5

A robust domain decomposition method for singularly perturbed parabolic reaction-diffusion systems with time delay

We consider the following coupled system of singularly perturbed parabolic reaction-diffusion problems with time delay

$$\begin{cases} \mathscr{L}\boldsymbol{u}(x,t) := \boldsymbol{\mathcal{L}}\boldsymbol{u}(x,t) + \boldsymbol{B}\boldsymbol{u}(x,t-\tau) = \boldsymbol{f}(x,t) & \text{in } Q := \Omega \times (0,T] = (0,1)\times(0,T], \\ \boldsymbol{u}(x,t) = \boldsymbol{\phi}(x,t) & \text{in } \Gamma_b = [0,1]\times[-\tau,0], \\ \boldsymbol{u}(0,t) = \boldsymbol{\gamma}_0(t), \boldsymbol{u}(1,t) = \boldsymbol{\gamma}_1(t) & \text{in } (0,T], \end{cases} \tag{5.0.1}$$

where $\boldsymbol{\mathcal{L}}\boldsymbol{u}(x,t) = \partial_t \boldsymbol{u}(x,t) - \boldsymbol{E}\partial_x^2 \boldsymbol{u}(x,t) + \boldsymbol{A}\boldsymbol{u}(x,t)$, $\boldsymbol{u} = (u_1, u_2)^T$, $\mathbf{E} = \text{diag}(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ and $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. For each $(x,t) \in \overline{Q}$, the coupling matrices $\boldsymbol{A} = (\, a_{ij}(x,t)\,), \boldsymbol{B} = (\, b_{ij}(x,t)\,)$ are assumed to satisfy

$$a_{ij}(x,t) \leq 0, \quad i \neq j, \quad \text{and} \quad b_{ij}(x,t) \leq 0, \quad i,j = 1,2, \tag{5.0.2}$$

$$a_{ii}(x,t) > 0, \quad \sum_{j=1}^{2}(a_{ij} + b_{ij})(x,t) \geq \alpha > 0, \quad i = 1,2. \tag{5.0.3}$$

Moreover, we assume that data $\boldsymbol{f} = (f_1, f_2)^T$, $\boldsymbol{\gamma}_0 = (\gamma_{01}, \gamma_{02})^T, \boldsymbol{\gamma}_1 = (\gamma_{11}, \gamma_{12})^T, \boldsymbol{\phi} = (\phi_1, \phi_2)^T$, and the coupling matrices $\boldsymbol{A}, \boldsymbol{B}$ satisfy sufficient regularity and compatibility conditions in order that there exists a unique solution $\boldsymbol{u} \in C^{4,2}(\overline{Q})^2$, cf. [72].

In this chapter, we extend our work in Chapters 2 and 4 to coupled systems of singularly perturbed time delay PDEs (5.0.1). We establish a priori bounds on the solution and its partial derivatives. Based on a priori estimates of the solution we decompose the problem domain into five overlapping subdomains. Introducing a uniform mesh in both space and time, on each subdomain we consider central difference approximation and Euler implicit approximation to discretize the model problem in the spatial and the time direction, respectively. We then formulate an iterative algorithm to approximate the solution of the model problem. The analysis of uniform convergence is conducted based on some auxiliary problems. We prove that the method is of second order accurate in space and first order accurate in time. Finally, numerical results are given to support our theoretical error bounds.

5.1 Derivative bounds

In order to set-up and to study convergence behavior of the method, we shall derive some a priori bounds. We need the following maximum principle results.

Lemma 5.1.1. *Suppose* $\boldsymbol{\psi}(x,t) \geq \mathbf{0}$ *for* $(x,t) \in \Gamma_b$ *and* $\boldsymbol{\psi}(0,t) \geq \mathbf{0}$, $\boldsymbol{\psi}(1,t) \geq \mathbf{0}$ *for* $t \in (0,T]$. *Then* $\boldsymbol{\mathcal{L}}\boldsymbol{\psi} \geq \mathbf{0}$ *in* Q *implies that* $\boldsymbol{\psi} \geq \mathbf{0}$ *in* $\overline{Q}$.

Proof. See [29]. □

Lemma 5.1.2. *Suppose* $\boldsymbol{\varphi}(x,t) \geq \mathbf{0}$ *for* $(x,t) \in \Gamma_b$ *and* $\boldsymbol{\varphi}(0,t) \geq \mathbf{0}$, $\boldsymbol{\varphi}(1,t) \geq \mathbf{0}$ *for* $t \in (0,T]$. *If* $\boldsymbol{\pounds}\boldsymbol{\varphi} \geq \mathbf{0}$ *in* Q *then* $\boldsymbol{\varphi}(x,t) \geq \mathbf{0}$ *in* $\overline{Q}$.

Proof. Supposing $\boldsymbol{\psi} = \boldsymbol{\varphi}$ in $[0,1] \times [-\tau, \tau]$, we have

$$\boldsymbol{\psi}(x,t) \geq \mathbf{0} \ \text{ for } (x,t) \in \Gamma_b \ \text{ and } \ \boldsymbol{\psi}(0,t) \geq \mathbf{0},\ \boldsymbol{\psi}(1,t) \geq \mathbf{0} \ \text{ for } \ t \in (0,\tau].$$

Also

$$\boldsymbol{\pounds}\boldsymbol{\psi}(x,t) \geq -\boldsymbol{B}(x,t)\boldsymbol{\varphi}(x,t-\tau) \geq \mathbf{0} \ \text{ for } (x,t) \in (0,1) \times (0,\tau],$$

as $b_{i,j} \leq 0$ and $\boldsymbol{\psi} \geq \mathbf{0}$ in $[0,1] \times [-\tau, 0]$. Hence, Lemma 5.1.1 gives $\boldsymbol{\varphi} = \boldsymbol{\psi} \geq \mathbf{0}$ in $[0,1] \times [0,\tau]$. Now we can show that $\boldsymbol{\varphi} \geq \mathbf{0}$ in $[0,1] \times [i\tau, (i+1)\tau]$, $i \geq 1$, using $\boldsymbol{\varphi} \geq \mathbf{0}$ in $[0,1] \times [(i-1)\tau, i\tau]$, and using the earlier argument. □

The following lemma proves that the first two time derivatives are bounded independent of the perturbation parameters.

Lemma 5.1.3. *Suppose* $\boldsymbol{u}$ *is the solution of problem (5.0.1). Then*

$$\|\partial_t^m u_n\|_{\overline{Q}} \leq C \quad \text{for } m = 0,1,2,\ n = 1,2. \tag{5.1.1}$$

Proof. Choosing a constant barrier function the result for $m = 0$ can be easily proved using the maximum principle for $\boldsymbol{\pounds}$. We now assume that (5.1.1) holds for $m = 0, \ldots, \kappa - 1$, $1 \leq \kappa \leq 2$. We need to establish result for $m = \kappa$. Defining

$\boldsymbol{\Psi} = \partial_t^{\kappa}\boldsymbol{u}$, it holds

$$\begin{cases} \boldsymbol{\mathcal{L}}\boldsymbol{\Psi}(x,t) := \partial_t\boldsymbol{\Psi}(x,t) - \boldsymbol{E}\partial_x^2\boldsymbol{\Psi}(x,t) + \boldsymbol{A}\boldsymbol{\Psi}(x,t) + \boldsymbol{B}\boldsymbol{\Psi}(x,t-\tau) \\ = \partial_t^{\kappa}\boldsymbol{f}(x,t) - \sum_{l=0}^{\kappa-1}\begin{pmatrix}\kappa \\ l\end{pmatrix}\boldsymbol{A}_t^{(\kappa-l)}\partial_t^l\boldsymbol{u}(x,t) - \sum_{l=0}^{\kappa-1}\begin{pmatrix}\kappa \\ l\end{pmatrix}\boldsymbol{B}_t^{(\kappa-l)}\partial_t^l\boldsymbol{u}(x,t-\tau) \\ := \boldsymbol{\Psi}_\kappa \quad \text{in } Q = (0,1)\times(0,T], \end{cases}$$

with $|\boldsymbol{\Psi}(x,t)| \leq \mathbf{C}$ for $(x,t) \in \Gamma_b$ and $(x,t) \in \{0,1\}\times(0,T]$. Now using hypothesis of induction, we obtain $|\boldsymbol{\Psi}_\kappa(x,t)| \leq \mathbf{C}$. Thus, we get the required result using the maximum principle for $\boldsymbol{\mathcal{L}}$ together with a constant barrier function. □

Next we write the solution of problem (5.0.1) as $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, where $\boldsymbol{v}$ is the regular part satisfying

$$\begin{cases} \boldsymbol{\mathcal{L}}\boldsymbol{v}(x,t) = \boldsymbol{f}(x,t) & \text{for } (x,t) \in Q, \\ \boldsymbol{v}(x,t) = \boldsymbol{\phi}(x,t) & \text{for } (x,t) \in \Gamma_b, \\ \boldsymbol{v}(x,t) = \boldsymbol{\chi}(x,t) & \text{for } (x,t) \in \{0,1\}\times(0,T], \end{cases} \tag{5.1.2}$$

with $\boldsymbol{\chi}$ satisfying

$$\begin{cases} \partial_t\boldsymbol{\chi}(x,t) + \boldsymbol{A}\boldsymbol{\chi}(x,t) + \boldsymbol{B}\boldsymbol{\chi}(x,t)(x,t-\tau) = \boldsymbol{f}(x,t), & (x,t) \in \{0,1\}\times(0,T], \\ \boldsymbol{\chi}(x,t) = \boldsymbol{\phi}(x,t) & (x,t) \in \Gamma_b, \end{cases} \tag{5.1.3}$$

and $\boldsymbol{w}$ is the singular part satisfying

$$\begin{cases} \boldsymbol{\mathcal{L}}\boldsymbol{w}(x,t) = \mathbf{0} & \text{for } (x,t) \in Q, \\ \boldsymbol{w}(x,t) = (\boldsymbol{u}-\boldsymbol{v})(x,t) & \text{for } (x,t) \in \Gamma_b \cup (\{0,1\}\times(0,T]). \end{cases} \tag{5.1.4}$$

Lemma 5.1.4. *The solution $\boldsymbol{v}$ of problem (5.1.2) satisfies*

$$\left\|\partial_t^k v_n\right\|_{\overline{Q}} \leq C \quad \textit{for } k = 0,1,2, \; n = 1,2. \tag{5.1.5}$$

$$\left\|\partial_x^k v_n\right\|_{\overline{Q}} \leq C(1+\varepsilon_n^{1-k/2}) \quad \textit{for } k = 1,2,3,4, \; n = 1,2. \tag{5.1.6}$$

Proof. The proof of (5.1.5) follows using agruments similar to Lemma 5.1.3. Thus, we only neeed to establish the bounds (5.1.6). Differentiating the equation in (5.1.2) twice with respect to x and defining $\boldsymbol{z}(x,t)=\partial_x^\kappa \boldsymbol{v}(x,t), \kappa=2,(x,t)\in Q$, it holds

$$\begin{aligned}|\boldsymbol{\mathcal{L}}\boldsymbol{z}(x,t)| &= \left|\partial_x^\kappa \boldsymbol{f}(x,t)-\textstyle\sum_{l=0}^{\kappa-1}\begin{pmatrix}\kappa\\ l\end{pmatrix}\boldsymbol{A}_x^{(\kappa-l)}\partial_x^l\boldsymbol{v}(x,t)-\sum_{l=0}^{\kappa}\begin{pmatrix}\kappa\\ l\end{pmatrix}\boldsymbol{B}_x^{(\kappa-l)}\partial_x^l\boldsymbol{v}(x,t-\tau)\right|\\ &\le (1+\|\boldsymbol{v}_x\|_{\overline{Q}})\mathbf{C}.\end{aligned} \tag{5.1.7}$$

Further, since the problem for $\boldsymbol{\chi}$ is independent of small parameters ε_1 and ε_2, so one can use classical arguments to prove that the derivatives of $\boldsymbol{\chi}$ are bounded independent of ε_1 and ε_2. Hence, we obtain $|\boldsymbol{z}(x,t)|\le \mathbf{C}$ for $(x,t)\in\Gamma_b$ and $(x,t)\in\{0,1\}\times(0,T]$. Now using the maximum principle for $\boldsymbol{\mathcal{L}}$ with barrier function $\boldsymbol{\varphi}=\mathbf{C}(1+\|\partial_x\boldsymbol{v}\|_{\overline{Q}})t$, we obtain

$$\left\|\partial_x^2\boldsymbol{v}\right\|_{\overline{Q}}\le \mathbf{C}_*(1+\|\partial_x\boldsymbol{v}\|_{\overline{Q}}) \tag{5.1.8}$$

Further, by using the argument given in [76], we get

$$\|\partial_x\boldsymbol{v}\|_{\overline{Q}}\le \mathbf{C}+\left\|\partial_x^2\boldsymbol{v}\right\|_{\overline{Q}}/2\mathbf{C}_* \tag{5.1.9}$$

From (5.1.8) and (5.1.9), we have

$$\left\|\partial_x^k v_n\right\|_{\overline{Q}}\le C \quad \text{for } k=1,2,\ n=1,2.$$

Next, before establishing bounds on $\left\|\partial_x^k v_n\right\|_{\overline{Q}}$, for $k=3,4,\ n=1,2$, we need to derive bounds on $\left\|\partial_{x,t}^{1,1}\boldsymbol{v}\right\|_{\overline{Q}}$ and $\left\|\partial_{x,t}^{2,1}\boldsymbol{v}\right\|_{\overline{Q}}$. For this, we differentiate the equation in (5.1.2) twice and once, respectively, with respect to x and t, and define $\tilde{\boldsymbol{z}}=\partial_{x,t}^{\kappa,1}\boldsymbol{v},\kappa=2$, to get

$$\begin{aligned}\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{z}}(x,t) &= \partial_{x,t}^{\kappa,1}\boldsymbol{f}(x,t)-\textstyle\sum_{l=0}^{\kappa}\begin{pmatrix}\kappa\\ l\end{pmatrix}\boldsymbol{A}_{x,t}^{(\kappa-l),1}\partial_x^l\boldsymbol{v}(x,t)-\sum_{l=0}^{\kappa-1}\begin{pmatrix}\kappa\\ l\end{pmatrix}\boldsymbol{A}_x^{(\kappa-l)}\partial_{x,t}^{l,1}\boldsymbol{v}(x,t)\\ &\quad-\textstyle\sum_{l=0}^{\kappa}\begin{pmatrix}\kappa\\ l\end{pmatrix}\boldsymbol{B}_{x,t}^{(\kappa-l),1}\partial_x^l\boldsymbol{v}(x,t-\tau)-\sum_{l=0}^{\kappa-1}\begin{pmatrix}\kappa\\ l\end{pmatrix}\boldsymbol{B}_x^{(\kappa-l)}\partial_{x,t}^{l,1}\boldsymbol{v}(x,t-\tau)\end{aligned}$$

which implies

$$|\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{z}}(x,t)|\le(1+\left\|\partial_{x,t}^{1,1}\boldsymbol{v}\right\|_{\overline{Q}})\mathbf{C}.$$

Again using the previous arguments we obtain $|\tilde{\boldsymbol{z}}(x,t)|\le\mathbf{C}$ for $(x,t)\in\{0,1\}\times(0,T]$ and $(x,t)\in\Gamma_b$. Now applying the maximum principle together with barrier function $\boldsymbol{\varphi}_1=\mathbf{C}t(1+\left\|\partial_{x,t}^{1,1}\boldsymbol{v}\right\|_{\overline{Q}})$, we get $\left\|\partial_{x,t}^{2,1}\boldsymbol{v}\right\|_{\overline{Q}}\le\mathbf{C}_*(1+\left\|\partial_{x,t}^{1,1}\boldsymbol{v}\right\|_{\overline{Q}})$. Again following the argument from [76], we get

$$\left\|\partial_{x,t}^{1,1}\boldsymbol{v}\right\|_{\overline{Q}} \le \mathbf{C} + \left\|\partial_{x,t}^{2,1}\boldsymbol{v}\right\|_{\overline{Q}}/2\mathbf{C}_* \tag{5.1.10}$$

Thus, we have $\left\|\partial_{x,t}^{1,1}\boldsymbol{v}\right\|_{\overline{Q}} \le \mathbf{C}$ and $\left\|\partial_{x,t}^{2,1}\boldsymbol{v}\right\|_{\overline{Q}} \le \mathbf{C}$.

Now from the $n^{th}, n = 1, 2$, equation of system (5.1.7) for $\kappa = 2$, with previously obtained bounds, we get $\|\partial_x^4 v_n\|_{\overline{Q}} \le c\varepsilon_n^{-1}$. So, we can apply the argument in [76] to get $\|\partial_x^3 v_n\|_{\overline{Q}} \le c\varepsilon_n^{-1/2}$. Hence, the proof is complete. □

Lemma 5.1.5. *Suppose $\boldsymbol{w}$ is the solution of problem (5.1.4). Then*

$$\left\|\partial_t^k w_n\right\|_{\overline{Q}} \le C\mathcal{B}_{\varepsilon_2}(x), \quad \textit{for} \;\; k = 0, 1, 2, \quad n = 1, 2.$$

Proof. The result for $k = 0$ holds trivially with the barrier function $\boldsymbol{\varphi} = \mathbf{C}e^{2\alpha t}\mathcal{B}_{\varepsilon_2}(x)$, by noting that $|\boldsymbol{w}(x,t)| \le \mathbf{C}$ for $(x,t) \in \Gamma_b$ and $(x,t) \in \{0,1\} \times (0,T]$. To apply induction here, we suppose that lemma holds for $k = 0, \ldots, m-1$, $1 \le m \le 2$. Now we will derive bounds for $k = m$. Defining $\tilde{\mathbf{g}} = \partial_t^m \boldsymbol{w}$, we have

$$\begin{cases} \boldsymbol{\mathcal{L}}\tilde{\mathbf{g}}(x,t) := \partial_t\tilde{\mathbf{g}}(x,t) - \boldsymbol{E}\partial_x^2\tilde{\mathbf{g}}(x,t) + \boldsymbol{A}\tilde{\mathbf{g}}(x,t) - \boldsymbol{B}\tilde{\mathbf{g}}(x,t-\tau) \\ = -\sum_{l=0}^{m-1}\begin{pmatrix} m \\ l \end{pmatrix}\boldsymbol{A}_t^{(m-l)}\partial_t^l\boldsymbol{w}(x,t) - \sum_{l=0}^{m-1}\begin{pmatrix} m \\ l \end{pmatrix}\boldsymbol{B}_t^{(m-l)}\partial_t^l\boldsymbol{w}(x,t-\tau) \\ := \tilde{\mathbf{g}}_m \;\; \text{in } Q = (0,1) \times (0,T] \end{cases}$$

with $|\tilde{\mathbf{g}}(x,t)| \le \mathbf{C}$ in Γ_b and $|\tilde{\mathbf{g}}(x,t)| \le \mathbf{C}$ for $(x,t) \in \{0,1\} \times (0,T]$, where the bounds on initial and boundary conditions follow from previous arguments. Using the induction hypothesis, we get $|\tilde{\mathbf{g}}_m(x,t)| \le \boldsymbol{\mathcal{L}}\boldsymbol{\varphi}(x,t)$. Hence, using the maximum principle for $\boldsymbol{\mathcal{L}}$ with the barrier function $\boldsymbol{\varphi}$ we obtain the required result. □

Lemma 5.1.6. *Suppose $\boldsymbol{w}$ is the solution of problem (5.1.4). Then*

$$|\partial_x^\kappa w_n(x,t)| \le \begin{cases} C\mathcal{B}_{\varepsilon_2}(x), \quad \kappa = 0, n = 1, 2, \\ C(\varepsilon_1^{-\kappa/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-\kappa/2}\mathcal{B}_{\varepsilon_2}(x)), \;\; \kappa = 1, 2, 3, 4, n = 1, \\ C\varepsilon_2^{-\kappa/2}\mathcal{B}_{\varepsilon_2}(x), \;\; \kappa = 1, 2, n = 2 \\ C\varepsilon_2^{-1}(\varepsilon_1^{-(\kappa-2)/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-(\kappa-2)/2}\mathcal{B}_{\varepsilon_2}(x)), \quad \kappa = 3, 4, n = 2, \end{cases}$$

for all $(x,t) \in \overline{Q}$.

Proof. The proof follows using the arguments similar to [29, Lemma 9]. □

Lemma 5.1.7. *Suppose that $\varepsilon_1 < \varepsilon_2$ and $\varepsilon_2 \le \alpha/2$. Then the solution $\boldsymbol{w} = (w_1, w_2)^T$*

of problem (5.1.4) can be expressed as $w_n = \widehat{w}_{n,\varepsilon_1} + \widehat{w}_{n,\varepsilon_2}, \quad n = 1, 2,$ *where*

$$|\widehat{w}_{n,\varepsilon_1}(x)| \leq \mathcal{B}_{\varepsilon_1}(x,t), \quad |\partial_x^2 \widehat{w}_{n,\varepsilon_1}(x,t)| \leq \varepsilon_n^{-1}\mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^\kappa \widehat{w}_{n,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x),$$

for all $(x,t) \in \overline{Q}$.

Proof. The decomposition in this lemma utilizes bounds on $\boldsymbol{w}$ obtained in the previous lemma. The lemma can be proved using the idea in [76, Lemma 5]. □

5.2 Domain decomposition method

We define the algorithm in the following way. We split the domain Q into five overlapping subdomains $Q_\rho = \Omega_\rho \times \omega$, $\rho = \ell\ell, \ell, m, r, rr$, with $\Omega_{\ell\ell} = (0, 4\sigma_1), \Omega_\ell = (\sigma_1, 4\sigma_2 - 3\sigma_1), \Omega_m = (\sigma_2, 1-\sigma_2), \Omega_r = (1 - 4\sigma_2 + 3\sigma_1, 1-\sigma_1)$, $\Omega_{rr} = (1-4\sigma_1, 1)$, and $\omega = (0, T]$, where the parameters σ_1 and σ_2 are considered to be

$$\sigma_2 = \min\left\{\frac{2}{13}, \frac{2\sqrt{\varepsilon_2}}{\sqrt{\alpha}}\ln N\right\} \quad \text{and} \quad \sigma_1 = \min\left\{\frac{\sigma_2}{4}, \frac{2\sqrt{\varepsilon_1}}{\sqrt{\alpha}}\ln N\right\}. \tag{5.2.1}$$

Suppose the discretization parameter $N = 2^n, n \geq 2$. For each subdomain $\overline{Q}_\rho = [d, c] \times [0, T]$, we define the corresponding discretized subdomain $\overline{Q}_\rho^{N,M} = \overline{\Omega}_\rho^N \times \overline{\omega}^M$, where $\overline{\Omega}_\rho^N = \{x_i |\ x_i = ih_\rho, i = 0, 1 \ldots, N,\ h_\rho = (c-d)/N\}$ and $\overline{\omega}^M = \{t_j |\ t_j = j\Delta t,\ j = 0, 1 \ldots, M,\ \Delta t = T/M\}$. We also define $Q_\rho^{N,M} = \overline{Q}_\rho^{N,M} \cap Q_\rho$, $\Omega_\rho^N = \overline{\Omega}_\rho^N \cap \Omega_\rho$ and $\omega^M = \overline{\omega}^M \cap \omega$. Further, we define $\overline{Q}_{\rho;p,q}^{N,m_\tau} = \overline{\Omega}_\rho^N \times \overline{\omega}_{p,q}^{m_\tau}$, where $\overline{\omega}_{p,q}^{m_\tau} = \{t_\ell |\ t_\ell = (p-1)\tau + \ell\triangle t, \ell = 0, \ldots, (q-p+1)m_\tau, \triangle t = \tau/m_\tau\}$. We also introduce the mesh $\Gamma_{b,\rho}^{N,m_\tau}$ on $\overline{\Omega}_\rho \times [-\tau, 0]$ as $\Gamma_{b,\rho}^{N,m_\tau} = \overline{\Omega}_\rho^N \times \overline{\omega}_{0,0}^{m_\tau}$. For $(x_i, t_j) \in Q_\rho^{N,M}$, $\rho = \ell\ell, \ell, m, r, rr$, the discrete problem is

$$[\boldsymbol{\mathfrak{L}}_\rho^{N,M} \boldsymbol{U}_\rho]_{i,j} := [\boldsymbol{\mathcal{L}}_\rho^{N,M} \boldsymbol{U}_\rho]_{i,j} + \mathbf{B}_{i,j} \boldsymbol{U}_{\rho;i,j-m_\tau} = \boldsymbol{f}_{i,j}, \tag{5.2.2}$$

where

$$[\boldsymbol{\mathcal{L}}_\rho^{N,M} \boldsymbol{U}_\rho]_{i,j} = [\delta_t \boldsymbol{U}_\rho]_{i,j} - \varepsilon[\delta_x^2 \boldsymbol{U}_\rho]_{i,j} + \mathbf{A}_{i,j} \boldsymbol{U}_{\rho;i,j} \tag{5.2.3}$$

with

$$[\delta_t \mathbf{X}]_{i,j} = (\mathbf{X}_{i,j} - \mathbf{X}_{i,j-1})/\Delta t$$

and

$$[\delta_x^2 \mathbf{X}]_{i,j} = (\mathbf{X}_{i,j-1} - 2\mathbf{X}_{i,j} + \mathbf{X}_{i,j+1})/h_\rho^2.$$

The iterative method for the approximate solution of problem (5.0.1) is defined as follows:

Step 1. Initialization: We shall start with initial approximation $\boldsymbol{U}^{[0]}(x_i,t_j)$, $(x_i,t_j)\in \overline{Q}^{N,M}$ defined as follows

$$\boldsymbol{U}^{[0]}(x_i,t_j)=\begin{cases}\mathbf{0}, & 0<x_i<1,\ 0<t_j\le T,\\ \boldsymbol{u}(x_i,t_j), & 0\le x_i\le 1,\ -\tau\le t_j\le 0,\\ \boldsymbol{u}(0,t_j), & (x_i,t_j)\in\{0\}\times\omega^M,\\ \boldsymbol{u}(1,t_j), & (x_i,t_j)\in\{1\}\times\omega^M.\end{cases}\tag{5.2.4}$$

Step 2. We get the better appriximatation $\boldsymbol{U}_\rho^{[k]}(x_i,t_j)$, $(x_i,t_j)\in Q_\rho^{N,M}$, $\rho=\ell\ell,\ell,m,r,rr$, $k\ge 1$, by solving the following discrete problems

$$\begin{cases}[\boldsymbol{\mathcal{L}}_{\ell\ell}^{N,M}\boldsymbol{U}_{\ell\ell}^{[k]}]_{i,j}+\boldsymbol{B}_{i,j}\boldsymbol{U}_{\ell\ell;i,j-m_\tau}^{[k]}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{\ell\ell}^{N,M},\\ \boldsymbol{U}_{\ell\ell}^{[k]}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in\Gamma_{b,\ell\ell}^{N,m_\tau},\\ \boldsymbol{U}_{\ell\ell}^{[k]}(0,t_j)=\boldsymbol{\gamma}_0(t_j),\ \boldsymbol{U}_{\ell\ell}^{[k]}(4\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}^{[k-1]}(4\sigma_1,t_j) & \text{for } t_j\in\omega^M,\end{cases}$$

$$\begin{cases}[\boldsymbol{\mathcal{L}}_{rr}^{N,M}\boldsymbol{U}_{rr}^{[k]}]_{i,j}+\boldsymbol{B}_{i,j}\boldsymbol{U}_{rr;i,j-m_\tau}^{[k]}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{rr}^{N,M},\\ \boldsymbol{U}_{rr}^{[k]}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in\Gamma_{b,rr}^{N,m_\tau},\\ \boldsymbol{U}_{rr}^{[k]}(1-4\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}^{[k-1]}(1-4\sigma_1,t_j),\ \boldsymbol{U}_{rr}^{[k]}(1,t_j)=\boldsymbol{\gamma}_1(t_j), & \text{for } t_j\in\omega^M,\end{cases}$$

$$\begin{cases}[\boldsymbol{\mathcal{L}}_{\ell}^{N,M}\boldsymbol{U}_{\ell}^{[k]}]_{i,j}+\boldsymbol{B}_{i,j}\boldsymbol{U}_{\ell;i,j-m_\tau}^{[k]}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{\ell}^{N,M},\\ \boldsymbol{U}_{\ell}^{[k]}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in\Gamma_{b,\ell}^{N,m_\tau},\\ \boldsymbol{U}_{\ell}^{[k]}(\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}_{\ell\ell}^{[k]}(\sigma_1,t_j),\ t_j\in\omega^M, & \\ \boldsymbol{U}_{\ell}^{[k]}(4\sigma_2-3\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}^{[k-1]}(4\sigma_2-3\sigma_1,t_j) & \text{for } t_j\in\omega^M,\end{cases}$$

$$\begin{cases}[\boldsymbol{\mathcal{L}}_{r}^{N,M}\boldsymbol{U}_{r}^{[k]}]_{i,j}+\boldsymbol{B}_{i,j}\boldsymbol{U}_{r;i,j-m_\tau}^{[k]}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{r}^{N,M},\\ \boldsymbol{U}_{r}^{[k]}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in\Gamma_{b,r}^{N,m_\tau},\\ \boldsymbol{U}_{r}^{[k]}(1-4\sigma_2+3\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}^{[k-1]}(1-4\sigma_2+3\sigma_1,t_j), & \text{for } t_j\in\omega^M,\\ \boldsymbol{U}_{r}^{[k]}(1-\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}_{rr}^{[k]}(1-\sigma_1,t_j), & \text{for } t_j\in\omega^M,\end{cases}$$

$$\begin{cases}[\boldsymbol{\mathcal{L}}_{m}^{N,M}\boldsymbol{U}_{m}^{[k]}]_{i,j}+\boldsymbol{B}_{i,j}\boldsymbol{U}_{m;i,j-m_\tau}^{[k]}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{m}^{N,M},\\ \boldsymbol{U}_{m}^{[k]}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in\Gamma_{b,m}^{N,m_\tau},\\ \boldsymbol{U}_{m}^{[k]}(\sigma_2,t_j)=\mathcal{T}_j\boldsymbol{U}_{\ell}^{[k]}(\sigma_2,t_j),\ \boldsymbol{U}_{m}^{[k]}(1-\sigma_2,t_j)=\mathcal{T}_j\boldsymbol{U}_{r}^{[k]}(1-\sigma_2,t_j), & \text{for } t_j\in\omega^M,\end{cases}$$

where $\mathcal{T}_j\boldsymbol{U}_\rho^{[k]}$ denotes the piecewise linear interpolant of $\boldsymbol{U}_\rho^{[k]}$ at given time step t_j on the mesh $\overline{\Omega}^N:=(\overline{\Omega}_{\ell\ell}^N\backslash\overline{\Omega}_\ell)\cup(\overline{\Omega}_\ell^N\backslash\overline{\Omega}_m)\cup\overline{\Omega}_m^N\cup(\overline{\Omega}_r^N\backslash\overline{\Omega}_m)\cup(\overline{\Omega}_{rr}^N\backslash\overline{\Omega}_r)$.

Step 3. To get $\boldsymbol{U}^{[k]}$, we combine $\boldsymbol{U}_{\rho}^{[k]}, \quad \rho = \ell\ell, \ell, m, r, rr$, in the following way

$$\boldsymbol{U}^{[k]}(x_i,t_j) = \begin{cases} \boldsymbol{U}_{\ell\ell}^{[k]}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{\ell\ell}^{N,M} \setminus \overline{Q}_{\ell}; \\ \boldsymbol{U}_{\ell}^{[k]}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{\ell}^{N,M} \setminus \overline{Q}_{m}; \\ \boldsymbol{U}_{m}^{[k]}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{m}^{N,M}; \\ \boldsymbol{U}_{r}^{[k]}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{r}^{N,M} \setminus \overline{Q}_{m}; \\ \boldsymbol{U}_{rr}^{[k]}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{rr}^{N,M} \setminus \overline{Q}_{r}. \end{cases} \tag{5.2.5}$$

Step 4. If

$$||\boldsymbol{U}^{[k+1]} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq \Upsilon$$

(desired tolerance Υ is achieved), then stop; otherwise get a still better approximation by repeating Step 2.

5.3 Error analysis

For each $\rho = \ell\ell, \ell, m, r, rr$, the operators $\boldsymbol{\mathcal{L}}_{\rho}^{N,M}$ and $\boldsymbol{\mathfrak{L}}_{\rho}^{N,M}$ satisfy the following discrete maximum principles.

Lemma 5.3.1. *Let the mesh function $\boldsymbol{Z}$ satisfies $\boldsymbol{Z}(x_i,t_j) \geq \mathbf{0}$ for $(x_i,t_j) \in \Gamma_{b,\rho}^{N,m_\tau}$ and $\boldsymbol{Z}(x_0,t_j) \geq \mathbf{0}$, $\boldsymbol{Z}(x_N,t_j) \geq \mathbf{0}$ for $t_j \in \omega^M$. Then $[\boldsymbol{\mathcal{L}}_{\rho}^{N,M}\boldsymbol{Z}]_{i,j} \geq \mathbf{0}$ for $(x_i,t_j) \in Q_{\rho}^{N,M}$ implies that $\boldsymbol{Z}(x_i,t_j) \geq \mathbf{0}$ for $(x_i,t_j) \in \overline{Q}_{\rho}^{N,M}$.*

Proof. We refer to [77, Lemma 9]. □

Lemma 5.3.2. *Let the mesh function $\boldsymbol{Z}$ satisfies $\boldsymbol{Z}(x_i,t_j) \geq \mathbf{0}$ for $(x_i,t_j) \in \Gamma_{b,\rho}^{N,m_\tau}$ and $\boldsymbol{Z}(x_0,t_j) \geq \mathbf{0}$, $\boldsymbol{Z}(x_N,t_j) \geq \mathbf{0}$ for $t_j \in \omega^M$. Then $[\boldsymbol{\mathfrak{L}}_{\rho}^{N,M}\boldsymbol{Z}]_{i,j} \geq \mathbf{0}$ for $(x_i,t_j) \in Q_{\rho}^{N,M}$ implies that $\boldsymbol{Z}(x_i,t_j) \geq \mathbf{0}$ for $(x_i,t_j) \in \overline{Q}_{\rho}^{N,M}$.*

Proof. Let $\boldsymbol{Y}(x_i,t_j) = \mathbf{Z}(x_i,t_j)$ for $(x_i,t_j) \in \overline{Q}_{\rho;0,1}^{N,m_\tau}, \rho = \ell\ell, \ell, m, r, rr$. Then, we have

$$\boldsymbol{Y}(x_i,t_j) \geq \mathbf{0} \ \text{ for } (x_i,t_j) \in \Gamma_{b,\rho}^{N,m_\tau} \ \text{ and } \ \boldsymbol{Y}(x_0,t_j) \geq \mathbf{0}, \ \boldsymbol{Y}(x_N,t_j) \geq \mathbf{0} \ \text{ for } \ t_j \in \omega_{1,1}^{m_\tau}.$$

Also

$$[\boldsymbol{\mathcal{L}}_{\rho}^{N,M}\boldsymbol{Y}]_{i,j} \geq -\mathbf{B}_{i,j}\boldsymbol{Y}(x_i,t_{j-m_\tau}) \geq \mathbf{0} \text{ for } (x_i,t_j) \in Q_{\rho;1,1}^{N,m_\tau}.$$

Therefore, using the discrete maximum principle for $\boldsymbol{\mathcal{L}}_{p}^{N,M}$, it follows

$$\mathbf{Z}(x_i,t_j) = \boldsymbol{Y}(x_i,t_j) \geq \mathbf{0} \text{ for } (x_i,t_j) \in \overline{Q}_{\rho;1,1}^{N,m_\tau}.$$

We now use $\mathbf{Z}(x_i,t_j)\geq \mathbf{0}$ for $(x_i,t_j)\in \overline{Q}^{N,m_\tau}_{\rho;s-1,s-1}$, and the previous arguments to deduce that

$$\mathbf{Z}(x_i,t_j)\geq \mathbf{0} \text{ for } (x_i,t_j)\in \overline{Q}^{N,m_\tau}_{\rho;s,s},\ s\geq 2.$$

□

We shall analyze the present method using the following auxiliary problems

$$\begin{cases} [\boldsymbol{\mathcal{L}}^{N,M}_{\ell\ell}\widetilde{\boldsymbol{U}}_{\ell\ell}]_{i,j}+\boldsymbol{B}_{i,j}\widetilde{\boldsymbol{U}}_{\ell\ell;i,j-m_\tau}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q^{N,M}_{\ell\ell},\\ \widetilde{\boldsymbol{U}}_{\ell\ell}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in \Gamma^{N,m_\tau}_{b,\ell\ell},\\ \widetilde{\boldsymbol{U}}_{\ell\ell}(0,t_j)=\boldsymbol{u}(0,t_j),\ \widetilde{\boldsymbol{U}}_{\ell\ell}(4\sigma_1,t_j)=\boldsymbol{u}(4\sigma_1,t_j) & \text{for } t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}^{N,M}_{rr}\widetilde{\boldsymbol{U}}_{rr}]_{i,j}+\boldsymbol{B}_{i,j}\widetilde{\boldsymbol{U}}_{rr;i,j-m_\tau}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q^{N,M}_{rr},\\ \widetilde{\boldsymbol{U}}_{rr}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in \Gamma^{N,m_\tau}_{b,rr},\\ \widetilde{\boldsymbol{U}}_{rr}(1-4\sigma_1,t_j)=\boldsymbol{u}(1-4\sigma_1,t_j),\ \widetilde{\boldsymbol{U}}_{rr}(1,t_j)=\boldsymbol{u}(1,t_j), & \text{for } t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}^{N,M}_{\ell}\widetilde{\boldsymbol{U}}_{\ell}]_{i,j}+\boldsymbol{B}_{i,j}\widetilde{\boldsymbol{U}}_{\ell;i,j-m_\tau}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q^{N,M}_{\ell},\\ \widetilde{\boldsymbol{U}}_{\ell}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in \Gamma^{N,m_\tau}_{b,\ell},\\ \widetilde{\boldsymbol{U}}_{\ell}(\sigma_1,t_j)=\boldsymbol{u}(\sigma_1,t_j),\ \widetilde{\boldsymbol{U}}_{\ell}(4\sigma_2-3\sigma_1,t_j)=\boldsymbol{u}(4\sigma_2-3\sigma_1,t_j) & \text{for } t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}^{N,M}_{r}\widetilde{\boldsymbol{U}}_{r}]_{i,j}+\boldsymbol{B}_{i,j}\widetilde{\boldsymbol{U}}_{r;i,j-m_\tau}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q^{N,M}_{r},\\ \widetilde{\boldsymbol{U}}_{r}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in \Gamma^{N,m_\tau}_{b,r},\\ \widetilde{\boldsymbol{U}}_{r}(1-4\sigma_2+3\sigma_1,t_j)=\boldsymbol{u}(1-4\sigma_2+3\sigma_1,t_j), & \text{for } t_j\in\omega^M,\\ \widetilde{\boldsymbol{U}}_{r}(1-\sigma_1,t_j)=\boldsymbol{u}(1-\sigma_1,t_j), & \text{for } t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}^{N,M}_{m}\widetilde{\boldsymbol{U}}_{m}]_{i,j}+\boldsymbol{B}_{i,j}\widetilde{\boldsymbol{U}}_{m;i,j-m_\tau}=\boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q^{N,M}_{m},\\ \widetilde{\boldsymbol{U}}_{m}(x_i,t_j)=\boldsymbol{\phi}(x_i,t_j) & \text{for } (x_i,t_j)\in \Gamma^{N,m_\tau}_{b,m},\\ \widetilde{\boldsymbol{U}}_{m}(\sigma_2,t_j)=\boldsymbol{u}(\sigma_2,t_j),\ \widetilde{\boldsymbol{U}}_{m}(1-\sigma_2,t_j)=\boldsymbol{u}(1-\sigma_2,t_j), & \text{for } t_j\in\omega^M. \end{cases}$$

Before continuing further, let us define

$$\zeta^{[k]}=\max\{||(\widetilde{\boldsymbol{U}}_{\ell\ell}-\mathcal{T}_j\boldsymbol{U}^{[k-1]})(4\sigma_1,t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_{rr}-\mathcal{T}_j\boldsymbol{U}^{[k-1]})(1-4\sigma_1,t_j)||_\infty,$$

$$||(\widetilde{\boldsymbol{U}}_{\ell}-\mathcal{T}_j\boldsymbol{U}^{[k-1]})(4\sigma_2-3\sigma_1,t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_{r}-\mathcal{T}_j\boldsymbol{U}^{[k-1]})(1-4\sigma_2+3\sigma_1,t_j)||_\infty\},$$

$$\zeta_{\sigma_1}=\max\left\{||(\widetilde{\boldsymbol{U}}_{\ell}-\widetilde{\boldsymbol{U}}_{\ell\ell})(\sigma_1,t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_{r}-\widetilde{\boldsymbol{U}}_{rr})(1-\sigma_1,t_j)||_\infty\right\},$$

$$\zeta_{\sigma_2}=\max\left\{||(\widetilde{\boldsymbol{U}}_{m}-\widetilde{\boldsymbol{U}}_{\ell})(\sigma_2,t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_{m}-\widetilde{\boldsymbol{U}}_{r})(1-\sigma_2,t_j)||_\infty\right\}$$

$$\zeta_{4\sigma_1}=\max\left\{||(\widetilde{\boldsymbol{U}}_{\ell\ell}-\mathcal{T}_j\widetilde{\boldsymbol{U}}_{\ell})(4\sigma_1,t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_{rr}-\mathcal{T}_j\widetilde{\boldsymbol{U}}_{r})(1-4\sigma_1,t_j)||_\infty\right\},$$

$$\zeta_{4\sigma_2-3\sigma_1}=\max\{||(\widetilde{\boldsymbol{U}}_{\ell}-\mathcal{T}_j\widetilde{\boldsymbol{U}}_{m})(4\sigma_2-3\sigma_1,t_j)||_\infty,$$

$$||(\widetilde{\boldsymbol{U}}_{r}-\mathcal{T}_j\widetilde{\boldsymbol{U}}_{m})(1-4\sigma_2+3\sigma_1,t_j)||_\infty\},$$

$$\widetilde{\boldsymbol{U}}(x_i,t_j) = \begin{cases} \widetilde{\boldsymbol{U}}_{\ell\ell}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{\ell\ell}^{N,M} \setminus \overline{Q}_{\ell}; \\ \widetilde{\boldsymbol{U}}_{\ell}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{\ell}^{N,M} \setminus \overline{Q}_{m}; \\ \widetilde{\boldsymbol{U}}_{m}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{m}^{N,M}; \\ \widetilde{\boldsymbol{U}}_{r}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{r}^{N,M} \setminus \overline{Q}_{m}; \\ \widetilde{\boldsymbol{U}}_{rr}(x_i,t_j), \ (x_i,t_j) \in \overline{Q}_{rr}^{N,M} \setminus \overline{Q}_{r}. \end{cases} \tag{5.3.1}$$

Lemma 5.3.3. *Suppose $\boldsymbol{u}$ is the solution of problem (5.0.1) and $\widetilde{\boldsymbol{U}}$ is as defined in (5.3.1). Then*

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}||_{\overline{Q}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N). \tag{5.3.2}$$

Proof. For $(x_i,t_j) \in Q_\rho^{N,M}, \rho = \ell\ell, rr$, we have

$$\begin{aligned} [|\boldsymbol{\mathcal{L}}_\rho^{N,M}(\boldsymbol{u}-\widetilde{\boldsymbol{U}}_\rho)]_{i,j}| = |[(\boldsymbol{\mathcal{L}}_\rho^{N,M} - \boldsymbol{\mathcal{L}}_\rho)\boldsymbol{u}]_{i,j}| &= |[(\delta_t - \partial_t)\boldsymbol{u}]_{i,j} + \boldsymbol{E}[(\partial_x^2 - \delta_x^2)\boldsymbol{u}]_{i,j}| \\ &\leq \boldsymbol{C}\Delta t + \boldsymbol{E}Ch_\rho^2||\partial_x^4\boldsymbol{u}(.,t_j)||_{[x_{i-1},x_{i+1}]} \\ &\leq \boldsymbol{C}(\Delta t + N^{-2}\ln^2 N) \end{aligned} \tag{5.3.3}$$

where we have used Taylor expansions and $h_\rho^2 \leq C\varepsilon_1 N^{-2}\ln^2 N$. Hence, using the discrete maximum principle for $\boldsymbol{\mathcal{L}}_\rho^{N,M}$, we get

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}_\rho||_{\overline{Q}_\rho^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N). \tag{5.3.4}$$

Next we get a bound for $|(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_\rho)(x_i,t_j)|, \ (x_i,t_j) \in Q_\rho^{N,M}, \ \rho = \ell, r$. For the same we use Taylor expansion and Lemma 5.1.3 to get

$$\begin{aligned} |[\mathcal{L}_{\rho,n}^{N,M}(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_\ell)]_{i,j}| &\leq |[\delta_t u_n - \partial_t u_n]_{i,j}| + \varepsilon_n|[\partial_x^2 u_n - \delta_x^2 u_n]_{i,j}| \\ &\leq C\Delta t + \varepsilon_n|[\partial_x^2 u_n - \delta_x^2 u_n]_{i,j}|, \ n = 1,2. \end{aligned}$$

For the term $\varepsilon_n[\partial_x^2 u_n - \delta_x^2 u_n]_{i,j}$, we use the decomposition $u_n = v_n + w_n$, $w_n = \widehat{w}_{n,\varepsilon_1} + \widehat{w}_{n,\varepsilon_2}$ to deduce

$$\begin{aligned} \varepsilon_n|[\partial_x^2 u_n - \delta_x^2 u_n]_{i,j}| &\leq \varepsilon_n(|[\partial_x^2 v_n - \delta_x^2 v_n]_{i,j}| + |[\partial_x^2 \widehat{w}_{n,\varepsilon_1} - \delta_x^2 \widehat{w}_{n,\varepsilon_1}]_{i,j}| + |[\partial_x^2 \widehat{w}_{n,\varepsilon_2} - \delta_x^2 \widehat{w}_{n,\varepsilon_2}]_{i,j}|) \\ &\leq C\varepsilon_n h_\rho^2\|\partial_x^4 v_n(.,t_j)\|_{[x_{i-1},x_{i+1}]} + C\varepsilon_n\|\partial_x^2 \widehat{w}_{n,\varepsilon_1}(.,t_j)\|_{[x_{i-1},x_{i+1}]} \\ &\quad + C\varepsilon_n h_\rho^2\|\partial_x^4 \widehat{w}_{n,\varepsilon_2}(.,t_j)\|_{[x_{i-1},x_{i+1}]} \end{aligned}$$

$$\leq CN^{-2}\ln^2 N \tag{5.3.5}$$

where we have used Taylor expansions, Lemmas 5.1.4 and 5.1.7, and the mesh width to get (5.3.5). Hence, the discrete maximum principle for $\boldsymbol{\mathcal{L}}_{\rho}^{N,M}, \rho = \ell, r$, gives

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}_{\rho}||_{\overline{Q}_{\rho}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N). \tag{5.3.6}$$

To estimate $|(\boldsymbol{u}-\widetilde{\boldsymbol{U}}_m)(x_i,t_j)|$, we need a bound on $|\mathcal{L}_{m,n}^{N,M}(\boldsymbol{u}-\widetilde{\boldsymbol{U}}_m)]|$, $(x_i,t_j) \in Q_m^{N,M}$. For the same we use the decomposition $u_n = v_n + w_n$ to split the truncation error as

$$\begin{aligned} |[\mathcal{L}_{m,n}^{N,M}(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_m)]_{i,j}| &\leq |[\delta_t u_n - \partial_t u_n]_{i,j}| + \varepsilon_n|[\partial_x^2 v_n - \delta_x^2 v_n]_{i,j}| + \varepsilon_n|[\partial_x^2 w_n - \delta_x^2 w_n]_{i,j}| \\ &\leq C\Delta t + C\varepsilon_n h_m^2 \|\partial_x^4 v_n(.,t_j)\|_{[x_{i-1},x_{i+1}]} + C\varepsilon_n\|\partial_x^4 w_n(.,t_j)\|_{[x_{i-1},x_{i+1}]} \\ &\leq C(\Delta t + N^{-2}), \end{aligned} \tag{5.3.7}$$

where we have used Taylor expansions, Lemmas 5.1.4 and 5.1.6 to get (5.3.7). Therefore, using the discrete maximum principle for $\boldsymbol{\mathcal{L}}_m^{N,M}$, we have

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}_m||_{\overline{Q}_m^{N,M}} \leq C(\Delta t + N^{-2}). \tag{5.3.8}$$

Combining (5.3.4), (5.3.6), and (5.3.8) yields

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}||_{\overline{Q}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N)$$

which is the desired result. □

Lemma 5.3.4. *Suppose $\boldsymbol{U}^{[k]}$ is the k^{th} iterate of the algorithm defined in Section 5.2 and $\widetilde{\boldsymbol{U}}$ is as defined in (5.3.1). Then*

$$||\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq C2^{-k} + C(\Delta t + N^{-2}\ln^2 N). \tag{5.3.9}$$

Proof. Letting

$$\boldsymbol{\psi}^{\pm}(x_i,t_j) := \frac{x_i}{4\sigma_1}\zeta^{[1]}\mathbf{1} \pm (\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]})(x_i,t_j),$$

where

$$\begin{cases} \left[\boldsymbol{\mathcal{L}}_{\ell\ell}^{N,M}\left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)\right]_{i,j} = \mathbf{0} & \text{for } (x_i,t_j) \in Q_{\ell\ell}^{N,M}, \\ \left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)(x_i,t_j) = \mathbf{0}, & \text{for } (x_i,t_j) \in \Gamma_{b,\ell\ell}^{N,m_\tau}, \\ \left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)(0,t_j) = \mathbf{0}, \left|\left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)(4\sigma_1,t_j)\right| \leq \zeta^{[1]}\mathbf{1}, & \text{for } t_j \in \omega^M, \end{cases}$$

we obtain

$$\begin{cases} \boldsymbol{\psi}^{\pm}(x_i,t_j) \geq \mathbf{0} & \text{for } (x_i,t_j) \in \Gamma_{b,\ell\ell}^{N,m_\tau}, \\ \boldsymbol{\psi}^{\pm}(0,t_j) = \mathbf{0}, \boldsymbol{\psi}^{\pm}(2\sigma,t_j) \geq \mathbf{0} & \text{for } t_j \in \omega^M, \end{cases}$$

and for $(x_i,t_j) \in Q_{\ell\ell}^{N,M}$,

$$[\boldsymbol{\mathcal{L}}_{\ell\ell}^{N,M}\boldsymbol{\psi}^{\pm}]_{i,j} \;\; = \;\; (\boldsymbol{A}_{i,j}+\boldsymbol{B}_{i,j})\frac{x_i}{4\sigma_1}\zeta^{[1]}\mathbf{1} \pm \mathbf{0} \geq \mathbf{0}.$$

Then, by the discrete maximum principle, we get

$$\left|(\widetilde{\boldsymbol{U}}_{\ell\ell}-\boldsymbol{U}_{\ell\ell}^{[1]})(x_i,t_j)\right| \leq \frac{x_i}{4\sigma_1}\zeta^{[1]}\mathbf{1} \qquad \text{for } (x_i,t_j) \in \overline{Q}_{\ell\ell}^{N,M}.$$

Hence

$$\left\|\widetilde{\boldsymbol{U}}_{\ell\ell}-\boldsymbol{U}_{\ell\ell}^{[1]}\right\|_{\overline{Q}_{\ell\ell}^{N,M}\setminus\overline{Q}_\ell} \leq \frac{1}{4}\zeta^{[1]}, \quad \text{as } x_i \leq \sigma_1. \tag{5.3.10}$$

Similarly

$$\left\|\widetilde{\boldsymbol{U}}_{rr}-\boldsymbol{U}_{rr}^{[1]}\right\|_{\overline{Q}_{rr}^{N,M}\setminus\overline{Q}_r} \leq \frac{1}{4}\zeta^{[1]}. \tag{5.3.11}$$

Next, we introduce the mesh function

$$\boldsymbol{\psi}^{\pm}(x_i,t_j) := \phi(x_i)\zeta^{[1]}\mathbf{1} + \zeta_{\sigma_1}\mathbf{1} \pm \left(\widetilde{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}\right)(x_i,t_j),$$

where

$$\phi(x) := \frac{-x^2+(13\sigma_2-11\sigma_1)x+12\sigma_2^2+24\sigma_1^2-37\sigma_1\sigma_2}{48(\sigma_2-\sigma_1)^2}, \; x \in [\sigma_1, 4\sigma_2-3\sigma_1],$$

is a monotonically increasing function with $\phi(\sigma_1)=1/4$, $\phi(4\sigma_2-3\sigma_1)=1$, $\phi(\sigma_2)=1/2$, $\phi>0$ in $\overline{\Omega}_\ell^N$, and $[\boldsymbol{\mathcal{L}}_\ell^{N,M}\phi\mathbf{1}] > \mathbf{0}$ in $Q_\ell^{N,M}$. Also

$$\begin{cases} \left[\boldsymbol{\mathcal{L}}_\ell^{N,M}\left(\widetilde{\boldsymbol{U}}_\ell-\boldsymbol{U}_\ell^{[1]}\right)\right]_{i,j} = \mathbf{0} & \text{for } (x_i,t_j) \in Q_\ell^{N,M}, \\ \left(\widetilde{\boldsymbol{U}}_\ell-\boldsymbol{U}_\ell^{[1]}\right)(x_i,t_j) = \mathbf{0}, & \text{for } (x_i,t_j) \in \Gamma_{b,\ell}^{N,m_\tau}, \\ \left|\left(\widetilde{\boldsymbol{U}}_\ell-\boldsymbol{U}_\ell^{[1]}\right)(\sigma_1,t_j)\right| \leq \zeta_{\sigma_1}\mathbf{1}+\frac{1}{4}\zeta^{[1]}\mathbf{1}, & \text{for } \; t_j \in \omega^M, \\ \left|\left(\widetilde{\boldsymbol{U}}_\ell-\boldsymbol{U}_\ell^{[1]}\right)(4\sigma_2-3\sigma_1,t_j)\right| \leq \zeta^{[1]}\mathbf{1}, & \text{for } t_j \in \omega^M, \end{cases}$$

where $(\sigma_1,t_j) \in \overline{Q}_{\ell\ell}^{N,M}$ is used to get

$$\left|\left(\widetilde{\boldsymbol{U}}_\ell-\boldsymbol{U}_\ell^{[1]}\right)(\sigma_1,t_j)\right| = \left|\left(\widetilde{\boldsymbol{U}}_\ell-\mathcal{I}_j\boldsymbol{U}_{\ell\ell}^{[1]}\right)(\sigma_1,t_j)\right| \leq \zeta_{\sigma_1}\mathbf{1}+\frac{1}{4}\zeta^{[1]}\mathbf{1}, \quad t_j \in \omega^M.$$

Hence, using the discrete maximum principle for $\boldsymbol{\mathcal{L}}_\ell^{N,M}$ we have

$$\left|\left(\widetilde{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}\right)(x_i,t_j)\right| \le \phi(x_i)\zeta^{[1]}\mathbf{1} + \zeta_{\sigma_1}\mathbf{1} \ \text{ for } (x_i,t_j) \in \overline{Q}_\ell^{N,M}.$$

Consequently

$$\left\|\widetilde{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}\right\|_{\overline{Q}_\ell^{N,M}\setminus\overline{Q}_m} \le \frac{1}{2}\zeta^{[1]} + \zeta_{\sigma_1}. \tag{5.3.12}$$

Similarly, we can obtain

$$\left\|\widetilde{\boldsymbol{U}}_r - \boldsymbol{U}_r^{[1]}\right\|_{\overline{Q}_r^{N,M}\setminus\overline{Q}_m} \le \frac{1}{2}\zeta^{[1]} + \zeta_{\sigma_1}. \tag{5.3.13}$$

Next, we obtain a bound for $\left\|\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right\|_{\overline{Q}_m^{N,M}}$. We have

$$\begin{cases} \left[\boldsymbol{\mathcal{L}}_m^{N,M}\left(\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right)\right]_{i,j} = \mathbf{0}, \quad \text{for } (x_i,t_j) \in Q_m^{N,M}, \\ \left(\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right)(x_i,t_j) = \mathbf{0}, \quad \text{for } (x_i,t_j) \in \Gamma_{b,m}^{N,m_\tau}, \\ \left|\left(\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right)(\sigma_2,t_j)\right| \le \zeta_{\sigma_2}\mathbf{1} + \frac{1}{2}\zeta^{[1]}\mathbf{1} + \zeta_{\sigma_1}\mathbf{1},\ t_j \in \omega^M, \\ \left|\left(\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right)(1-\sigma_2,t_j)\right| \le \zeta_{\sigma_2}\mathbf{1} + \frac{1}{2}\zeta^{[1]}\mathbf{1} + \zeta_{\sigma_1}\mathbf{1},\ t_j \in \omega^M, \end{cases}$$

as $(\sigma_2,t_j) \in \overline{Q}_\ell^{N,M}$ and $(1-\sigma_2,t_j) \in \overline{Q}_r^{N,M}$. Now use the discrete maximum principle for $\boldsymbol{\mathcal{L}}_m^{N,M}$ to get

$$\left\|\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right\|_{\overline{Q}_m^{N,M}} \le \frac{1}{2}\zeta^{[1]} + \zeta_{\sigma_1} + \zeta_{\sigma_2}. \tag{5.3.14}$$

Letting $\phi^{[k]} = \left\|\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[k]}\right\|_{\overline{Q}^{N,M}}$, from (5.3.10)-(5.3.14), it holds

$$\phi^{[1]} \le \frac{1}{2}\zeta^{[1]} + \zeta_{\sigma_1} + \zeta_{\sigma_2}.$$

To estimate $\phi^{[2]}$ we need a bound on $\zeta^{[2]}$. We have

$$\zeta^{[2]} = \max\{||(\widetilde{\boldsymbol{U}}_{\ell\ell} - \mathcal{T}_j\boldsymbol{U}^{[1]})(4\sigma_1,t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_{rr} - \mathcal{T}_j\boldsymbol{U}^{[1]})(1-4\sigma_1,t_j)||_\infty,$$

$$||(\widetilde{\boldsymbol{U}}_\ell - \mathcal{T}_j\boldsymbol{U}^{[1]})(4\sigma_2-3\sigma_1,t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_r - \mathcal{T}_j\boldsymbol{U}^{[1]})(1-4\sigma_2+3\sigma_1,t_j)||_\infty\}.$$

Using a triangle inequality, the stability of the operator $\mathcal{T}_j$, (5.3.12), (5.3.13), and (5.3.14), we can bound each of the terms involved. We get

$$\zeta^{[2]} \le \frac{1}{2}\zeta^{[1]} + \eta, \quad \eta = \zeta_{\sigma_1} + \zeta_{\sigma_2} + \zeta_{4\sigma_1} + \zeta_{4\sigma_2-3\sigma_1},$$

Hence, we have

$$\max\left\{\phi^{[1]},\zeta^{[2]}\right\} \le \frac{1}{2}\zeta^{[1]} + \eta.$$

We use the similar argument to get

$$\max\left\{\phi^{[k]},\zeta^{[k+1]}\right\}\leq\frac{1}{2}\zeta^{[k]}+\eta.$$

So $\zeta^{[k]}\leq 2^{-(k-1)}\zeta^{[1]}+2\eta$. Therefore

$$\phi^{[k]}\leq 2^{-k}\zeta^{[1]}+2\eta. \tag{5.3.15}$$

Use Lemma 5.1.3 to get $\zeta^{[1]}\leq C$. Next, we have to find a bound on η. As $(\sigma_1,t_j)\in\overline{Q}_{\ell\ell}^{N,M}$, $(1-\sigma_1,t_j)\in\overline{Q}_{rr}^{N,M}$, $(\sigma_2,t_j)\in\overline{Q}_{\ell}^{N,M}$, $(1-\sigma_2,t_j)\in\overline{Q}_{r}^{N,M}$, we use Lemma 5.3.3 to deduce $\zeta_{\sigma_1}+\zeta_{\sigma_2}\leq C(\Delta t+N^{-2}\ln^2 N)$. To get an estimate for $\zeta_{4\sigma_1}$, proceed as follows

$$\begin{aligned}\left|\left(\boldsymbol{u}-\mathcal{T}_j\widetilde{\boldsymbol{U}}_\ell\right)(4\sigma_1,t_j)\right|&\leq\left|\left(\boldsymbol{u}-\mathcal{T}_j\boldsymbol{u}\right)(4\sigma_1,t_j)\right|+\left|\mathcal{T}_j\left(\boldsymbol{u}-\widetilde{\boldsymbol{U}}_\ell\right)(4\sigma_1,t_j)\right| \quad (5.3.16)\\&\leq\left|\left(\boldsymbol{u}-\mathcal{T}_j\boldsymbol{u}\right)(4\sigma_1,t_j)\right|+\boldsymbol{C}(\Delta t+N^{-2}\ln^2 N),\end{aligned}$$

where stability of $\mathcal{T}_j$ and Lemma 5.3.3 are used for the second term. Further, using $u_n=v_n+w_n, w_n=\widehat{w}_{n,\varepsilon_1}+\widehat{w}_{n,\varepsilon_2}$ and standard interpolation error estimate we get

$$\begin{aligned}|(u_n-\mathcal{T}_ju_n)(4\sigma_1,t_j)|&\leq|(v_n-\mathcal{T}_jv_n)(4\sigma_1,t_j)|+|(\widehat{w}_{n,\varepsilon_1}-\mathcal{T}_j\widehat{w}_{n,\varepsilon_1})(4\sigma_1,t_j)|\\&\quad+|(\widehat{w}_{n,\varepsilon_2}-\mathcal{T}_j\widehat{w}_{n,\varepsilon_2})(4\sigma_1,t_j)|,\\&\leq Ch_\ell^2||\partial_x^2v(.,t_j)||_{[x_i,x_{i+1}]}+||\widehat{w}_{n,\varepsilon_1}(.,t_j)||_{[x_i,x_{i+1}]}\\&\quad+Ch_\ell^2||\partial_x^2\widehat{w}_{n,\varepsilon_2}(.,t_j)||_{[x_i,x_{i+1}]},\\&\leq CN^{-2}+C\|\mathcal{B}_{\varepsilon_1}\|_{[x_i,x_{i+1}]}+Ch_\ell^2\varepsilon_2^{-1}\|\mathcal{B}_{\varepsilon_2}\|_{[x_i,x_{i+1}]},\\&\leq CN^{-2}\ln^2 N, n=1,2,\ t_j\in\omega^M.\end{aligned}$$

Similarly

$$|(\boldsymbol{u}-\mathcal{T}_j\widetilde{\boldsymbol{U}}_r)(1-4\sigma_1,t_j)|\leq\boldsymbol{C}(\Delta t+N^{-2}\ln^2 N),\ t_j\in\omega^M.$$

To bound $\zeta_{4\sigma_2-3\sigma_1}$, proceed as follows

$$\begin{aligned}\left|\left(\boldsymbol{u}-\mathcal{T}_j\widetilde{\boldsymbol{U}}_m\right)(4\sigma_2-3\sigma_1,t_j)\right|&\leq\left|\left(\boldsymbol{u}-\mathcal{T}_j\boldsymbol{u}\right)(4\sigma_2-3\sigma_1,t_j)\right|+\left|\mathcal{T}_j\left(\boldsymbol{u}-\widetilde{\boldsymbol{U}}_m\right)(4\sigma_2-3\sigma_1,t_j)\right|\\&\leq\boldsymbol{C}N^{-2}+\boldsymbol{C}(\Delta t+N^{-2}\ln^2 N),\quad t_j\in\omega^M,\end{aligned}$$

where previous arguments are used to prove

$$|(u_n-\mathcal{T}_j u_n)(4\sigma_2-3\sigma_1,t_j)| \le |(v_n-\mathcal{T}_j v_n)(4\sigma_2-3\sigma_1,t_j)|+|(w_n-\mathcal{T}_j w_n)(4\sigma_2-3\sigma_1,t_j)|$$

$$\le Ch_m^2||\partial_x^2 v(.,t_j)||_{[x_i,x_{i+1}]} + C||w(.,t_j)||_{[x_i,x_{i+1}]}$$
$$\le CN^{-2},\ n=1,2,\ t_j\in\omega^M.$$

Similarly, it holds

$$|(\boldsymbol{u}-\mathcal{T}_j\widetilde{\boldsymbol{U}}_m)(1-4\sigma_2+3\sigma_1,t_j)| \le \boldsymbol{C}(\Delta t+N^{-2}\ln^2 N),\ t_j\in\omega^M.$$

We now combine the bounds for ζ_{σ_1}, ζ_{σ_2}, $\zeta_{4\sigma_1}$, and $\zeta_{4\sigma_2-3\sigma_1}$ to get $\eta \le C(\Delta t + N^{-2}\ln^2 N)$. Thus, we have the proof the lemma. □

On combining Lemmas 5.3.3 and 5.3.4, we have the following main convergence result for the proposed method.

Theorem 5.3.5. *Suppose* $\boldsymbol{u}$ *is the solution of problem (5.0.1) and* $\boldsymbol{U}^{[k]}$ *is its approximation obtained by the* k^{th} *iterate of the method. Then*

$$||\boldsymbol{u}-\boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \le C2^{-k} + C(\Delta t+N^{-2}\ln^2 N). \tag{5.3.17}$$

5.4 Numerical results

To validate the convergence result of Theorem 5.3.5, we shall provide numerical results for the following two test problems.

Example 5.4.1. Consider the problem

$$\begin{cases} \frac{\partial \mathbf{u}(x,t)}{\partial t} - \mathbf{E}\frac{\partial^2\mathbf{u}(x,t)}{\partial x^2} + \mathbf{A}\mathbf{u}(x,t)+\mathbf{B}\mathbf{u}(x,t-1)=\mathbf{f}(x,t) & (x,t)\in Q:=\Omega\times(0,2],\\ \mathbf{u}(x,0)=\mathbf{0} & (x,t)\in[0,1]\times[-1,0],\\ \mathbf{u}(0,t)=\mathbf{0},\mathbf{u}(1,t)=\mathbf{0} & t\in(0,2], \end{cases}$$

with

$$\mathbf{A}=\begin{pmatrix} 3(1+x)^2 & -(1+x^3)\\ -2\cos(\pi x/4) & 4\exp(1-x)\end{pmatrix},\ \mathbf{B}=\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix},\ \mathbf{f}=\begin{pmatrix}\cos(\pi x/2)\\ x\end{pmatrix}.$$

Example 5.4.2. Consider the problem

$$\begin{cases} \frac{\partial \mathbf{u}(x,t)}{\partial t} - \mathbf{E}\frac{\partial^2 \mathbf{u}(x,t)}{\partial x^2} + \mathbf{A}\mathbf{u}(x,t) + \mathbf{B}\mathbf{u}(x,t-1) = \mathbf{f}(x,t) & (x,t) \in Q := \Omega \times (0,2], \\ \mathbf{u}(x,t) = \boldsymbol{\phi}(x,t) & (x,t) \in [0,1] \times [-1,0], \\ \mathbf{u}(0,t) = \boldsymbol{\gamma}_0(t), \mathbf{u}(1,t) = \boldsymbol{\gamma}_1(t) & t \in (0,2], \end{cases}$$

with

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}.$$

where $f_1, f_2, \boldsymbol{\phi}, \boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ are so that

$$u_1(x,t) = (x+1)te^{-t} + t(\phi_1(x) + \phi_2(x) - 2),$$

$$u_2(x,t) = (t - t^2)(\phi_2(x) - 1) + \varepsilon_1(1 - e^{-t})(\phi_1(x) - 1),$$

with

$$\phi_i(x) = \frac{e^{-x/\sqrt{\varepsilon_i}} + e^{-(1-x)/\sqrt{\varepsilon_i}}}{1 + e^{-1/\sqrt{\varepsilon_i}}}, \quad i = 1, 2.$$

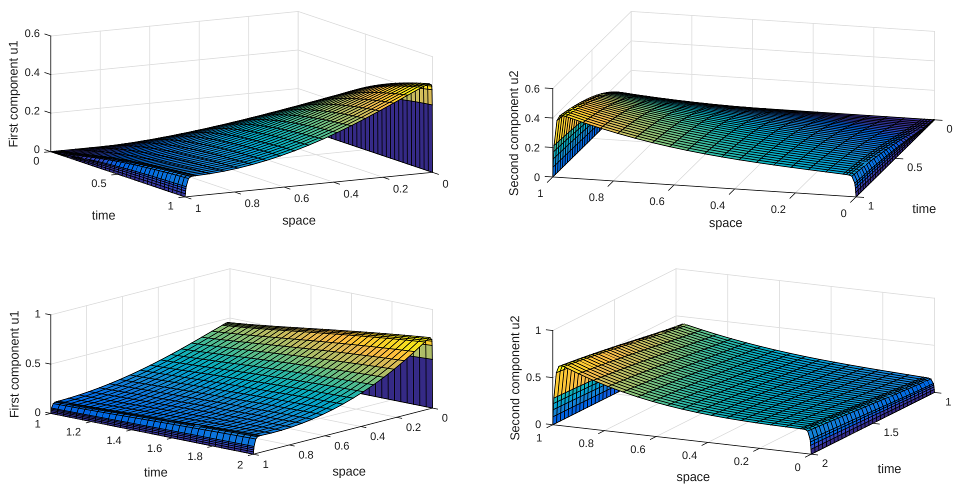

Figure 5.1: Approximate solution plot for Example 5.4.1 taking parameters $\varepsilon_1 = 10^{-6}, \varepsilon_2 = 10^{-4}$ with $N = 32, M = 64$. The left figure is for the 1st component and the right figure is for the 2nd component.

As we know the exact solution of the test problem in Example 5.4.2, we get the errors by

$$E_{n,\boldsymbol{\varepsilon}}^{N,\Delta t} = ||u_n - U_{n,\boldsymbol{\varepsilon}}^{N,\Delta t}||_{\overline{Q}^{N,M}}, n = 1, 2,$$

where $U_{n,\boldsymbol{\varepsilon}}^{N,\Delta t}$ denotes the nth component of the approximate solution obtained after terminating iterative process. For the test problem in Example 5.4.1, we compare two approximate solutions $U_{n,\boldsymbol{\varepsilon}}^{N,\Delta t}$ and $U_{n,\boldsymbol{\varepsilon}}^{2N,\Delta t/4}$ obtained with time step sizes Δt and

Table 5.1: Errors $E^{N,\Delta t}_{n,\varepsilon_1}$ and $E^{N,\Delta t}_{n}$, and convergence rates $\varrho_{n,\varepsilon_1}$ and ϱ^{unif}_{n} for Example 5.4.1.

ε_1	$N=2^5$ $\Delta t=1/4$	$N=2^6$ $\Delta t=1/4^2$	$N=2^7$ $\Delta t=1/4^3$	$N=2^8$ $\Delta=1/4^4$	$N=2^9$ $\Delta t=1/4^5$
	1.939e-02	7.351e-03	2.151e-03	5.620e-04	1.421e-04
10^{-1}	1.399	1.773	1.936	1.983	
	1.587e-02	5.930e-03	1.757e-03	4.608e-04	1.167e-04
	1.420	1.755	1.931	1.982	
	2.349e-02	8.179e-03	2.281e-03	5.882e-04	1.482e-04
10^{-2}	1.522	1.843	1.955	1.989	
	2.047e-02	7.211e-03	2.049e-03	5.312e-04	1.341e-04
	1.505	1.815	1.947	1.986	
	2.407e-02	8.296e-03	2.287e-03	5.873e-04	1.478e-04
10^{-3}	1.537	1.859	1.962	1.990	
	2.222e-02	7.696e-03	2.158e-03	5.575e-04	1.406e-04
	1.529	1.834	1.953	1.988	
	2.527e-02	8.615e-03	2.374e-03	6.098e-04	1.535e-04
10^{-4}	1.553	1.860	1.961	1.990	
	2.289e-02	7.924e-03	2.224e-03	5.781e-04	1.468e-04
	1.530	1.833	1.944	1.977	
	2.703e-02	1.006e-02	3.170e-03	9.147e-04	2.467e-04
10^{-5}	1.425	1.666	1.793	1.890	
	2.307e-02	7.977e-03	2.236e-03	5.805e-04	1.474e-04
	1.532	1.835	1.945	1.977	
	2.703e-02	1.006e-02	3.170e-03	9.145e-04	2.589e-04
10^{-6}	1.425	1.667	1.793	1.820	
	2.312e-02	8.025e-03	2.240e-03	5.812e-04	1.476e-04
	1.527	1.841	1.946	1.977	
	2.703e-02	1.007e-02	3.169e-03	9.145e-04	2.589e-04
10^{-7}	1.425	1.667	1.793	1.820	
	2.447e-02	8.792e-03	2.624e-03	7.101e-04	1.856e-04
	1.477	1.745	1.886	1.936	
	2.703e-02	1.008e-02	3.169e-03	9.144e-04	2.589e-04
10^{-8}	1.424	1.669	1.793	1.820	
	2.548e-02	1.006e-02	2.954e-03	8.406e-04	2.312e-04
	1.341	1.768	1.813	1.863	
$E_1^{N,\Delta t}$	2.703e-02	1.008e-02	3.170e-03	9.147e-04	2.589e-04
ϱ_1^{unif}	1.424	1.669	1.793	1.820	
$E_2^{N,\Delta t}$	2.548e-02	1.006e-02	2.954e-03	8.406e-04	2.312e-04
ϱ_2^{unif}	1.341	1.768	1.813	1.863	

$\Delta t/4$ in each subdomain, respectively, and using N and $2N$ spatial mesh intervals in each subdomain, respectively; where subdomain parameters σ_1 and σ_2 defined with parameter N are same used for both the approximate solutions. We get the estimate of the errors as follows

Table 5.2: Errors $E_{n,\varepsilon_1}^{N,\Delta t}$ and $E_n^{N,\Delta t}$, and convergence rates $\varrho_{n,\varepsilon_1}$ and ϱ_n^{unif} for Example 5.4.2.

ε_1	$N=2^5$ $\Delta t=1/4$	$N=2^6$ $\Delta t=1/4^2$	$N=2^7$ $\Delta t=1/4^3$	$N=2^8$ $\Delta=1/4^4$	$N=2^9$ $\Delta t=1/4^5$
	5.083e-02	1.451e-02	3.786e-03	9.573e-04	2.400e-04
10^{-1}	1.809	1.938	1.984	1.996	
	3.880e-02	9.859e-03	2.476e-03	6.197e-04	1.550e-04
	1.976	1.994	1.998	2.000	
	6.200e-02	1.759e-02	4.580e-03	1.157e-03	2.901e-04
10^{-2}	1.817	1.941	1.985	1.996	
	8.297e-02	2.105e-02	5.282e-03	1.322e-03	3.305e-04
	1.979	1.995	1.999	2.000	
	6.625e-02	1.878e-02	4.888e-03	1.235e-03	3.095e-04
10^{-3}	1.819	1.941	1.985	1.996	
	8.581e-02	2.176e-02	5.460e-03	1.366e-03	3.416e-04
	1.979	1.995	1.999	2.000	
	6.805e-02	1.924e-02	5.008e-03	1.265e-03	3.170e-04
10^{-4}	1.822	1.942	1.985	1.996	
	8.645e-02	2.193e-02	5.501e-03	1.376e-03	3.442e-04
	1.979	1.995	1.999	2.000	
	6.867e-02	1.939e-02	5.041e-03	1.272e-03	3.188e-04
10^{-5}	1.824	1.944	1.986	1.997	
	8.666e-02	2.199e-02	5.516e-03	1.380e-03	3.451e-04
	1.979	1.995	1.999	2.000	
	6.901e-02	1.949e-02	5.066e-03	1.279e-03	3.205e-04
10^{-6}	1.824	1.944	1.986	1.997	
	8.672e-02	2.201e-02	5.522e-03	1.382e-03	3.455e-04
	1.978	1.995	1.999	2.000	
	6.914e-02	1.952e-02	5.075e-03	1.281e-03	3.211e-04
10^{-7}	1.824	1.944	1.986	1.996	
	8.674e-02	2.201e-02	5.523e-03	1.382e-03	3.456e-04
	1.978	1.995	1.999	2.000	
	6.919e-02	1.954e-02	5.079e-03	1.282e-03	3.214e-04
10^{-8}	1.824	1.944	1.986	1.996	
	8.675e-02	2.201e-02	5.524e-03	1.382e-03	3.456e-04
	1.978	1.995	1.999	2.000	
$E_1^{N,\Delta t}$	6.919e-02	1.954e-02	5.079e-03	1.282e-03	3.214e-04
ϱ_1^{unif}	1.824	1.944	1.986	1.996	
$E_2^{N,\Delta t}$	8.675e-02	2.201e-02	5.524e-03	1.382e-03	3.456e-04
ϱ_2^{unif}	1.978	1.995	1.999	2.000	

$$E_{n,\boldsymbol{\varepsilon}}^{N,\Delta t} = ||\, U_{n,\boldsymbol{\varepsilon}}^{N,\Delta t} - U_{n,\boldsymbol{\varepsilon}}^{2N,\Delta t/4}||_{\overline{Q}^{N,M}},\ n = 1,2.$$

After that, for some non-negative integer m, we use fixed value $\varepsilon_1 = 10^{-m}$ to get

$$E_{n,\varepsilon_1}^{N,\Delta t} = \max\{E_{n,(\varepsilon_1,1)}^{N,\Delta t}, E_{n,(\varepsilon_1,10^{-1})}^{N,\Delta t}, \ldots, E_{n,(\varepsilon_1,10^{-m})}^{N,\Delta t}\}.$$

Table 5.3: Iteration counts for fixed $\varepsilon_1 = 10^{-8}$ in Example 5.4.1.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	3	3	4	5	6
1	2	2	4	5	6
2	2	3	4	5	6
3	3	4	5	6	7
4	3	4	5	5	6
5	3	3	3	3	3
6	2	2	2	2	2
7	1	1	1	1	1
8	1	1	1	1	1

Table 5.4: Iteration counts for fixed $\varepsilon_1 = 10^{-8}$ in Example 5.4.2.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	4	4	5	6	7
1	2	4	5	6	7
2	3	4	5	6	7
3	4	5	6	7	7
4	4	5	5	6	6
5	3	3	3	3	3
6	2	2	2	2	2
7	1	1	1	1	1
8	1	1	1	1	1

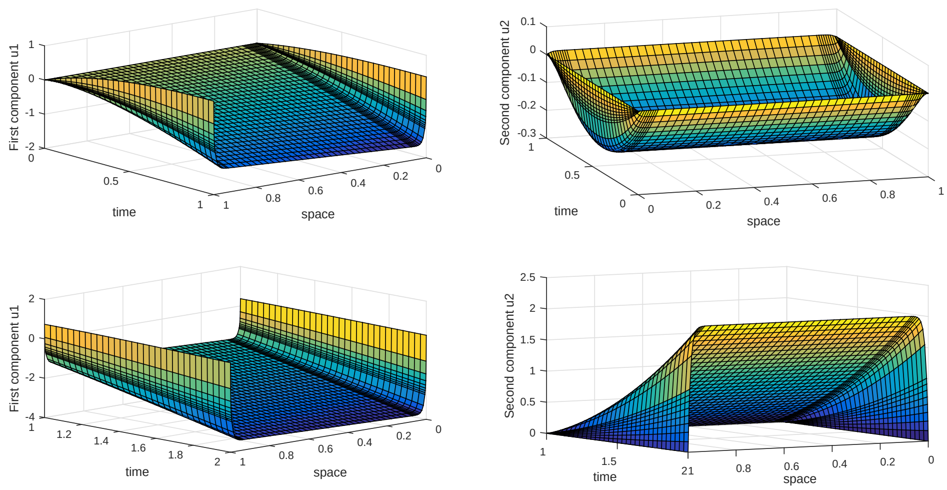

Figure 5.2: Approximate solution plot for Example 5.4.2 taking parameters $\varepsilon_1 = 10^{-6}, \varepsilon_2 = 10^{-4}$ with $N = 32, M = 64$. The left figure is for the 1st component and the right figure is for the 2nd component.

We compute the uniform errors by $E_n^{N,\Delta t} = \max_{\varepsilon_1} E_{n,\varepsilon_1}^{N,\Delta t}$. The rates of convergence are obtained with

$$\varrho_{n,\varepsilon_1} = \log_2(E_{n,\varepsilon_1}^{N,\Delta t}/E_{n,\varepsilon_1}^{2N,\Delta t/4}), \qquad \varrho_n^{unif} = \log_2(E_n^{N,\Delta t}/E_n^{2N,\Delta t/4}).$$

The stopping criteria of the method is chosen to be

$$||\boldsymbol{U}^{[k+1]} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq N^{-2}. \tag{5.4.1}$$

For various values of ε_1, N, and Δt, we provide the numerical results for Examples 5.4.1 and 5.4.2 in Tables 5.1 and 5.2, respectively. For each ε_1, the first and second rows represent the errors $E_{1,\varepsilon_1}^{N,\Delta t}$ and rates of convergence $\varrho_{1,\varepsilon_1}$ for the first component, and the third and fourth rows represent the errors $E_{2,\varepsilon_1}^{N,\Delta t}$ and rates of convergence $\varrho_{2,\varepsilon_1}$ for the second component. At the end of each table we provide uniform errors and uniform rates of convergence for each component of the solution. From Tables 5.1 and 5.2, we see that the numerical results are totally in agreement with Theorem 5.3.5. For fixed $\varepsilon_1 = 10^{-8}$, Tables 5.3 and 5.4 display the number of iterations that the proposed method needs to give satisfactory numerical approximations. Note that only few iterations are required by the proposed method and the number of iteration reduces to one when the parameters are small and of same magnitude.

Figures 5.1 and 5.2 display both the components of the approximate solutions for Examples 5.4.1 and 5.4.2, receptively. Clearly, one can see the layers at both the boundaries.

5.5 Conclusions

In this paper, we have developed and analyzed a robust domain decomposition method of SWR type for a coupled system of singularly perturbed parabolic reaction-diffusion problems with delay in time. We used information obtained from derivative bounds to partition the domain into five overlapping subdomains. On each subdomain the problem is discretized by employing a scheme in which central difference scheme is used for the spatial derivative and the backward Euler scheme is used for the time derivative. We established the discrete maximum principle and based on that the method is shown to be convergent independent of both the small parameters. In the end, some numerical experiments are conducted to confirm the applicability and robustness of the method.