to discretize in space. On each subdomain, the discretization in time is based on the backward Euler approximation with a uniform mesh. Introducing some auxiliary problems and splitting the iteration error from the discretization error, the convergence is proven to be independent of the perturbation parameters. More precisely, the method is shown to be almost second order convergent in space and first order convergent in time. Finally, some numerical results are given in support of the theory.

6.1 Domain decomposition method

For accelerating the convergence of the iterative process, motivated by the idea in Chapters 2 and 3, we divide the domain Q into three subdomains $Q_p = \Omega_p \times (0,T]$, $p = \ell, c, r$, where $\Omega_\ell = (0, 2\tau_2)$, $\Omega_c = (\tau_2, 1-\tau_2)$, $\Omega_r = (1-2\tau_2, 1)$ with

$$\tau_2 = \min\left\{\frac{1}{4},\ \ 2\sqrt{\frac{\varepsilon_2}{\alpha}}\ln N\right\}. \tag{6.1.1}$$

As we know that the both components of the solution have boundary layers of width $O(\sqrt{\varepsilon_2}\ln(1/\varepsilon_2))$ and the component u_1 also has sublayers of width $O(\sqrt{\varepsilon_1}\ln(1/\varepsilon_1))$, so to resolve them on left and right subdomains $Q_p, p = \ell, r$, we consider a non-uniform mesh in space and a uniform mesh in time, whereas on Q_c (where the solution behaves smoothly) we consider a uniform mesh in both space and time. On $\overline{\Omega}_\ell$ we define the mesh $\overline{\Omega}_\ell^N = \{x_i\}_0^N$, where

$$x_i = \begin{cases} \frac{4}{N} i\tau_1, & i = 0, \dots, \frac{N}{4}, \\ \tau_1 + \frac{4}{N}(i - \frac{N}{4})(\tau_2 - \tau_1), & i = \frac{N}{4} + 1, \dots, \frac{N}{2}, \\ \tau_2 + \frac{2}{N}(i - \frac{N}{2})\tau_2, & i = \frac{N}{2} + 1, \dots, N, \end{cases} \tag{6.1.2}$$

where

$$\tau_1 = \min\left\{\frac{\tau_2}{2},\ \ 2\sqrt{\frac{\varepsilon_1}{\alpha}}\ln N\right\}. \tag{6.1.3}$$

On $\overline{\Omega}_r$ we define the mesh $\overline{\Omega}_r^N = \{x_i\}_0^N$, where

$$x_i = \begin{cases} (1-2\tau_2) + \frac{2}{N} i\tau_2, & i = 0, \dots, \frac{N}{2}, \\ (1-\tau_2) + \frac{4}{N}(i - \frac{N}{2})(\tau_2 - \tau_1), & i = \frac{N}{2} + 1, \dots, \frac{3N}{4}, \\ (1-\tau_1) + \frac{4}{N}(i - \frac{3N}{4})\tau_1, & i = \frac{3N}{4} + 1, \dots, N. \end{cases} \tag{6.1.4}$$

On $\overline{\Omega}_c$, we define the mesh

$$\overline{\Omega}_c^N = \{x_i |\ x_i = \tau_2 + ih_c,\ i = 0, \dots, N,\ h_c = (1-2\tau_2)/N\}$$

We divide the interval $[0,T]$ into M subintervals of equal length Δt and introduce uniform mesh $\overline{\omega}^M = \{t_j |\ t_j = j\Delta t,\ j = 0, \dots, M,\ \Delta t = T/M\}$. We define $\Omega_p^N =$

$\overline{\Omega}_p^N \cap \Omega_p$ and $\omega^M = \overline{\omega}^M \cap (0, T]$. On each subdomain $Q_p^{N,M}$, $p = \ell, c, r$ we define the discretization

$$[\boldsymbol{\mathcal{L}}_p^{N,M} \boldsymbol{U}_p]_{i,j} = \boldsymbol{f}_{i,j}, \tag{6.1.5}$$

where

$$[\boldsymbol{\mathcal{L}}_p^{N,M} \boldsymbol{U}_p]_{i,j} = \begin{pmatrix} [\delta_t U_{1,p}]_{i,j} - \varepsilon_1 [\delta_x^2 U_{1,p}]_{i,j} + a_{11;ij} U_{1,p;i,j} + a_{12;i,j} U_{2,p;i,j} \\ [\delta_t U_{2,p}]_{i,j} - \varepsilon_2 [\delta_x^2 U_{2,p}]_{i,j} + a_{21;i,j} U_{1,p;i,j} + a_{22;ij} U_{2,p;i,j} \end{pmatrix}, \tag{6.1.6}$$

with

$$[\delta_x^2 U_{n,p}]_{i,j} = \frac{2}{h_i + h_{i+1}} \left(\frac{U_{n,p;i+1,j} - U_{n,p;i,j}}{h_{i+1}} - \frac{U_{n,p;i,j} - U_{n,p;i-1,j}}{h_i} \right), \quad \text{for } n = 1, 2,$$

and

$$[\delta_t U_{n,p}]_{i,j} := (U_{n,p;i,j} - U_{n,p;i,j-1})/\Delta t, \qquad \text{for } n = 1, 2.$$

After the discretization on each subdomain, we define the iterative process as follows.

Step 1. Initial Approximation: we take the initial approximation as follows

$$\boldsymbol{U}^{[0]}(x_i, t_j) = \begin{cases} \boldsymbol{0}, & 0 < x_i < 1,\ 0 < t_j \le T, \\ \boldsymbol{u}(x_i, 0), & \text{for } \ x_i \in \overline{\Omega}, \\ \boldsymbol{u}(0, t_j), & \text{for } \ t_j \in \omega^M, \\ \boldsymbol{u}(1, t_j), & \text{for } \ t_j \in \omega^M, \end{cases} \tag{6.1.7}$$

Step 2. For each $k \ge 1$, we find the k^{th} approximation $\boldsymbol{U}_p^{[k]}$, $p = \ell, c, r$, by solving following problems

$$\begin{cases} [\boldsymbol{\mathcal{L}}_\ell^{N,M} \boldsymbol{U}_\ell^{[k]}]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in Q_\ell^{N,M}, \\ \boldsymbol{U}_\ell^{[k]}(x_i, 0) = \boldsymbol{\varphi}(x_i) & \text{for } x_i \in \overline{\Omega}_\ell^N, \\ \boldsymbol{U}_\ell^{[k]}(0, t_j) = \boldsymbol{g}_0(t_j),\ \boldsymbol{U}_\ell^{[k]}(2\tau_2, t_j) = \mathcal{T}_{t_j} \boldsymbol{U}^{[k-1]}(2\tau_2, t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_r^{N,M} \boldsymbol{U}_r^{[k]}]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in Q_r^{N,M}, \\ \boldsymbol{U}_r^{[k]}(x_i, 0) = \boldsymbol{\varphi}(x_i) & \text{for } x_i \in \overline{\Omega}_r^N, \\ \boldsymbol{U}_r^{[k]}(1 - 2\tau_2, t_j) = \mathcal{T}_{t_j} \boldsymbol{U}^{[k-1]}(1 - 2\tau_2, t_j),\ \boldsymbol{U}_r^{[k]}(1, t_j) = \boldsymbol{g}_1(t_j), & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_c^{N,M} \boldsymbol{U}_c^{[k]}]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in Q_c^{N,M}, \\ \boldsymbol{U}_c^{[k]}(x_i, 0) = \boldsymbol{\varphi}(x_i) & \text{for } x_i \in \overline{\Omega}_c^N, \\ \boldsymbol{U}_c^{[k]}(\tau_2, t_j) = \mathcal{T}_{t_j} \boldsymbol{U}_\ell^{[k]}(\tau_2, t_j),\ \boldsymbol{U}_c^{[k]}(1 - \tau_2, t_j) = \mathcal{T}_{t_j} \boldsymbol{U}_r^{[k]}(1 - \tau_2, t_j), & \text{for } t_j \in \omega^M, \end{cases}$$

where $\mathcal{T}_{t_j} \boldsymbol{U}^{[k]}$ is used to denote the piecewise linear interpolant of the k^{th} iterate

$\boldsymbol{U}^{[k]}$ at time level t_j on $\overline{\Omega}^N := (\overline{\Omega}_\ell^N \backslash \overline{\Omega}_c) \cup \overline{\Omega}_c^N \cup (\overline{\Omega}_r^N \backslash \overline{\Omega}_c)$.

Step 3. To compute the solution of problem (6.0.1) in the original domain, we merge the solutions obtained in Step 2 in the following way

$$\boldsymbol{U}^{[k]}(x_i,t_j) = \begin{cases} \boldsymbol{U}_\ell^{[k]}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_\ell^{N,M} \setminus \overline{Q}_c; \\ \boldsymbol{U}_c^{[k]}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_c^{N,M}; \\ \boldsymbol{U}_r^{[k]}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_r^{N,M} \setminus \overline{Q}_c. \end{cases} \tag{6.1.8}$$

Step 4. Termination: The iterative process is terminated if condition

$$||\,\boldsymbol{U}^{[k]} - \boldsymbol{U}^{[k-1]}||_{\overline{Q}^{N,M}} \leq tol \tag{6.1.9}$$

is true, otherwise repeat Step 2 and continue the iterative process until the desired accuracy is not obtained.

It is easy to prove that for each $p = \ell, c, r$, the operator $\boldsymbol{\mathcal{L}}_p^{N,M}$ satisfies the following discrete maximum principle.

Lemma 6.1.1. *Suppose the mesh function $\boldsymbol{U}_p, p = \ell, c, r,$ satisfies $\boldsymbol{U}_{p;i,0} \geq \mathbf{0}$ for $x_i \in \overline{\Omega}_p^N$ and $\boldsymbol{U}_{p;0,j} \geq \mathbf{0}$ and $\boldsymbol{U}_{p;N,j} \geq \mathbf{0}$ for $t_j \in \omega^M$. Then $[\boldsymbol{\mathcal{L}}_p^{N,M}\boldsymbol{U}_p]_{ij} \geq \mathbf{0}$ for $(x_i,t_j) \in Q_p^{N,M}$, implies $\boldsymbol{U}_{p;i,j} \geq \mathbf{0}$ for $(x_i,t_j) \in \overline{Q}_p^{N,M}$.*

6.2 Error analysis

In this section we provide error analysis of the proposed method, where the following bounds will be used.

Lemma 6.2.1. *The solution $\boldsymbol{u}$ of (4.0.1) satisfies*

$$\|\partial_t^\ell u_i\|_{\overline{Q}} \leq C, \quad for \ \ \ell = 0,1,2, \quad i = 1,2.$$

$$\|\partial_x^s v_1\|_{\overline{Q}} \leq C(1+\varepsilon_1^{1-s/2}), \qquad \|\partial_x^s v_2\|_{\overline{Q}} \leq C(1+\varepsilon_2^{1-s/2}), \quad s = 0,\ldots,4,$$

$$|\partial_x^3 v_1| \leq C\{\varepsilon_2^{-1} + \varepsilon_1^{-1}\mathcal{B}_{\varepsilon_1}(x)\},$$

$$|w_1(x,t)| \leq C\mathcal{B}_{\varepsilon_2}(x), \qquad |w_2(x)| \leq C\mathcal{B}_{\varepsilon_2}(x),$$

$$|\partial_x^s w_1(x,t)| \leq C(\varepsilon_1^{-s/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x)), \quad |\partial_x^s w_2(x,t)| \leq C\varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x), \ \ s = 1,2,$$

$$|\partial_x^s w_1(x,t)| \leq C(\varepsilon_1^{-s/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x)), \quad s = 3,4,$$

$$|\partial_x^s w_2(x,t)| \leq C\varepsilon_2^{-1}(\varepsilon_1^{-(s-2)/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-(s-2)/2}\mathcal{B}_{\varepsilon_2}(x)), \quad s = 3,4,$$

for $(x,t) \in \overline{Q}$ *and* $\mathcal{B}_{\varepsilon_i}(x) = \exp(-x\sqrt{\alpha/\varepsilon_i}) + \exp(-(1-x)\sqrt{\alpha/\varepsilon_i}),\ x \in \overline{\Omega}, i = 1,2.$

Proof. See [29, 78]. □

Lemma 6.2.2. *Suppose that* $\varepsilon_1 < \varepsilon_2$ *and* $\varepsilon_2 \leq \alpha/2$. *Then* $\boldsymbol{w} = (w_1, w_2)^T$ *is decomposed as follows*

$$w_1 = \widehat{w}_{1,\varepsilon_1} + \widehat{w}_{1,\varepsilon_2}, \qquad w_2 = \widehat{w}_{2,\varepsilon_1} + \widehat{w}_{2,\varepsilon_2}, \tag{6.2.1}$$

$$w_1 = \widetilde{w}_{1,\varepsilon_1} + \widetilde{w}_{1,\varepsilon_2}, \qquad w_2 = \widetilde{w}_{2,\varepsilon_1} + \widetilde{w}_{2,\varepsilon_2}, \tag{6.2.2}$$

where

$$|\widehat{w}_{1,\varepsilon_1}(x)| \leq \mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^2\widehat{w}_{1,\varepsilon_1}(x,t)| \leq \varepsilon_1^{-1}\mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^4\widehat{w}_{1,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x), \tag{6.2.3}$$

$$|\widehat{w}_{2,\varepsilon_1}(x,t)| \leq \mathcal{B}_{\varepsilon_1}(x), \ |\partial_x^2\widehat{w}_{2,\varepsilon_1}(x,t)| \leq \varepsilon_2^{-1}\mathcal{B}_{\varepsilon_1}(x), \ |\partial_x^4\widehat{w}_{2,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x), \tag{6.2.4}$$

and

$$|\widetilde{w}_{1,\varepsilon_1}(x)| \leq \mathcal{B}_{\varepsilon_1}(x,t), \quad |\partial_x^2\widetilde{w}_{1,\varepsilon_1}(x,t)| \leq \varepsilon_1^{-1}\mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^3\widetilde{w}_{1,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-3/2}\mathcal{B}_{\varepsilon_2}(x), \tag{6.2.5}$$

$$|\widetilde{w}_{2,\varepsilon_1}(x,t)| \leq \mathcal{B}_{\varepsilon_1}(x), \ |\partial_x^2\widetilde{w}_{2,\varepsilon_1}(x,t)| \leq \varepsilon_2^{-1}\mathcal{B}_{\varepsilon_1}(x), \ |\partial_x^3\widetilde{w}_{2,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-3/2}\mathcal{B}_{\varepsilon_2}(x), \tag{6.2.6}$$

for all $(x,t) \in \overline{Q}$.

Proof. This lemma can be proved using arguments in [29] □

Next we define the auxiliary mesh functions $\overline{\boldsymbol{U}}_p,\ p = \ell, m, r$, satisfying

$$\begin{cases} [\boldsymbol{\mathcal{L}}_\ell^{N,M}\overline{\boldsymbol{U}}_\ell]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j) \in Q_\ell^{N,M}, \\ \overline{\boldsymbol{U}}_\ell(x_i,0) = \boldsymbol{u}(x_i,0) & \text{for } x_i \in \overline{\Omega}_\ell^N, \\ \overline{\boldsymbol{U}}_\ell(0,t_j) = \boldsymbol{u}(0,t_j), \overline{\boldsymbol{U}}_\ell(2\tau_2,t_j) = \boldsymbol{u}(2\tau_2,t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_r^{N,M}\overline{\boldsymbol{U}}_r]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j) \in Q_r^{N,M}, \\ \overline{\boldsymbol{U}}_r(x_i,0) = \boldsymbol{u}(x_i,0) & \text{for } x_i \in \overline{\Omega}_r^N, \\ \overline{\boldsymbol{U}}_r(1-2\tau_2,t_j) = \boldsymbol{u}(1-2\tau_2,t_j), \overline{\boldsymbol{U}}_r(1,t_j) = \boldsymbol{u}(1,t_j), & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_c^{N,M}\overline{\boldsymbol{U}}_c]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j) \in Q_c^{N,M}, \\ \overline{\boldsymbol{U}}_c(x_i,0) = \boldsymbol{u}(x_i,0) & \text{for } x_i \in \overline{\Omega}_c^N, \\ \overline{\boldsymbol{U}}_c(\tau_2,t_j) = \boldsymbol{u}(\tau_2,t_j), \overline{\boldsymbol{U}}_c(1-\tau_2,t_j) = \boldsymbol{u}(1-\tau_2,t_j), & \text{for } t_j \in \omega^M, \end{cases}$$

where $\pounds_p^{N,M},\ p = \ell, c, r$, are as defined in Section 6.1, and $\boldsymbol{u}$ is the exact solution of (6.0.1). Also we define

$$\overline{\boldsymbol{U}}(x_i,t_j) = \begin{cases} \overline{\boldsymbol{U}}_\ell(x_i,t_j),\ (x_i,t_j) \in \overline{Q}_\ell^{N,M} \setminus \overline{Q}_c; \\ \overline{\boldsymbol{U}}_c(x_i,t_j),\ (x_i,t_j) \in \overline{Q}_c^{N,M}; \\ \overline{\boldsymbol{U}}_r(x_i,t_j),\ (x_i,t_j) \in \overline{Q}_r^{N,M} \setminus \overline{Q}_c. \end{cases} \tag{6.2.7}$$

We now apply a triangle inequality to split the error into two parts, the discretization error and the iteration error, as follows

$$||\boldsymbol{u} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq ||\boldsymbol{u} - \overline{\boldsymbol{U}}||_{\overline{Q}^{N,M}} + ||\overline{\boldsymbol{U}} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}}. \tag{6.2.8}$$

The following lemma presents a bound for the discretization error $||\mathbf{u} - \overline{\boldsymbol{U}}||_{\overline{Q}^{N,M}}$.

Lemma 6.2.3. *Let $\boldsymbol{u}$ be the solution of problem (6.0.1) and $\overline{\boldsymbol{U}}$ is as described in (6.2.7). Then*

$$||\boldsymbol{u} - \overline{\boldsymbol{U}}||_{\overline{Q}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N). \tag{6.2.9}$$

Proof. For simplicity we consider $\tau_2 = (2\sqrt{\varepsilon_2}\ln N)/\sqrt{\alpha}$ and $\tau_1 = (2\sqrt{\varepsilon_1}\ln N)/\sqrt{\alpha}$, that is ε_1 and ε_2 are small and of different magnitude. This is the most interesting case of problem (6.0.1), as overlapping layers occur in this case.

We split the truncation error using the solution decomposition $u_n = v_n + w_n$. We get

$$[\pounds_{\ell,n}^{N,M}(\boldsymbol{u} - \overline{\boldsymbol{U}}_\ell)]_{i,j} = [\delta_t u_n - \partial_t u_n]_{i,j} + \varepsilon_n[\partial_x^2 v_n - \delta_x^2 v_n]_{i,j} + \varepsilon_n[\partial_x^2 w_n - \delta_x^2 w_n]_{i,j} \tag{6.2.10}$$

By Taylor expansion and Lemma 6.2.1, for the first term on the right-hand side, we get

$$|[\delta_t u_n - \partial_t u_n]_{i,j}| \leq C\Delta t||\partial_t^2 u_n(x_i,.)||_{[t_{j-1},t_j]} \leq C\Delta t \tag{6.2.11}$$

For the second term on the right-hand side, by Taylor expansions we get

$$\varepsilon_n|[\partial_x^2 v_n - \delta_x^2 v_n]_{i,j}| \leq \begin{cases} \frac{\varepsilon_n}{3}(h_{i+1} + h_i)||\partial_x^3 v_n||_{\overline{Q}_\ell}, \quad x_i = \tau_1, \tau_2, \\ \frac{\varepsilon_n}{12}h_i^2||\partial_x^4 v_n||_{\overline{Q}_\ell}, \quad \text{otherwise}. \end{cases} \tag{6.2.12}$$

Since $h_{i+1} + h_i \leq CN^{-1}$ and $h_i^2 \leq CN^{-2}$, we have

$$\varepsilon_n|[\partial_x^2 v_n - \delta_x^2 v_n]_{i,j}| \leq \begin{cases} C\sqrt{\varepsilon_n}N^{-1}, \quad x_i = \tau_1, \tau_2, \\ CN^{-2}, \quad \text{otherwise}. \end{cases}$$

We use argument in [78, Theorem 1] to get the sharp estimate at $x_i = \tau_2$ as follows

$$\begin{aligned}\varepsilon_1|[\partial_x^2 v_1 - \delta_x^2 v_1]_{i,j}| &\le \frac{\varepsilon_1}{3}(h_{i+1}+h_i)||\partial_x^3 v_1||_{\overline{Q}_\ell}\\ &\le C\varepsilon_1 N^{-1}(\varepsilon_2^{-1/2}+\varepsilon_1^{-1/2}\mathcal{B}_{\varepsilon_1}(x))\\ &\le C\varepsilon_1\varepsilon_2^{-1/2}N^{-1},\end{aligned}$$

as one can prove that $\varepsilon_1^{-1/2}\mathcal{B}_{\varepsilon_1}(x) \le 2\varepsilon_2^{-1/2}\exp(-x\sqrt{\alpha/\varepsilon_2}) \le 2\varepsilon_2^{-1/2}$ for $\sqrt{\varepsilon_2/\alpha} \le x \le 1/2$ and $2\sqrt{\varepsilon_1} < \sqrt{\varepsilon_2}$.

For the third term on the right-hand side, we consider the following four cases:

Case I: For $x_i \in (0, \tau_1)$, we use $h_i \le C\sqrt{\varepsilon_1}N^{-1}\ln N$ and Lemma 6.2.1 to get

$$\varepsilon_n|[\partial_x^2 w_n - \delta_x^2 w_n]_{i,j}| \le C\varepsilon_n h_i^2\|\partial_x^4 w_n\|_{\overline{Q}} \le CN^{-2}\ln^2 N.$$

Case II: For $x_i \in (\tau_1, \tau_2)$, we use the decomposition $w_n = \widehat{w}_{n,\varepsilon_1} + \widehat{w}_{n,\varepsilon_2}$ to get

$$\varepsilon_n|[\partial_x^2 w_n - \delta_x^2 w_n]_{i,j}| \le \varepsilon_n|[\partial_x^2\widehat{w}_{n,\varepsilon_1} - \delta_x^2\widehat{w}_{n,\varepsilon_1}]_{i,j}| + \varepsilon_n|[\partial_x^2\widehat{w}_{n,\varepsilon_2} - \delta_x^2\widehat{w}_{n,\varepsilon_2}]_{i,j}|.$$

Using Lemma 6.2.2 and mesh width $h_i \le C\sqrt{\varepsilon_2}N^{-1}\ln N$, we get

$$\begin{aligned}\varepsilon_n|[\partial_x^2 w_n - \delta_x^2 w_n]_{i,j}| &\le C\varepsilon_n(\|\partial_x^2\widehat{w}_{n,\varepsilon_1}(.,t_j)\|_{[x_{i-1},x_{i+1}]} + h_i^2\|\partial_x^4\widehat{w}_{n,\varepsilon_2}(.,t_j)\|_{[x_{i-1},x_{i+1}]})\\ &\le C\mathcal{B}_{\varepsilon_1}(\tau_1) + CN^{-2}\ln^2 N\\ &\le CN^{-2} + CN^{-2}\ln^2 N\\ &\le CN^{-2}\ln^2 N.\end{aligned}$$

Case III: For $x_i \in [\tau_2, 2\tau_2)$, we use Lemma 6.2.1 to get

$$\varepsilon_n|[\partial_x^2 w_n - \delta_x^2 w_n]_{i,j}| \le \varepsilon_n\|\partial_x^2 w_n(.,t_j)\|_{[x_{i-1},x_{i+1}]} \le C\|\mathcal{B}_{\varepsilon_2}\|_{[x_{i-1},x_{i+1}]}$$

Now $x_i \in [\tau_2, 2\tau_2)$, implies that $\|\mathcal{B}_{\varepsilon_2}\|_{[x_{i-1},x_{i+1}]} = \mathcal{B}_{\varepsilon_2}(x_{i-1})$ and $x_{i-1} \ge \tau_2 - 4\tau_2/N$. Hence

$$\begin{aligned}\mathcal{B}_{\varepsilon_2}(x_{i-1}) &\le 2e^{(-\tau_2+4\tau_2/N)\sqrt{\alpha/\varepsilon_2}} = 2e^{-\tau_2\sqrt{\alpha/\varepsilon_2}}e^{(4\tau_2/N)\sqrt{\alpha/\varepsilon_2}}\\ &= 2e^{-2\ln N}e^{8N^{-1}\ln N} \le CN^{-2}.\end{aligned}$$

Case IV: For $x_i = \tau_1$ we use the decomposition similar to Case II and proceed as follows

$$\varepsilon_1|[\partial_x^2 w_1 - \delta_x^2 w_1]_{i,j}| \le \varepsilon_1|[\partial_x^2\widetilde{w}_{1,\varepsilon_1} - \delta_x^2\widetilde{w}_{1,\varepsilon_1}]_{i,j}| + \varepsilon_1|[\partial_x^2\widetilde{w}_{1,\varepsilon_2} - \delta_x^2\widetilde{w}_{1,\varepsilon_2}]_{i,j}|$$

$$\begin{aligned}
&\leq C\varepsilon_1\|\partial_x^2\widetilde{w}_{1,\varepsilon_1}(.,t_j)\|_{[x_{i-1},x_{i+1}]}+C\varepsilon_1(h_i+h_{i+1})\|\partial_x^3\widetilde{w}_{1,\varepsilon_2}(.,t_j)\|_{[x_{i-1},x_{i+1}]}\\
&\leq CN^{-2}+C\varepsilon_1\frac{(\tau_1+\tau_2)}{N}(\varepsilon_2^{-3/2}\|\mathcal{B}_{\varepsilon_2}\|_{[x_{i-1},x_{i+1}]})\\
&\leq CN^{-2}+C\frac{\varepsilon_1\ln N}{\varepsilon_2 N}\exp(-\tau_1\sqrt{\alpha/\varepsilon_2})\\
&\leq CN^{-2}+C\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}N}\frac{\sqrt{\varepsilon_1}\ln N}{\sqrt{\varepsilon_2}}\exp(-2\sqrt{\varepsilon_1/\varepsilon_2}\ln N)\\
&\leq CN^{-2}+C\varepsilon_1^{1/2}\varepsilon_2^{-1/2}N^{-1}.
\end{aligned}$$

Similarly we can obtain

$$\varepsilon_2|[\partial_x^2 w_2-\delta_x^2 w_2]_{i,j}|\leq CN^{-2}+C\varepsilon_1^{-1/2}\varepsilon_2^{1/2}N^{-1}.$$

Collecting the above bounds, we get the following inequality

$$|[\pounds_{\ell,n}^{N,M}(\boldsymbol{u}-\overline{\boldsymbol{U}}_\ell)]_{i,j}|\leq C\begin{cases}\Delta t+N^{-2}\ln^2 N, & x_i\neq\tau_1,\tau_2, t_j\in\overline{\omega}^M, n=1,2,\\ \Delta t+\varepsilon_1^{1/2}\varepsilon_2^{-1/2}N^{-1}+N^{-2}, & x_i=\tau_1, t_j\in\overline{\omega}^M, n=1,\\ \Delta t+\varepsilon_1^{-1/2}\varepsilon_2^{1/2}N^{-1}+N^{-2}, & x_i=\tau_1, t_j\in\overline{\omega}^M, n=2,\\ \Delta t+\varepsilon_1\varepsilon_2^{-1/2}N^{-1}+N^{-2}, & x_i=\tau_2, t_j\in\overline{\omega}^M, n=1,\\ \Delta t+\varepsilon_2^{1/2}N^{-1}+N^{-2}, & x_i=\tau_2, t_j\in\overline{\omega}^M, n=2.\end{cases} \tag{6.2.13}$$

Now using the idea of [78] we define the barrier function

$$\mathbf{g}:=C\Delta t(1,1)^T+CN^{-2}\ln^2 N(1+\phi_{\varepsilon_1}+\phi_{\varepsilon_2})(1,1)^T$$

where ϕ_{ε_1} and ϕ_{ε_2} are as follows

$$\phi_{\varepsilon_1}(x,t)=\begin{cases}\frac{x}{\tau_1}, & (x,t)\in[0,\tau_1]\times[0,T],\\ 1, & (x,t)\in[\tau_1,2\tau_2]\times[0,T],\end{cases} \tag{6.2.14}$$

and

$$\phi_{\varepsilon_2}(x,t)=\begin{cases}\frac{x}{\tau_2}, & (x,t)\in[0,\tau_2]\times[0,T],\\ 1, & (x,t)\in[\tau_2,2\tau_2]\times[0,T].\end{cases} \tag{6.2.15}$$

Notice the fact that

$$-[\delta_x^2\phi_{\varepsilon_1}]_{i,j}\geq\begin{cases}\frac{\alpha N}{8\sqrt{\varepsilon_1\varepsilon_2}\ln^2 N} & \text{if } x_i=\tau_1,\\ 0 & \text{otherwise.}\end{cases}$$

and

$$-[\delta_x^2 \phi_{\varepsilon_2}]_{i,j} \geq \begin{cases} \frac{\sqrt{\alpha}N}{\sqrt{\varepsilon_2}\ln N} & \text{if } x_i = \tau_2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, use (6.2.13) and choose C (in $\mathbf{g}$) sufficiently large, independent of ε_1 and ε_2, such that

$$|[\pounds_{\ell,n}^{N,M}(\boldsymbol{u} - \overline{\boldsymbol{U}}_\ell)]_{i,j}| \leq [\pounds_{\ell,n}^{N,M}\mathbf{g}]_{i,j} \quad \text{for} \quad i = 1, \ldots, N-1; j = 1, \ldots, M; n = 1, 2$$

and

$$|[(\boldsymbol{u} - \overline{\boldsymbol{U}}_\ell)]_{i,j}| \leq [\mathbf{g}]_{i,j} \quad \text{for} \quad (x_i, t_j) \in \partial\overline{Q}_\ell^{N,M},$$

where $\partial\overline{Q}_\ell^{N,M} = (\overline{\Omega}_\ell^N \times \{0\}) \cup (\{0, 2\tau_2\} \times \omega^M)$. Hence, by the discrete maximum principle we get

$$||\boldsymbol{u} - \overline{\boldsymbol{U}}_\ell||_{\overline{Q}_\ell^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N). \tag{6.2.16}$$

Similarly, we can prove that

$$||\boldsymbol{u} - \overline{\boldsymbol{U}}_r||_{\overline{Q}_r^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N). \tag{6.2.17}$$

The bound on $|(\boldsymbol{u} - \overline{\boldsymbol{U}}_c)(x_i, t_j)|$, $(x_i, t_j) \in Q_c^{N,M}$ can be easily established using Taylor expansions and (6.2.10). We get

$$\begin{aligned} [\pounds_{c,n}^{N,M}(\boldsymbol{u} - \overline{\boldsymbol{U}}_c)]_{i,j} &\leq C(\Delta t + \varepsilon_n h_c^2 \|\partial_x^4 v_n(., t_j)\|_{[x_{i-1}, x_i]} + \varepsilon_n \|\partial_x^2 w_n(., t_j)\|_{[x_{i-1}, x_i]}) \\ &\leq C(\Delta t + N^{-2}). \end{aligned} \tag{6.2.18}$$

Hence, using the discrete maximum principle we get

$$||\boldsymbol{u} - \overline{\boldsymbol{U}}_c||_{\overline{Q}_c^{N,M}} \leq C(\Delta t + N^{-2}). \tag{6.2.19}$$

Then, combining the bounds given in (6.2.16), (6.2.17) and (6.2.19) on $\overline{Q}_p^{N,M}$, $p = \ell, c, r$, we get the required result. □

The following lemma presents a bound for the iteration error term on the right-hand side of (6.2.8).

Lemma 6.2.4. *Let $\boldsymbol{U}^{[k]}$ be the k^{th} approximation to the exact solution of problem (6.0.1) and let $\overline{\boldsymbol{U}}$ be as described in (6.2.7). Then*

$$||\overline{\boldsymbol{U}} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq C2^{-k} + C(\Delta t + N^{-2}\ln^2 N). \tag{6.2.20}$$

Proof. First, we introduce some notation

$$\theta^{[k]} = \max\{||(\overline{\boldsymbol{U}}_\ell - \mathcal{T}_{t_j}\boldsymbol{U}^{[k-1]})(2\tau_2, t_j)||_\infty, ||(\overline{\boldsymbol{U}}_r - \mathcal{T}_{t_j}\boldsymbol{U}^{[k-1]})(1-2\tau_2, t_j)||_\infty\},$$

$$\theta_{2\tau_2} = \max\left\{||(\overline{\boldsymbol{U}}_\ell - \mathcal{T}_{t_j}\overline{\boldsymbol{U}}_c)(2\tau_2, t_j)||_\infty, ||(\overline{\boldsymbol{U}}_r - \mathcal{T}_{t_j}\overline{\boldsymbol{U}}_c)(1-2\tau_2, t_j)||_\infty\right\},$$

$$\theta_{\tau_2} = \max\left\{||(\overline{\boldsymbol{U}}_c - \overline{\boldsymbol{U}}_\ell)(\tau_2, t_j)||_\infty, ||(\overline{\boldsymbol{U}}_c - \overline{\boldsymbol{U}}_r)(1-\tau_2, t_j)||_\infty\right\}.$$

Now define

$$\boldsymbol{\varphi}^\pm(x_i, t_j) = \frac{x_i}{2\tau_2}\theta^{[1]}\mathbf{1} \pm (\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]})(x_i, t_j),$$

where $\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}$ satisfies

$$\begin{cases} [\boldsymbol{\mathcal{L}}_\ell^{N,M}(\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]})_{i,j}] = \mathbf{0} & \text{for } (x_i, t_j) \in Q_\ell^{N,M}, \\ (\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]})(x_i, 0) = \mathbf{0} & \text{for } x_i \in \overline{\Omega}_\ell^N, \\ (\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]})(0, t_j) = \mathbf{0}, |(\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]})(2\tau_2, t_j)| \le \theta^{[1]}\mathbf{1} & \text{for } t_j \in \omega^M. \end{cases}$$

We now use Lemma 6.1.1 to get

$$|(\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]})(x_i, t_j)| \le \frac{x_i}{2\tau_2}\theta^{[1]}\mathbf{1} \quad \text{for } (x_i, t_j) \in \overline{Q}_\ell^{N,M}.$$

Hence

$$||\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}||_{\overline{Q}_\ell^{N,M}\setminus\overline{Q}_c} \le \frac{\theta^{[1]}}{2}, \quad \text{as } x_i \le \tau_2. \tag{6.2.21}$$

Similarly, we have

$$||\overline{\boldsymbol{U}}_r - \boldsymbol{U}_r^{[1]}||_{\overline{Q}_r^{N,M}\setminus\overline{Q}_c} \le \frac{\theta^{[1]}}{2}. \tag{6.2.22}$$

Next

$$[\boldsymbol{\mathcal{L}}_c^{N,M}(\overline{\boldsymbol{U}}_c - \boldsymbol{U}_c^{[1]})]_{i,j} = \mathbf{0} \;\text{ for } (x_i, t_j) \in Q_c^{N,M},$$

$$(\overline{\boldsymbol{U}}_c - \boldsymbol{U}_c^{[1]})(x_i, 0) = \mathbf{0} \;\text{ for } x_i \in \overline{\Omega}_c^N.$$

As $(\tau_2, t_j) \in \overline{Q}_\ell^{N,M}$ and $(1-\tau_2, t_j) \in \overline{Q}_r^{N,M}$, we have

$$\begin{aligned} |(\overline{\boldsymbol{U}}_c - \boldsymbol{U}_c^{[1]})(\tau_2, t_j)| &= |(\overline{\boldsymbol{U}}_c - \mathcal{T}_{t_j}\boldsymbol{U}_\ell^{[1]})(\tau_2, t_j)| \\ &\le |(\overline{\boldsymbol{U}}_c - \overline{\boldsymbol{U}}_\ell)(\tau_2, t_j)| + |(\overline{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]})(\tau_2, t_j)| \\ &\le \theta_{\tau_2}\mathbf{1} + \frac{\theta^{[1]}}{2}\mathbf{1}, \quad \text{for } t_j \in \omega^M, \end{aligned}$$

and

$$|(\overline{\boldsymbol{U}}_c - \boldsymbol{U}_c^{[1]})(1-\tau_2, t_j)| = |(\overline{\boldsymbol{U}}_c - \mathcal{T}_{t_j}\boldsymbol{U}_r^{[1]})(1-\tau_2, t_j)|$$

$$\leq |(\overline{\boldsymbol{U}}_c - \overline{\boldsymbol{U}}_r)(1-\tau_2, t_j)| + |(\overline{\boldsymbol{U}}_r - \boldsymbol{U}_r^{[1]})(1-\tau_2, t_j)|$$

$$\leq \theta_{\tau_2}\mathbf{1} + \frac{\theta^{[1]}}{2}\mathbf{1}, \quad \text{for } t_j \in \omega^M,$$

Then, using the discrete maximum principle of Lemma 6.1.1, it holds

$$||\overline{\boldsymbol{U}}_c - \boldsymbol{U}_c^{[1]}||_{\overline{Q}_c^{N,M}} \leq \theta_{\tau_2} + \frac{\theta^{[1]}}{2}. \tag{6.2.23}$$

Thus, using (6.2.21), (6.2.22) and (6.2.23), we have

$$||\overline{\boldsymbol{U}} - \boldsymbol{U}^{[1]}||_{\overline{Q}^{N,M}} \leq \theta_{\tau_2} + \frac{\theta^{[1]}}{2}. \tag{6.2.24}$$

Next, for bounding $||\overline{\boldsymbol{U}} - \boldsymbol{U}^{[2]}||_{\overline{Q}^{N,M}}$ we have to obtain a bound for $\theta^{[2]}$. Using a triangle inequality and stability of operator $\mathcal{T}_{t_j}$, we have

$$|(\overline{\boldsymbol{U}}_\ell - \mathcal{T}_{t_j}\boldsymbol{U}^{[1]})(2\tau_2, t_j)| \leq |(\overline{\boldsymbol{U}}_\ell - \mathcal{T}_{t_j}\overline{\boldsymbol{U}}_c)(2\tau_2, t_j)| + |\mathcal{T}_{t_j}(\overline{\boldsymbol{U}}_c - \boldsymbol{U}^{[1]})(2\tau_2, t_j)|$$

$$\leq \theta_{2\tau_2}\mathbf{1} + \theta_{\tau_2}\mathbf{1} + \frac{\theta^{[1]}}{2}\mathbf{1}, \quad \text{for } t_j \in \omega^M,$$

and

$$|(\overline{\boldsymbol{U}}_r - \mathcal{T}_{t_j}\boldsymbol{U}^{[1]})(1-2\tau_2, t_j)| \leq |(\overline{\boldsymbol{U}}_r - \mathcal{T}_{t_j}\overline{\boldsymbol{U}}_c)(1-2\tau_2, t_j)| + |\mathcal{T}_{t_j}(\overline{\boldsymbol{U}}_c - \boldsymbol{U}^{[1]})(1-2\tau_2, t_j)|$$

$$\leq \theta_{2\tau_2}\mathbf{1} + \theta_{\tau_2}\mathbf{1} + \frac{\theta^{[1]}}{2}\mathbf{1}, \quad \text{for } t_j \in \omega^M.$$

Hence

$$\theta^{[2]} \leq \theta_{2\tau_2} + \theta_{\tau_2} + \frac{\theta^{[1]}}{2}.$$

So, we have

$$\max\{||\overline{\boldsymbol{U}} - \boldsymbol{U}^{[1]}||_{\overline{Q}^{N,M}}, \theta^{[2]}\} \leq \frac{\theta^{[1]}}{2} + \mu, \quad \mu = \theta_{2\tau_2} + \theta_{\tau_2}.$$

Then, repeating the above procedure for further iterations, we have

$$\max\{||\overline{\boldsymbol{U}} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}}, \theta^{[k+1]}\} \leq \frac{\theta^{[k]}}{2} + \mu.$$

Simplifying the above inequality we get $\theta^{[k]} \leq 2^{-(k-1)}\theta^{[1]} + 2\mu$. Hence

$$||\overline{\boldsymbol{U}} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq 2^{-k}\theta^{[1]} + 2\mu. \tag{6.2.25}$$

Note that $\theta^{[1]} \leq C$. Also, from Lemma 6.2.3, $\theta_{\tau_2} \leq C(\Delta t + N^{-2}\ln^2 N)$, as $(\tau_2, t_j) \in$

$\overline{Q}_\ell^{N,M}$ and $(1-\tau_2, t_j) \in \overline{Q}_r^{N,M}$. Next we obtain a bound on $\theta_{2\tau_2}$. For bounding $\theta_{2\tau_2}$, use a triangle inequality to get

$$|(\overline{\boldsymbol{U}}_\ell - \mathcal{T}_{t_j}\overline{\boldsymbol{U}}_c)(2\tau_2, t_j)| = |(\boldsymbol{u} - \mathcal{T}_{t_j}\overline{\boldsymbol{U}}_c)(2\tau_2, t_j)|$$

$$\leq |(\boldsymbol{u} - \mathcal{T}_{t_j}\boldsymbol{u})(2\tau_2, t_j)| + |\mathcal{T}_{t_j}(\boldsymbol{u} - \overline{\boldsymbol{U}}_c)(2\tau_2, t_j)|.$$

By stability of $\mathcal{T}_{t_j}$ and Lemma 6.2.3, we have

$$\left|\mathcal{T}_{t_j}\left(\boldsymbol{u} - \overline{\boldsymbol{U}}_c\right)(2\tau_2, t_j)\right| \leq \boldsymbol{C}(\Delta t + N^{-2}\ln^2 N), \quad t_j \in \omega^M.$$

To bound the interpolation error we use solution decomposition $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ of Lemma 6.2.1 to get

$$|(\boldsymbol{u} - \mathcal{T}_{t_j}\boldsymbol{u})(2\tau_2, t_j)| \leq |(\boldsymbol{v} - \mathcal{T}_{t_j}\boldsymbol{v})(2\tau_2, t_j)| + |(\boldsymbol{w} - \mathcal{T}_{t_j}\boldsymbol{w})(2\tau_2, t_j)|$$

$$\leq \boldsymbol{C}h_c^2||\partial_x^2\boldsymbol{v}(., t_j)||_{[x_i, x_{i+1}]} + \boldsymbol{C}||\boldsymbol{w}(., t_j)||_{[x_i, x_{i+1}]}$$

$$\leq \boldsymbol{C}N^{-2}, \ t_j \in \omega^M.$$

Similarly

$$|(\boldsymbol{u} - \mathcal{T}_{t_j}\overline{\boldsymbol{U}}_c)(1 - 2\tau_2, t_j)| \leq \boldsymbol{C}(\Delta t + N^{-2}\ln^2 N), \ t_j \in \omega^M.$$

Hence

$$\theta_{2\tau_2} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Combine the bounds for θ_{τ_2} and $\theta_{2\tau_2}$ to get $\mu \leq C(\Delta t + N^{-2}\ln^2 N)$. This proves the lemma. □

We finally use Lemma 6.2.3 and Lemma 6.2.4 together with inequality (6.2.8), to prove the following main result of this chapter.

Theorem 6.2.5. *Let $\boldsymbol{U}^{[k]}$ be the k^{th} approximation to the exact solution of problem (6.0.1). Then*

$$||\boldsymbol{u} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq C(2^{-k} + \Delta t + N^{-2}\ln^2 N). \tag{6.2.26}$$

6.3 Numerical results

To validate the theoretical error bound in the previous section, we now present the numerical results for two test problems.

Example 6.3.1. Consider the following coupled system of singularly perturbed problems [32]

$$\begin{cases} \partial_t \boldsymbol{u} - \boldsymbol{E}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} & in\ Q := (0,1) \times (0,1], \\ \boldsymbol{u}(x,0) = \boldsymbol{0} & in\ [0,1], \\ \boldsymbol{u}(0,t) = \boldsymbol{0}, \boldsymbol{u}(1,t) = \boldsymbol{0} & in\ (0,1], \end{cases} \tag{6.3.1}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 2(1+x)^2 & -(1+x^3) \\ -2\cos(\pi x/4) & 2.2e^{1-x} \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} \cos(\pi x/2) \\ x \end{pmatrix}.$$

The exact solution of this problem is not known. Choosing $tol = N^{-2}$, we denote the final computed solution by $\boldsymbol{U}^{N,\Delta t}$. We now use double mesh differences to estimate errors for different value of $\boldsymbol{\varepsilon}, N$ and Δt by $E_{n,\boldsymbol{\varepsilon}}^{N,\Delta t} = ||U_n^{N,\Delta t} - U_n^{2N,\Delta t/4}||_{\overline{Q}^{N,M}}$, $n = 1, 2$, where $U_n^{2N,\Delta t/4}$ denotes componentwise approximate solution computed using $2N$ spatial mesh intervals and time step $\Delta t/4$ in each subdomain.

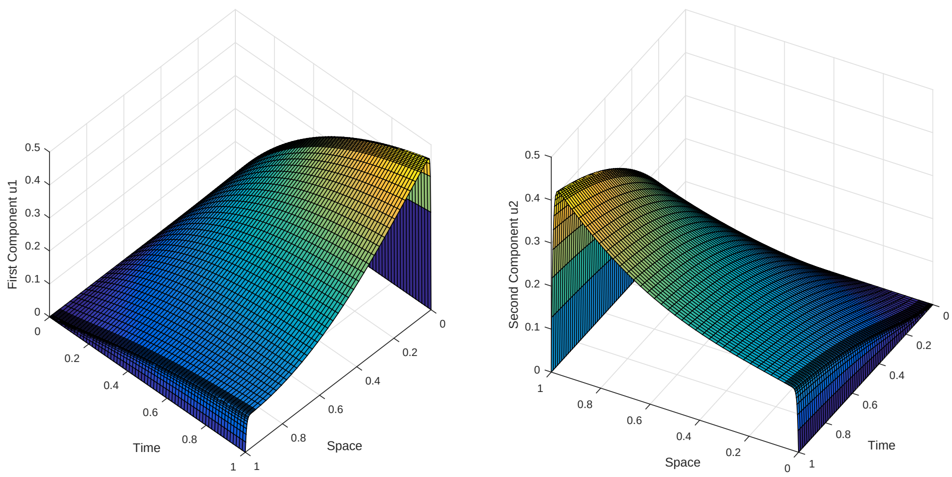

Figure 6.1: Numerical solution for $\varepsilon_1 = 10^{-5}, \varepsilon_2 = 10^{-4}$ with $N = 32, M = 64$ in Example 6.3.1. The first and second components are depicted in the left and right figures, respectively.

Fixing ε_1 by taking $\varepsilon_1 = 10^{-n}$, for some non-negative integer n, we compute

$$E_{n,\varepsilon_1}^{N,\Delta t} = \max\{E_{n,(\varepsilon_1,1)}^{N,\Delta t}, E_{n,(\varepsilon_1,10^{-1})}^{N,\Delta t}, \ldots, E_{n,(\varepsilon_1,10^{-n})}^{N,\Delta t}\}.$$

We now compute the uniform error as $E_n^{N,\Delta t} = \max_{\varepsilon_1} E_{n,\varepsilon_1}^{N,\Delta t}$, and the rates of conver-

Table 6.1: Errors $E_{n,\varepsilon_1}^{N,\Delta t}$ and $E_n^{N,\Delta t}$, and convergence rates $\rho_{n,\varepsilon_1}^{N,\Delta t}$ and $\rho_n^{N,\Delta t}$ for Example 6.3.1.

ε_1	$N=2^5$ $\Delta t=1/4$	$N=2^6$ $\Delta t=1/4^2$	$N=2^7$ $\Delta t=1/4^3$	$N=2^8$ $\Delta=1/4^4$	$N=2^9$ $\Delta t=1/4^5$
	2.184e-02	7.624e-03	2.155e-03	5.565e-04	1.403e-04
10^{-1}	1.519	1.823	1.953	1.988	
	1.795e-02	6.017e-03	1.645e-03	4.224e-04	1.064e-04
	1.577	1.872	1.961	1.990	
	2.620e-02	8.537e-03	2.318e-03	5.923e-04	1.489e-04
10^{-2}	1.618	1.881	1.969	1.992	
	2.405e-02	7.499e-03	2.010e-03	5.121e-04	1.287e-04
	1.681	1.899	1.973	1.993	
	2.902e-02	9.030e-03	2.423e-03	6.174e-04	1.551e-04
10^{-3}	1.684	1.898	1.973	1.993	
	2.721e-02	8.390e-03	2.241e-03	5.703e-04	1.432e-04
	1.698	1.905	1.974	1.994	
	3.067e-02	1.071e-02	3.190e-03	9.070e-04	2.539e-04
10^{-4}	1.518	1.747	1.814	1.837	
	2.789e-02	8.613e-03	2.343e-03	6.141e-04	1.595e-04
	1.695	1.878	1.932	1.945	
	3.074e-02	1.070e-02	3.187e-03	9.062e-04	2.545e-04
10^{-5}	1.522	1.748	1.814	1.832	
	2.810e-02	8.662e-03	2.354e-03	6.160e-04	1.600e-04
	1.698	1.880	1.934	1.945	
	3.076e-02	1.070e-02	3.186e-03	9.060e-04	2.545e-04
10^{-6}	1.523	1.748	1.814	1.832	
	2.816e-02	8.678e-03	2.357e-03	6.167e-04	1.601e-04
	1.698	1.880	1.935	1.945	
	3.076e-02	1.070e-02	3.186e-03	9.059e-04	2.544e-04
10^{-7}	1.524	1.748	1.814	1.832	
	2.818e-02	9.236e-03	2.467e-03	6.234e-04	1.602e-04
	1.609	1.904	1.985	1.961	
	3.077e-02	1.070e-02	3.186e-03	9.059e-04	2.544e-04
10^{-8}	1.524	1.748	1.814	1.832	
	2.880e-02	9.679e-03	2.694e-03	7.263e-04	1.925e-04
	1.573	1.845	1.891	1.916	
$E_1^{N,\Delta t}$	3.077e-02	1.071e-02	3.190e-03	9.070e-04	2.545e-04
$\rho_1^{N,\Delta t}$	1.523	1.747	1.814	1.833	
$E_2^{N,\Delta t}$	2.880e-02	9.679e-03	2.694e-03	7.263e-04	1.925e-04
$\rho_2^{N,\Delta t}$	1.573	1.845	1.891	1.916	

gence by $$\rho_{n,\varepsilon_1}^{N,\Delta t} = \log_2\left(\frac{E_{n,\varepsilon_1}^{N,\Delta t}}{E_{n,\varepsilon_1}^{2N,\Delta t/4}}\right), \qquad \rho_n^{N,\Delta t} = \log_2\left(\frac{E_n^{N,\Delta t}}{E_n^{2N,\Delta t/4}}\right).$$

Table 6.1 shows the componentwise maximum errors when $\varepsilon_1 = 10^{-n}$ is fixed and ε_2 belongs to the set $S = \{\varepsilon_2 = 10^{-m} : m = 0, 1, \ldots, n\}$, with the corresponding convergence rates, and the componentwise uniform errors with corresponding uniform

Table 6.2: Iteration counts k_1 (k_2) for the proposed method (for the method given in Chapter 4) taking fixed $\varepsilon_1 = 10^{-8}$ for Example 6.3.1.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	5 (3)	6 (4)	8 (5)	9 (6)	11 (6)
1	3 (2)	3 (3)	4 (4)	4 (5)	4 (6)
2	2 (2)	2 (3)	2 (5)	2 (6)	2 (7)
3	1 (3)	1 (4)	1 (5)	1 (6)	1 (7)
4	1 (3)	1 (4)	1 (5)	1 (5)	1 (6)
5	1 (3)	1 (3)	1 (3)	1 (3)	1 (3)
6	1 (2)	1 (2)	1 (2)	1 (2)	1 (2)
7	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
8	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)

convergence rates. For each value of ε_1, the 1^{st} and 2^{nd} rows represent the results corresponding to the first component, and the 3^{rd} and 4^{th} ones correspond to the second component. Table 6.1 also indicates that the numerical results are in agreement with the theoretical result in Theorem 6.2.5. In Table 6.2, we give iteration counts for the proposed method and also in round brackets iteration counts for the method developed in Chapter 4. From this table, we see that a less number of iterations are required to reach convergence for the proposed method than those required for the method developed in Chapter 4.

Example 6.3.2. Consider the following coupled system of singularly perturbed problems [35]

$$\begin{cases} \partial_t \boldsymbol{u} - \boldsymbol{E}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } Q := (0,1) \times (0,1], \\ \boldsymbol{u}(x,0) = \boldsymbol{0} \quad \text{in } [0,1], \quad \boldsymbol{u}(0,t) = \boldsymbol{g}_0(t), \; \boldsymbol{u}(1,t) = \boldsymbol{g}_1(t) \quad \text{in } (0,1], \end{cases} \tag{6.3.2}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix},$$

where $f_1, f_2, \boldsymbol{g}_0$ and $\boldsymbol{g}_1$ are taken in order that

$$u_1(x,t) = t(\varphi_1(x) + \varphi_2(x) - 2) + (t + xt)e^{-t},$$

$$u_2(x,t) = \varepsilon_1(1 - e^{-t})(\varphi_1(x) - 1) + (t - t^2)(\varphi_2(x) - 1),$$

with

$$\varphi_i(x) = \frac{e^{-x/\sqrt{\varepsilon_i}} + e^{-(1-x)/\sqrt{\varepsilon_i}}}{1 + e^{-1/\sqrt{\varepsilon_i}}}, \quad i = 1, 2,$$

is the exact solution.

Table 6.3: Errors $E_{n,\varepsilon_1}^{N,\Delta t}$ and $E_n^{N,\Delta t}$, and convergence rates $\rho_{n,\varepsilon_1}^{N,\Delta t}$ and $\rho_n^{N,\Delta t}$ for Example 6.3.2.

ε_1	$N=2^5$ $\Delta t=1/4$	$N=2^6$ $\Delta t=1/4^2$	$N=2^7$ $\Delta t=1/4^3$	$N=2^8$ $\Delta=1/4^4$	$N=2^9$ $\Delta t=1/4^5$
	6.206e-02	1.766e-02	4.590e-03	1.159e-03	2.906e-04
10^{-1}	1.813	1.944	1.985	1.996	
	3.404e-02	8.475e-03	2.108e-03	5.262e-04	1.315e-04
	2.006	2.007	2.002	2.001	
	7.733e-02	2.210e-02	5.722e-03	1.444e-03	3.618e-04
10^{-2}	1.807	1.949	1.987	2.000	
	8.324e-02	2.130e-02	5.350e-03	1.339e-03	3.348e-04
	1.967	1.993	1.998	2.000	
	8.297e-02	2.375e-02	6.144e-03	1.550e-03	3.884e-04
10^{-3}	1.804	1.951	1.987	1.997	
	8.969e-02	2.308e-02	5.808e-03	1.454e-03	3.637e-04
	1.958	1.991	1.998	1.999	
	8.517e-02	2.432e-02	6.279e-03	1.582e-03	3.959e-04
10^{-4}	1.808	1.954	1.989	1.998	
	9.202e-02	2.373e-02	5.977e-03	1.497e-03	3.744e-04
	1.955	1.990	1.997	1.999	
	8.626e-02	2.463e-02	6.359e-03	1.603e-03	4.014e-04
10^{-5}	1.808	1.953	1.988	1.997	
	9.286e-02	2.398e-02	6.041e-03	1.513e-03	3.784e-04
	1.953	1.989	1.997	1.999	
	8.709e-02	2.479e-02	6.391e-03	1.611e-03	4.036e-04
10^{-6}	1.813	1.956	1.988	1.997	
	9.312e-02	2.407e-02	6.064e-03	1.519e-03	3.799e-04
	1.952	1.989	1.997	1.999	
	8.740e-02	2.494e-02	6.435e-03	1.626e-03	4.091e-04
10^{-7}	1.809	1.955	1.984	1.991	
	9.321e-02	2.410e-02	6.072e-03	1.521e-03	3.804e-04
	1.951	1.989	1.997	1.999	
	8.747e-02	2.499e-02	6.465e-03	1.635e-03	4.115e-04
10^{-8}	1.808	1.951	1.983	1.990	
	9.323e-02	2.411e-02	6.075e-03	1.522e-03	3.806e-04
	1.951	1.989	1.997	1.999	
$E_1^{N,\Delta t}$	8.747e-02	2.499e-02	6.465e-03	1.635e-03	4.115e-04
$\rho_1^{N,\Delta t}$	1.808	1.951	1.983	1.990	
$E_2^{N,\Delta t}$	9.323e-02	2.411e-02	6.075e-03	1.522e-03	3.806e-04
$\rho_2^{N,\Delta t}$	1.951	1.989	1.997	1.999	

For this test problem, we compute solution error by $E_{n,\boldsymbol{\varepsilon}}^{N,\Delta t} = ||u_n - U_n^{N,\Delta t}||_{\overline{Q}^{N,M}}$. For different values of $\varepsilon_1, \varepsilon_2, N, \Delta t$, the errors $E_{n,\varepsilon_1}^{N,\Delta t}, E_n^{N,\Delta t}$ and rates of convergence are computed as described for Example 6.3.1. In Table 6.3, we present the maximum errors $E_{n,\varepsilon_1}^{N,\Delta t}$ and the uniform errors $E_n^{N,\Delta t}$ for some values of ε_1, N, and Δt, with

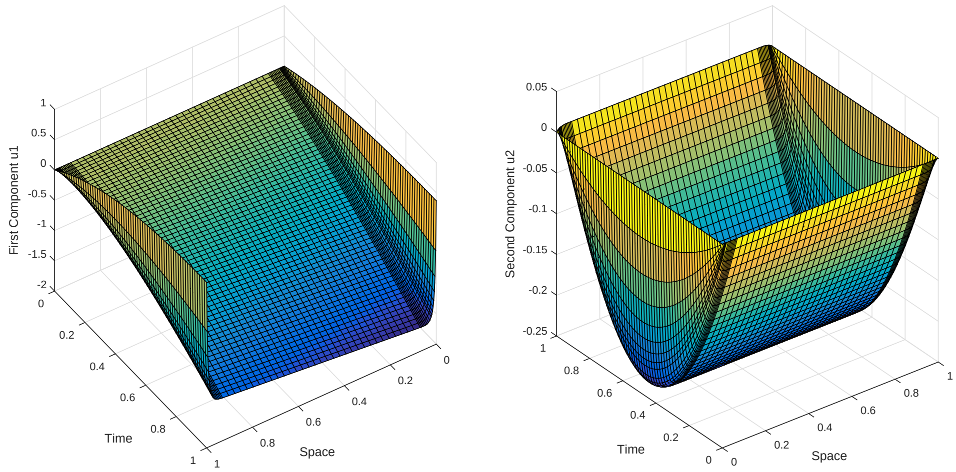

Figure 6.2: Numerical solution for $\varepsilon_1 = 10^{-5}, \varepsilon_2 = 10^{-4}$ with $N = 32, M = 64$ in Example 6.3.2. The first and second components are depicted in the left and right figures, respectively.

Table 6.4: Iteration counts k_1 (k_2) for the proposed method (for the method given in Chapter 4) taking fixed $\varepsilon_1 = 10^{-8}$ for Example 6.3.2.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	4 (2)	6 (3)	7 (4)	9 (5)	11 (6)
1	3 (2)	4 (3)	4 (3)	4 (4)	5 (5)
2	2 (2)	2 (3)	2 (4)	2 (5)	2 (6)
3	1 (2)	1 (3)	1 (4)	1 (5)	1 (6)
4	1 (3)	1 (4)	1 (4)	1 (5)	1 (6)
5	1 (3)	1 (3)	1 (3)	1 (3)	1 (3)
6	1 (2)	1 (2)	1 (2)	1 (2)	1 (2)
7	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)
8	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)

the corresponding convergence rates $\rho_{n,\varepsilon_1}^{N,\Delta t}$ and the uniform convergence rates $\rho_n^{N,\Delta t}$. From this table, we can clearly see that the results are according to the error estimate in Theorem 6.2.5. Table 6.4 gives iteration counts for the proposed method and also in round brackets iteration counts for the method given in Chapter 4. It can be seen from Table 6.4 that the proposed method is more efficient than the method developed in Chapter 4, considering number of iterations required within the same convergence criteria. The used CPU time in seconds for the proposed method for Examples 6.3.1 and 6.3.2 is given in Table 6.5. These results are computed using MATLAB software installed on a laptop equipped with an Intel(R) Core(TM) i3-3227U CPU with 1.90GHz speed and 8 GB RAM running on a 64 bit windows8 operating system.

Table 6.5: The used CPU time in seconds for Examples 6.3.1 and 6.3.2 with $\varepsilon_1 = 10^{-8}$, $\varepsilon_2 = 10^{-7}$.

	$N = 2^5$ $\Delta t = 0.25$	$N = 2^6$ $\Delta t = 0.25/4$	$N = 2^7$ $\Delta t = 0.25/4^2$	$N = 2^8$ $\Delta t = 0.25/4^3$
Example 6.3.1	0.315356	0.420821	1.530240	27.976567
Example 6.3.2	0.378820	0.754524	3.590116	35.250618

6.4 Conclusions

An improved domain decomposition method of SWR type is designed and analyzed for solving coupled system of singularly perturbed time-dependent reaction-diffusion problems. The method is based upon partitioning the computational domain into three overlapping subdomains and distinguishes itself from other domain decomposition methods in the usage of non-uniform meshes rather than uniform meshes in boundary layer subdomains. In each subdomain, a combination of the central difference approximation in space and the backward Euler difference approximation in time is used to solve subdomain problems locally. We then merge the local solutions obtained on overlapping subdomains to get the final solution, which provides uniformly convergent approximation of almost second order in space and first order in time.