4

A robust domain decomposition method for singularly perturbed parabolic reaction-diffusion systems

We consider the following coupled system of time dependent singularly perturbed reaction-diffusion problems

$$\begin{cases} \boldsymbol{\mathcal{L}u} := \partial_t \boldsymbol{u} - \boldsymbol{E}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} & \text{in } Q := \Omega \times (0,T] = (0,1) \times (0,T], \\ \boldsymbol{u}(x,0) = \boldsymbol{0} & \text{in } \overline{\Omega}, \\ \boldsymbol{u}(0,t) = \boldsymbol{\gamma}_0(t), \boldsymbol{u}(1,t) = \boldsymbol{\gamma}_1(t) & \text{in } (0,T], \end{cases} \tag{4.0.1}$$

where $\boldsymbol{f} = (f_1, f_2)^T$, $\boldsymbol{u} = (u_1, u_2)^T$, $\mathbf{E} = \text{diag}(\varepsilon_1, \varepsilon_2)$ and $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. For each $(x,t) \in \overline{Q}$, the coupling matrix $\boldsymbol{A} = (\, a_{ij}(x,t)\,)$ is assumed to satisfy

$$a_{ij}(x,t) \leq 0, \quad i \neq j, \tag{4.0.2}$$

$$a_{ii}(x,t) > 0, \quad \sum_{j=1}^{2} a_{ij}(x,t) \geq \alpha > 0, \quad i = 1,2. \tag{4.0.3}$$

Further, sufficient regularity and compatibility conditions on the data of (4.0.1) are assumed in order that $\boldsymbol{u} \in C^{4,2}(\overline{Q})^2$, cf. [29, 72]. With these assumptions problem (4.0.1) exhibits a unique solution having overlapping layers at $x = 0$ and $x = 1$ of $O(\sqrt{\varepsilon_i}\ln(1/\varepsilon_i)), i = 1,2$; see [29]. Such systems arise in the modeling of various physical phenomena, for instance in the model for turbulent interaction of waves and currents [73], in the nonlinear model for predator-prey population dynamics [74], and investigation of diffusion processes complicated by chemical reactions in electro analytic chemistry [75].

In the present chapter, we consider the most interesting and more challenging case of the coupled system of singularly perturbed parabolic reaction-diffusion equations where different diffusion parameters are present in each equation, which in general, can have different order of magnitude. We first describe a domain decomposition method of SWR type to solve problem (4.0.1) numerically. The discrete SWR method invokes the decomposition of the original computational domain into five overlapping subdomains. In the iterative steps of the Schwarz method, we employ the central difference scheme in spatial direction and the backward Euler scheme in time direction to solve the problem on each subdomain. We present the error analysis of the method, which is based on some auxiliary problems that allows to prove the uniform convergence in two steps, splitting the discretization error and the iteration error. The numerical approximations obtained from the method are shown to be almost second order (due to the logarithmic factor) uniformly convergent in spatial direction and first order convergent in time direction. At the end, some numerical experiments are conducted to support the theory.

4.1 Derivative bounds

In this section, we obtain bounds on the derivatives needed for convergence analysis of the proposed method. We decompose the solution $\boldsymbol{u}$ as $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, where $\boldsymbol{v}$ is the solution of the following problem

$$\begin{cases} \boldsymbol{\mathcal{L}v} = \boldsymbol{f} & \text{in } Q, \\ \boldsymbol{v}(0,t) = \boldsymbol{z}(0,t), \boldsymbol{v}(1,t) = \boldsymbol{z}(1,t) & \text{in } (0,T], \\ \boldsymbol{v}(x,0) = \boldsymbol{0} & \text{in } \overline{\Omega} \end{cases}$$

with $\boldsymbol{z}$ satisfying

$$\begin{cases} \partial_t \boldsymbol{z} + \boldsymbol{A}\boldsymbol{z} = \boldsymbol{f}, & (x,t) \in \{0,1\} \times (0,T], \\ \boldsymbol{z}(x,0) = \boldsymbol{0}, & x \in \{0,1\}, \end{cases}$$

and $\boldsymbol{w}$ is the solution of the following problem

$$\begin{cases} \boldsymbol{\mathcal{L}w} = \boldsymbol{0} & \text{in } Q, \\ \boldsymbol{w}(0,t) = (\boldsymbol{u}-\boldsymbol{v})(0,t), \boldsymbol{w}(1,t) = (\boldsymbol{u}-\boldsymbol{v})(1,t) & \text{in } (0,T], \\ \boldsymbol{w}(x,0) = \boldsymbol{0} & \text{in } \overline{\Omega}. \end{cases}$$

Here, $\boldsymbol{v}$ is the regular part of the solution and $\boldsymbol{w}$ is the layer part. Now using the ideas in Section 2 of [29], we can prove that the first two derivatives of the solution with respect to time variable are bounded independent of the perturbation parameters. Further, we can obtain bounds on the derivatives of $\boldsymbol{v}$ and $\boldsymbol{w}$ as given in the following lemma.

Lemma 4.1.1. *The solution* $\boldsymbol{u}$ *of (4.0.1) satisfies*

$$\|\partial_t^\ell u_i\|_{\overline{Q}} \le C, \quad \text{for } \ell = 0,1,2, \quad i = 1,2.$$

$$\|\partial_x^s v_1\|_{\overline{Q}} \le C(1+\varepsilon_1^{1-s/2}), \qquad \|\partial_x^s v_2\|_{\overline{Q}} \le C(1+\varepsilon_2^{1-s/2}), \quad s = 0,\dots,4,$$

$$|w_1(x,t)| \le C\mathcal{B}_{\varepsilon_2}(x), \qquad |w_2(x)| \le C\mathcal{B}_{\varepsilon_2}(x),$$

$$|\partial_x^s w_1(x,t)| \le C(\varepsilon_1^{-s/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x)), \quad |\partial_x^s w_2(x,t)| \le C\varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x), \ s = 1,2,$$

$$|\partial_x^s w_1(x,t)| \le C(\varepsilon_1^{-s/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x)), \quad s = 3,4,$$

$$|\partial_x^s w_2(x,t)| \le C\varepsilon_2^{-1}(\varepsilon_1^{-(s-2)/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-(s-2)/2}\mathcal{B}_{\varepsilon_2}(x)), \quad s = 3,4,$$

for all $(x,t) \in \overline{Q}$.

A further decomposition of the layer part given by the following lemma, which we can establish using the arguments of [29, Lemma 10], is also needed for convergence analysis of the proposed method.

Lemma 4.1.2. *Suppose that* $\varepsilon_1 < \varepsilon_2$ *and* $\varepsilon_2 \le \alpha/2$. *Then* $\boldsymbol{w} = (w_1, w_2)^T$ *is decomposed as follows*

$$w_1 = \widehat{w}_{1,\varepsilon_1} + \widehat{w}_{1,\varepsilon_2}, \qquad w_2 = \widehat{w}_{2,\varepsilon_1} + \widehat{w}_{2,\varepsilon_2}, \tag{4.1.1}$$

where

$$|\widehat{w}_{1,\varepsilon_1}(x)| \le \mathcal{B}_{\varepsilon_1}(x,t), \quad |\partial_x^2 \widehat{w}_{1,\varepsilon_1}(x,t)| \le \varepsilon_1^{-1}\mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^4 \widehat{w}_{1,\varepsilon_2}(x,t)| \le \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x), \tag{4.1.2}$$

$$|\widehat{w}_{2,\varepsilon_1}(x,t)| \le \mathcal{B}_{\varepsilon_1}(x), \ |\partial_x^2 \widehat{w}_{2,\varepsilon_1}(x,t)| \le \varepsilon_2^{-1}\mathcal{B}_{\varepsilon_1}(x), \ |\partial_x^4 \widehat{w}_{2,\varepsilon_2}(x,t)| \le \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x), \tag{4.1.3}$$

for all $(x,t) \in \overline{Q}$.

4.2 Domain decomposition method

To set up the method for solving problem (4.0.1), we decompose the computational domain domain Q into five subdomains Q_p, $p = \ell\ell, \ell, m, r, rr$ that are overlapping (see figure 4.4 on page 65). Here, we use behavior of the solution to define the decomposition in the following way

$$Q_p = \Omega_p \times (0,T], \quad p = \ell\ell, \ell, m, r, rr,$$

where $\Omega_{\ell\ell} = (0, 4\sigma_1), \Omega_\ell = (\sigma_1, 4\sigma_2 - 3\sigma_1), \Omega_m = (\sigma_2, 1-\sigma_2), \Omega_r = (1-4\sigma_2+3\sigma_1, 1-\sigma_1)$, and $\Omega_{rr} = (1-4\sigma_1, 1)$ with σ_1 and σ_2 (see [76]) defined as follows

$$\sigma_2 = \min\left\{\frac{4}{26}, \frac{2\sqrt{\varepsilon_2}}{\sqrt{\alpha}}\ln N\right\} \quad \text{and} \quad \sigma_1 = \min\left\{\frac{\sigma_2}{4}, \frac{2\sqrt{\varepsilon_1}}{\sqrt{\alpha}}\ln N\right\}. \tag{4.2.1}$$

Suppose $N = 2^n, n \geq 2$. On a given subdomain $Q_p = \Omega_p \times (0, T]$, where $\Omega_p = (b, a)$, we introduce a rectangular mesh $\Omega_p^N \times \omega^M$, where $\overline{\Omega}_p^N = \{x_i = ih_p,\ i = 0, 1\ldots, N,\ h_p = (a-b)/N\}$ and $\overline{\omega}^M = \{t_j = j\Delta t,\ j = 0, 1\ldots, M,\ \Delta t = T/M\}$ with $\Omega_p^N = \overline{\Omega}_p^N \cap \Omega_p$ and $\omega^M = \overline{\omega}^M \cap (0, T]$. On each subdomain $Q_p^{N,M}$ we consider the central difference scheme for the spatial discretization and the backward Euler scheme for the time discretization. We consider

$$[\boldsymbol{L}_p^{N,M}\boldsymbol{U}_p]_{i,j} = \boldsymbol{f}_{i,j} \tag{4.2.2}$$

with discrete operator defined as follows

$$[\boldsymbol{L}_p^{N,M}\boldsymbol{U}_p]_{i,j} = \begin{pmatrix} [\delta_t U_{p,1}]_{i,j} - \varepsilon_1[\delta_x^2 U_{p,1}]_{i,j} + a_{11;ij}U_{p,1;i,j} + a_{12;i,j}U_{p,2;i,j} \\ [\delta_t U_{p,2}]_{i,j} - \varepsilon_2[\delta_x^2 U_{p,2}]_{i,j} + a_{22;ij}U_{p,2;i,j} + a_{21;i,j}U_{p,1;i,j} \end{pmatrix}, \tag{4.2.3}$$

where

$$[\delta_t V_{p,k}]_{i,j} = \frac{V_{p,k;i,j} - V_{p,k;i,j-1}}{\Delta t}, \quad \text{for } k = 1, 2,$$

and

$$[\delta_x^2 V_{p,k}]_{i,j} = \frac{V_{p,k;i-1,j} - 2V_{p,k;i,j} + V_{p,k;i+1,j}}{h_p^2}, \quad \text{for } k = 1, 2.$$

After describing the discretization on each subdomain, the algorithm is given below.

Step 1. Initialization: Defining $\overline{Q}^{N,M} := (\overline{Q}_{\ell\ell}^{N,M}\backslash\overline{Q}_\ell) \cup (\overline{Q}_\ell^{N,M}\backslash\overline{Q}_m) \cup \overline{Q}_m^N \cup (\overline{Q}_r^{N,M}\backslash\overline{Q}_m) \cup (\overline{Q}_{rr}^{N,M}\backslash\overline{\Omega}_r)$, we consider $\boldsymbol{U}^{[0]}(x_i, t_j)$ for $(x_i, t_j) \in \overline{Q}^{N,M}$ to be

$$\boldsymbol{U}^{[0]}(0, t_j) = \boldsymbol{u}(0, t_j), \quad \boldsymbol{U}^{[0]}(1, t_j) = \boldsymbol{u}(1, t_j) \quad \text{for } t_j \in \omega^M,$$

$$\boldsymbol{U}^{[0]}(x_i, 0) = \boldsymbol{u}(x_i, 0) \quad \text{for } x_i \in \overline{\Omega}^N, \qquad \boldsymbol{U}^{[0]}(x_i, t_j) := \boldsymbol{0}, \ \ x_i \in (0,1),\ t_j \in (0, T].$$

Step 2. On subdomains $Q_p^{N,M}, p = \ell\ell, \ell, m, r, rr$, we compute $\boldsymbol{U}_p^{[k]},\ p = \ell\ell, \ell, m, r, rr$, by solving

$$\begin{cases} [\boldsymbol{L}_{\ell\ell}^{N,M}\boldsymbol{U}_{\ell\ell}^{[k]}]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in Q_{\ell\ell}^{N,M}, \\ \boldsymbol{U}_{\ell\ell}^{[k]}(x_i, 0) = 0 & \text{for } x_i \in \overline{\Omega}_{\ell\ell}^N, \\ \boldsymbol{U}_{\ell\ell}^{[k]}(0, t_j) = \boldsymbol{\gamma}_0(t_j),\ \boldsymbol{U}_{\ell\ell}^{[k]}(4\sigma_1, t_j) = \mathcal{T}_j\boldsymbol{U}^{[k-1]}(4\sigma_1, t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{rr}^{N,M}\boldsymbol{U}_{rr}^{[k]}]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{rr}^{N,M},\\ \boldsymbol{U}_{rr}^{[k]}(x_i,0)=0 & \text{for } x_i\in\overline{\Omega}_{rr}^{N},\\ \boldsymbol{U}_{rr}^{[k]}(1-4\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}^{[k-1]}(1-4\sigma_1,t_j),\ \boldsymbol{U}_{rr}^{[k]}(1,t_j)=\boldsymbol{\gamma}_1(t_j), & \text{for } t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{r}^{N,M}\boldsymbol{U}_{r}^{[k]}]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{r}^{N,M},\\ \boldsymbol{U}_{r}^{[k]}(x_i,0)=0 & \text{for } x_i\in\overline{\Omega}_{r}^{N},\\ \boldsymbol{U}_{r}^{[k]}(1-4\sigma_2+3\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}^{[k-1]}(1-4\sigma_2+3\sigma_1,t_j), & \text{for } t_j\in\omega^M,\\ \boldsymbol{U}_{r}^{[k]}(1-\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}_{rr}^{[k]}(1-\sigma_1,t_j), & \text{for } t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{\ell}^{N,M}\boldsymbol{U}_{\ell}^{[k]}]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{\ell}^{N,M},\\ \boldsymbol{U}_{\ell}^{[k]}(x_i,0)=0 & \text{for } x_i\in\overline{\Omega}_{\ell}^{N},\\ \boldsymbol{U}_{\ell}^{[k]}(\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}_{\ell\ell}^{[k]}(\sigma_1,t_j), & \text{for } t_j\in\omega^M,\\ \boldsymbol{U}_{\ell}^{[k]}(4\sigma_2-3\sigma_1,t_j)=\mathcal{T}_j\boldsymbol{U}^{[k-1]}(4\sigma_2-3\sigma_1,t_j) & \text{for } t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{m}^{N,M}\boldsymbol{U}_{m}^{[k]}]_{i,j} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j)\in Q_{m}^{N,M},\\ \boldsymbol{U}_{m}^{[k]}(x_i,0)=0 & \text{for } x_i\in\overline{\Omega}_{m}^{N},\\ \boldsymbol{U}_{m}^{[k]}(\sigma_2,t_j)=\mathcal{T}_j\boldsymbol{U}_{\ell}^{[k]}(\sigma_2,t_j), & \text{for } t_j\in\omega^M,\\ \boldsymbol{U}_{m}^{[k]}(1-\sigma_2,t_j)=\mathcal{T}_j\boldsymbol{U}_{r}^{[k]}(1-\sigma_2,t_j), & \text{for } t_j\in\omega^M. \end{cases}$$

Step 3. On combining the solutions $\boldsymbol{U}_p^{[k]}$, $p=\ell\ell,\ell,m,r,rr$, we get $\boldsymbol{U}^{[k]}$ given by

$$\boldsymbol{U}^{[k]}(x_i,t_j)=\begin{cases} \boldsymbol{U}_{\ell\ell}^{[k]}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{\ell\ell}^{N,M}\setminus\overline{Q}_{\ell};\\ \boldsymbol{U}_{\ell}^{[k]}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{\ell}^{N,M}\setminus\overline{Q}_{m};\\ \boldsymbol{U}_{m}^{[k]}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{m}^{N,M};\\ \boldsymbol{U}_{r}^{[k]}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{r}^{N,M}\setminus\overline{Q}_{m};\\ \boldsymbol{U}_{rr}^{[k]}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{rr}^{N,M}\setminus\overline{Q}_{r}. \end{cases} \tag{4.2.4}$$

Step 4. Rule for stopping: when

$$||\boldsymbol{U}^{[k+1]}-\boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}}\le\Upsilon$$

is satisfied, where Υ is prescribed accuracy, then we stop the iteration process; otherwise we go to Step 2.

It is easy to verify that the following discrete maximum principle holds for the operator $\boldsymbol{L}_p^{N,M}$.

Lemma 4.2.1. *Suppose* $\boldsymbol{Z}_{p;0,j}\ge\boldsymbol{0}$ *and* $\boldsymbol{Z}_{p;N,j}\ge\boldsymbol{0}$ *for* $t_j\in\omega^M$, *and* $\boldsymbol{Z}_{p;i,0}\ge\boldsymbol{0}$ *for* $x_i\in\overline{\Omega}_p^N$. *If* $[\boldsymbol{L}_p^{N,M}\boldsymbol{Z}_p]_{ij}\ge\boldsymbol{0}$ *for* $(x_i,t_j)\in Q_p^{N,M}$, *then* $\boldsymbol{Z}_{p;i,j}\ge\boldsymbol{0}$ *for* $(x_i,t_j)\in\overline{Q}_p^{N,M}$.

4.3 Error analysis

We provide uniform convergence analysis of our method in this section. For analysis we consider $\widetilde{\boldsymbol{U}}_p,\ p=\ell\ell,\ell,m,r,rr$, satisfying

$$\begin{cases} [\boldsymbol{L}_{\ell\ell}^{N,M}\widetilde{\boldsymbol{U}}_{\ell\ell}]_{i,j}=\boldsymbol{f}_{i,j}, & (x_i,t_j)\in Q_{\ell\ell}^{N,M},\\ \widetilde{\boldsymbol{U}}_{\ell\ell}(x_i,0)=0, & x_i\in\overline{\Omega}_{\ell\ell}^N,\\ \widetilde{\boldsymbol{U}}_{\ell\ell}(0,t_j)=\boldsymbol{u}(0,t_j),\widetilde{\boldsymbol{U}}_{\ell\ell}(4\sigma_1,t_j)=\boldsymbol{u}(4\sigma_1,t_j), & t_j\in\omega^M,\end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{rr}^{N,M}\widetilde{\boldsymbol{U}}_{rr}]_{i,j}=\boldsymbol{f}_{i,j}, & (x_i,t_j)\in Q_{rr}^{N,M},\\ \widetilde{\boldsymbol{U}}_{rr}(x_i,0)=0, & x_i\in\overline{\Omega}_{rr}^N,\\ \widetilde{\boldsymbol{U}}_{rr}(1-4\sigma_1,t_j)=\boldsymbol{u}(1-4\sigma_1,t_j),\widetilde{\boldsymbol{U}}_{rr}(1,t_j)=\boldsymbol{u}(1,t_j), & t_j\in\omega^M,\end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{r}^{N,M}\widetilde{\boldsymbol{U}}_{r}]_{i,j}=\boldsymbol{f}_{i,j}, & (x_i,t_j)\in Q_{r}^{N,M},\\ \widetilde{\boldsymbol{U}}_{r}(x_i,0)=0, & x_i\in\overline{\Omega}_{r}^N,\\ \widetilde{\boldsymbol{U}}_{r}(1-4\sigma_2+3\sigma_1,t_j)=\boldsymbol{u}(1-4\sigma_2+3\sigma_1,t_j), & t_j\in\omega^M\\ \widetilde{\boldsymbol{U}}_{r}(1-4\sigma_1,t_j)=\boldsymbol{u}(1-\sigma_1,t_j), & t_j\in\omega^M,\end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{\ell}^{N,M}\widetilde{\boldsymbol{U}}_{\ell}]_{i,j}=\boldsymbol{f}_{i,j}, & (x_i,t_j)\in Q_{\ell}^{N,M},\\ \widetilde{\boldsymbol{U}}_{\ell}(x_i,0)=0, & x_i\in\overline{\Omega}_{\ell}^N,\\ \widetilde{\boldsymbol{U}}_{\ell}(\sigma_1,t_j)=\boldsymbol{u}(\sigma_1,t_j),\widetilde{\boldsymbol{U}}_{\ell}(4\sigma_2-3\sigma_1,t_j)=\boldsymbol{u}(4\sigma_2-3\sigma_1,t_j), & t_j\in\omega^M,\end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{m}^{N,M}\widetilde{\boldsymbol{U}}_{m}]_{i,j}=\boldsymbol{f}_{i,j}, & (x_i,t_j)\in Q_{m}^{N,M},\\ \widetilde{\boldsymbol{U}}_{m}(x_i,0)=0, & x_i\in\overline{\Omega}_{m}^N,\\ \widetilde{\boldsymbol{U}}_{m}(\sigma_2,t_j)=\boldsymbol{u}(\sigma_2,t_j),\widetilde{\boldsymbol{U}}_{m}(1-\sigma_2,t_j)=\boldsymbol{u}(1-\sigma_2,t_j), & t_j\in\omega^M,\end{cases}$$

where $\boldsymbol{L}_p^{N,M}$ is the operator that was defined in Section 4.2, and $\boldsymbol{u}$ is the exact solution of problem (4.0.1).

Next we define

$$\widetilde{\boldsymbol{U}}(x_i,t_j)=\begin{cases} \widetilde{\boldsymbol{U}}_{\ell\ell}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{\ell\ell}^{N,M}\setminus\overline{Q}_{\ell};\\ \widetilde{\boldsymbol{U}}_{\ell}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{\ell}^{N,M}\setminus\overline{\Omega}_{m};\\ \widetilde{\boldsymbol{U}}_{m}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{m}^{N,M};\\ \widetilde{\boldsymbol{U}}_{r}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{r}^{N,M}\setminus\overline{Q}_{m};\\ \widetilde{\boldsymbol{U}}_{rr}(x_i,t_j),\ (x_i,t_j)\in\overline{Q}_{rr}^{N,M}\setminus\overline{Q}_{r}.\end{cases} \tag{4.3.1}$$

Now we can separate the discretization error and iteration error as follows

$$||\boldsymbol{u}-\boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}}\le||\boldsymbol{u}-\widetilde{\boldsymbol{U}}||_{\overline{Q}^{N,M}}+||\widetilde{\boldsymbol{U}}-\boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}}. \tag{4.3.2}$$

Thus, we can separately bound both the terms. The following lemma gives a bound for the first term.

Lemma 4.3.1. *Suppose $\boldsymbol{u}$ is the exact solution of problem (4.0.1) and $\widetilde{\boldsymbol{U}}$ is as defined in (4.3.1). Then*

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}||_{\overline{Q}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N). \tag{4.3.3}$$

Proof. To simplify our presentation we consider $\sigma_2 = (2\sqrt{\varepsilon_2}\ln N)/\sqrt{\alpha}$ and $\sigma_1 = (2\sqrt{\varepsilon_1}\ln N)/\sqrt{\alpha}$, that is ε_1 and ε_2 are small and of different magnitude. This is the most interesting case of problem (4.0.1), as overlapping layers occur in this case.

Note that

$$[\boldsymbol{L}_p^{N,M}(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_p)] = [\boldsymbol{L}_p^{N,M}\boldsymbol{u} - \boldsymbol{L}\boldsymbol{u}] = [\delta_t \boldsymbol{u} - \partial_t \boldsymbol{u}] + \boldsymbol{E}[\partial_x^2 \boldsymbol{u} - \delta_x^2 \boldsymbol{u}].$$

For $(x_i, t_j) \in Q_{\ell\ell}^{N,M}$, $n = 1, 2$, Taylor expansions give

$$|[L_{\ell\ell,n}^{N,M}(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_{\ell\ell})]_{i,j}| \leq C\Delta t||\partial_t^2 u_n(x_i, .)||_{[t_{j-1},t_j]} + C\varepsilon_n h_{\ell\ell}^2||\partial_x^4 u_n(., t_j)||_{[x_{i-1},x_{i+1}]}.$$

Using bounds on the derivatives in Lemma 4.1.1 and $h_{\ell\ell} \leq C\sqrt{\varepsilon_1}N^{-1}\ln N$, we get

$$|[L_{\ell\ell,n}^{N,M}(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_{\ell\ell})]_{i,j}| \leq C(\Delta t + N^{-2}\ln^2 N).$$

Hence, using Lemma 4.2.1 we obtain

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}_{\ell\ell}||_{Q_{\ell\ell}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Similarly

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}_{rr}||_{Q_{rr}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Now for $(x_i, t_j) \in Q_{\ell}^{N,M}$, $n = 1, 2$,

$$[L_{\ell,n}^{N,M}(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_{\ell})]_{i,j}| = [\delta_t u_n - \partial_t u_n]_{i,j} + \varepsilon_n[\partial_x^2 u_n - \delta_x^2 u_n]_{i,j}.$$

By Taylor expansion and Lemma 4.1.1, the first term is bounded as follows

$$|[\delta_t u_n - \partial_t u_n]_{i,j}| \leq C\Delta t, \quad (x_i, t_j) \in Q_{\ell}^{N,M}, n = 1, 2.$$

We use $u_n = v_n + w_n$ and $w_n = \widehat{w}_{n,\varepsilon_1} + \widehat{w}_{n,\varepsilon_2}$ to bound the second term. We obtain

$$\varepsilon_n|[\partial_x^2 u_n - \delta_x^2 u_n]_{i,j}| \leq \varepsilon_n|[\partial_x^2 v_n - \delta_x^2 v_n]_{i,j}| + \varepsilon_n|[\partial_x^2 \widehat{w}_{n,\varepsilon_1} - \delta_x^2 \widehat{w}_{n,\varepsilon_1}]_{i,j}| + \varepsilon_n|[\partial_x^2 \widehat{w}_{n,\varepsilon_2} - \delta_x^2 \widehat{w}_{n,\varepsilon_2}]_{i,j}|.$$

Using Taylor expansions, Lemmas 4.1.1 and 4.1.2, and the bound on the mesh width we have

$$\varepsilon_n|[\partial_x^2 u_n - \delta_x^2 u_n]_{i,j}| \leq C\varepsilon_n h_\ell^2 \|\partial_x^4 v_n(.,t_j)\|_{[x_{i-1},x_{i+1}]} + C\varepsilon_n \|\partial_x^2 \widehat{w}_{n,\varepsilon_1}(.,t_j)\|_{[x_{i-1},x_{i+1}]}$$
$$+ C\varepsilon_n h_\ell^2 \|\partial_x^4 \widehat{w}_{n,\varepsilon_2}(.,t_j)\|_{[x_{i-1},x_{i+1}]}$$
$$\leq CN^{-2}\ln^2 N.$$

Therefore, using Lemma 4.2.1 we get

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}_\ell||_{\overline{Q}_\ell^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

In a similar way

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}_r||_{\overline{Q}_r^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

We again use $u_n = v_n + w_n$, and consider Taylor expansions and Lemma 4.1.1 to get

$$[L_{m,n}^{N,M}(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_m)]_{i,j}| \leq |[\delta_t u_n - \partial_t u_n]_{i,j}| + \varepsilon_n|[\partial_x^2 v_n - \delta_x^2 v_n]_{i,j}| + \varepsilon_n|[\partial_x^2 w_n - \delta_x^2 w_n]_{i,j}|.$$
$$\leq C\Delta t + C\varepsilon_n h_m^2 \|\partial_x^4 v_n(.,t_j)\|_{[x_{i-1},x_{i+1}]} + C\varepsilon_n \|\partial_x^2 w_n(.,t_j)\|_{[x_{i-1},x_{i+1}]}$$
$$\leq C(\Delta t + N^{-2}).$$

So, we use Lemma 4.2.1 to get

$$||\boldsymbol{u} - \widetilde{\boldsymbol{U}}_m||_{\overline{Q}_m^{N,M}} \leq C(\Delta t + N^{-2}).$$

Hence, combining the bounds on subdomains $\overline{Q}_p^{N,M}$, $p = \ell\ell, \ell, m, r, rr$, we have the desired result. □

The following lemma provides a bound for the iteration error $||\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}}$.

Lemma 4.3.2. *Suppose $\widetilde{\boldsymbol{U}}$ is as defined in (4.3.1) and $\boldsymbol{U}^{[k]}$ is the k^{th} iterate of the algorithm defined in Section 4.2. Then*

$$||\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq C2^{-k} + C(\Delta t + N^{-2}\ln^2 N). \tag{4.3.4}$$

Proof. Before we proceed to prove this lemma, we define

$$\vartheta^{[k]} = \max\{||(\widetilde{\boldsymbol{U}}_{\ell\ell} - \mathcal{T}_j\boldsymbol{U}^{[k-1]})(4\sigma_1, t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_{rr} - \mathcal{T}_j\boldsymbol{U}^{[k-1]})(1-4\sigma_1, t_j)||_\infty,$$
$$||(\widetilde{\boldsymbol{U}}_\ell - \mathcal{T}_j\boldsymbol{U}^{[k-1]})(4\sigma_2 - 3\sigma_1, t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_r - \mathcal{T}_j\boldsymbol{U}^{[k-1]})(1-4\sigma_2+3\sigma_1, t_j)||_\infty\},$$
$$\vartheta_{\sigma_1} = \max\left\{||(\widetilde{\boldsymbol{U}}_\ell - \widetilde{\boldsymbol{U}}_{\ell\ell})(\sigma_1, t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_r - \widetilde{\boldsymbol{U}}_{rr})(1-\sigma_1, t_j)||_\infty\right\},$$

$$\vartheta_{\sigma_2} = \max\left\{||(\widetilde{\boldsymbol{U}}_m - \widetilde{\boldsymbol{U}}_\ell)(\sigma_2, t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_m - \widetilde{\boldsymbol{U}}_r)(1-\sigma_2, t_j)||_\infty\right\},$$

$$\vartheta_{4\sigma_1} = \max\left\{||(\widetilde{\boldsymbol{U}}_{\ell\ell} - \mathcal{T}_j\widetilde{\boldsymbol{U}}_\ell)(4\sigma_1, t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_{rr} - \mathcal{T}_j\widetilde{\boldsymbol{U}}_r)(1-4\sigma_1, t_j)||_\infty\right\},$$

$$\vartheta_{4\sigma_2-3\sigma_1} = \max\left\{||(\widetilde{\boldsymbol{U}}_\ell - \mathcal{T}_j\widetilde{\boldsymbol{U}}_m)(4\sigma_2-3\sigma_1, t_j)||_\infty, ||(\widetilde{\boldsymbol{U}}_r - \mathcal{T}_j\widetilde{\boldsymbol{U}}_m)(1-4\sigma_2+3\sigma_1, t_j)||_\infty\right\}.$$

Now

$$\left[\boldsymbol{L}_{\ell\ell}^{N,M}\left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)\right] = \boldsymbol{0} \qquad \text{in } Q_{\ell\ell}^{N,M}, \qquad \left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)(x_i, 0) = \boldsymbol{0}, \quad x_i \in \overline{\Omega}^N,$$

$$\left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)(0, t_j) = \boldsymbol{0}, \qquad \left|\left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)(4\sigma_1, t_j)\right| \le \vartheta^{[1]}\boldsymbol{1}, \qquad t_j \in \omega^M.$$

So, using Lemma 4.2.1 with

$$\boldsymbol{\psi}^{\pm}(x_i, t_j) := \frac{x_i}{4\sigma_1}\vartheta^{[1]}\boldsymbol{1} \pm (\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]})(x_i, t_j),$$

we obtain

$$\left|(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]})(x_i, t_j)\right| \le \frac{x_i}{4\sigma_1}\vartheta^{[1]}\boldsymbol{1} \qquad \text{for } (x_i, t_j) \in \overline{Q}_{\ell\ell}^{N,M}.$$

Hence

$$\left\|\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right\|_{\overline{Q}_{\ell\ell}^{N,M}\setminus\overline{Q}_\ell} \le \frac{1}{4}\vartheta^{[1]}. \tag{4.3.5}$$

Similarly we can prove that

$$\left\|\widetilde{\boldsymbol{U}}_{rr} - \boldsymbol{U}_{rr}^{[1]}\right\|_{\overline{Q}_{rr}^{N,M}\setminus\overline{Q}_r} \le \frac{1}{4}\vartheta^{[1]}. \tag{4.3.6}$$

Next

$$\left[\boldsymbol{L}_{\ell}^{N,M}\left(\widetilde{\boldsymbol{U}}_{\ell} - \boldsymbol{U}_{\ell}^{[1]}\right)\right] = \boldsymbol{0} \qquad \text{in } Q_{\ell}^{N,M}, \qquad \left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)(x_i, 0) = \boldsymbol{0}, \quad x_i \in \overline{\Omega}^N,$$

$$\left|\left(\widetilde{\boldsymbol{U}}_{\ell} - \boldsymbol{U}_{\ell}^{[1]}\right)(4\sigma_2 - 3\sigma_1, t_j)\right| \le \vartheta^{[1]}\boldsymbol{1}, \quad t_j \in \omega^M.$$

As (σ_1, t_j) is the mesh point of $\overline{Q}_{\ell\ell}^{N,M}$, we have

$$\begin{aligned}
\left|\left(\widetilde{\boldsymbol{U}}_{\ell} - \boldsymbol{U}_{\ell}^{[1]}\right)(\sigma_1, t_j)\right| &= \left|\left(\widetilde{\boldsymbol{U}}_{\ell} - \mathcal{T}_j\boldsymbol{U}_{\ell\ell}^{[1]}\right)(\sigma_1, t_j)\right| \\
&\le \left|\left(\widetilde{\boldsymbol{U}}_{\ell} - \widetilde{\boldsymbol{U}}_{\ell\ell}\right)(\sigma_1, t_j)\right| + \left|\left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \boldsymbol{U}_{\ell\ell}^{[1]}\right)(\sigma_1, t_j)\right| \\
&\le \vartheta_{\sigma_1}\boldsymbol{1} + \frac{1}{4}\vartheta^{[1]}\boldsymbol{1}, \quad t_j \in \omega^M.
\end{aligned}$$

Defining

$$\phi(x) := \frac{-x^2 + (13\sigma_2 - 11\sigma_1)x + 12\sigma_2^2 + 24\sigma_1^2 - 37\sigma_1\sigma_2}{48(\sigma_2 - \sigma_1)^2},\ x \in [\sigma_1, 4\sigma_2 - 3\sigma_1],$$

we have $\phi(\sigma_1) = 1/4$, $\phi(4\sigma_2 - 3\sigma_1) = 1$, $\phi > 0$ in $\overline{\Omega}_\ell^N$, and $[\boldsymbol{L}_\ell^{N,M}\phi\mathbf{1}] > \mathbf{0}$ in $Q_\ell^{N,M}$. Also note that φ is a monotonically increasing function and $\varphi(\sigma_2) = 1/2$. Hence, we can use Lemma 4.2.1 with

$$\boldsymbol{\psi}^{\pm}(x_i, t_j) := \phi(x_i)\vartheta^{[1]}\mathbf{1} + \vartheta_{\sigma_1}\mathbf{1} \pm \left(\widetilde{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}\right)(x_i, t_j),$$

to get

$$\left|\left(\widetilde{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}\right)(x_i, t_j)\right| \le \phi(x_i)\vartheta^{[1]}\mathbf{1} + \vartheta_{\sigma_1}\mathbf{1} \ \text{ for } (x_i, t_j) \in \overline{Q}_\ell^{N,M}.$$

Consequently

$$\left\|\widetilde{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}\right\|_{\overline{Q}_\ell^{N,M}\setminus\overline{Q}_m} \le \frac{1}{2}\vartheta^{[1]} + \vartheta_{\sigma_1}. \tag{4.3.7}$$

Similar arguments can be used for proving

$$\left\|\widetilde{\boldsymbol{U}}_r - \boldsymbol{U}_r^{[1]}\right\|_{\overline{Q}_r^{N,M}\setminus\overline{Q}_m} \le \frac{1}{2}\vartheta^{[1]} + \vartheta_{\sigma_1}. \tag{4.3.8}$$

We have

$$\left[\boldsymbol{L}_m^{N,M}\left(\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right)\right] = \mathbf{0} \qquad \text{in } Q_m^{N,M}, \qquad \left(\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right)(x_i, 0) = \mathbf{0}, \quad x_i \in \overline{\Omega}^N.$$

As (σ_2, t_j) and $(1-\sigma_2, t_j)$, respectively, are the mesh points of $\overline{Q}_\ell^{N,M}$ and $\overline{Q}_r^{N,M}$, we have

$$\left|\left(\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right)(\sigma_2, t_j)\right| \le \left|\left(\widetilde{\boldsymbol{U}}_m - \widetilde{\boldsymbol{U}}_\ell\right)(\sigma_2, t_j)\right| + \left|\left(\widetilde{\boldsymbol{U}}_\ell - \boldsymbol{U}_\ell^{[1]}\right)(\sigma_2, t_j)\right|$$

$$\le \vartheta_{\sigma_2}\mathbf{1} + \frac{1}{2}\vartheta^{[1]}\mathbf{1} + \vartheta_{\sigma_1}\mathbf{1}, \quad t_j \in \omega^M,$$

$$\left|\left(\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right)(1-\sigma_2, t_j)\right| \le \left|\left(\widetilde{\boldsymbol{U}}_m - \widetilde{\boldsymbol{U}}_r\right)(1-\sigma_2, t_j)\right| + \left|\left(\widetilde{\boldsymbol{U}}_r - \boldsymbol{U}_r^{[1]}\right)(1-\sigma_2, t_j)\right|$$

$$\le \vartheta_{\sigma_2}\mathbf{1} + \frac{1}{2}\vartheta^{[1]}\mathbf{1} + \vartheta_{\sigma_1}\mathbf{1}, \quad t_j \in \omega^M.$$

Therefore, use Lemma 4.2.1 to get

$$\left\|\widetilde{\boldsymbol{U}}_m - \boldsymbol{U}_m^{[1]}\right\|_{\overline{Q}_m^{N,M}} \le \frac{1}{2}\vartheta^{[1]} + \vartheta_{\sigma_1} + \vartheta_{\sigma_2}. \tag{4.3.9}$$

From (4.3.5), (4.3.6), (4.3.7), (4.3.8), and (4.3.9), we conclude that

$$\left\|\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[1]}\right\|_{\overline{Q}^{N,M}} \leq \frac{1}{2}\vartheta^{[1]} + \vartheta_{\sigma_1} + \vartheta_{\sigma_2}.$$

For bounding $\|\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[2]}\|_{\overline{Q}^{N,M}}$ we will require a bound on $\vartheta^{[2]}$. For the same, we use a triangle inequality, the stability of the operator $\mathcal{T}_j$, (4.3.7), (4.3.8), and (4.3.9) to get

$$\left|\left(\widetilde{\boldsymbol{U}}_{\ell\ell} - \mathcal{T}_j \boldsymbol{U}^{[1]}\right)(4\sigma_1, t_j)\right| \leq \vartheta_{4\sigma_1}\mathbf{1} + \frac{1}{2}\vartheta^{[1]}\mathbf{1} + \vartheta_{\sigma_1}\mathbf{1}, \quad t_j \in \omega^M,$$

$$\left|\left(\widetilde{\boldsymbol{U}}_{rr} - \mathcal{T}_j \boldsymbol{U}^{[1]}\right)(1 - 4\sigma_1, t_j)\right| \leq \vartheta_{4\sigma_1}\mathbf{1} + \frac{1}{2}\vartheta^{[1]}\mathbf{1} + \vartheta_{\sigma_1}\mathbf{1}, \quad t_j \in \omega^M,$$

$$\left|\left(\widetilde{\boldsymbol{U}}_{\ell} - \mathcal{T}_j \boldsymbol{U}^{[1]}\right)(4\sigma_2 - 3\sigma_1, t_j)\right| \leq \vartheta_{4\sigma_2 - 3\sigma_1}\mathbf{1} + \frac{1}{2}\vartheta^{[1]}\mathbf{1} + \vartheta_{\sigma_1}\mathbf{1} + \vartheta_{\sigma_2}\mathbf{1}, \quad t_j \in \omega^M,$$

$$\left|\left(\widetilde{\boldsymbol{U}}_{r} - \mathcal{T}_j \boldsymbol{U}^{[1]}\right)(1 - 4\sigma_2 + 3\sigma_1, t_j)\right| \leq \vartheta_{4\sigma_2 - 3\sigma_1}\mathbf{1} + \frac{1}{2}\vartheta^{[1]}\mathbf{1} + \vartheta_{\sigma_1}\mathbf{1} + \vartheta_{\sigma_2}\mathbf{1}, \quad t_j \in \omega^M.$$

Thus

$$\vartheta^{[2]} \leq \frac{1}{2}\vartheta^{[1]} + \vartheta_{\sigma_1} + \vartheta_{\sigma_2} + \vartheta_{4\sigma_1} + \vartheta_{4\sigma_2 - 3\sigma_1}.$$

Hence, letting $\varphi = \vartheta_{\sigma_1} + \vartheta_{\sigma_2} + \vartheta_{4\sigma_1} + \vartheta_{4\sigma_2 - 3\sigma_1}$, we have

$$\max\left\{\vartheta^{[2]}, \left\|\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[1]}\right\|_{\overline{Q}^{N,M}}\right\} \leq \varphi + \vartheta^{[1]}/2.$$

We repeat the arguments to get

$$\max\left\{\vartheta^{[k+1]}, \left\|\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[k]}\right\|_{\overline{Q}^{N,M}}\right\} \leq \varphi + \vartheta^{[k]}/2.$$

Consequently $\vartheta^{[k]} \leq 2\varphi + \vartheta^{[1]}/2^{(k-1)}$, and hence

$$\left\|\widetilde{\boldsymbol{U}} - \boldsymbol{U}^{[k]}\right\|_{\overline{Q}^{N,M}} \leq 2\varphi + \vartheta^{[1]}/2^k. \tag{4.3.10}$$

Now using Lemma 4.1.1, we can show that $\vartheta^{[1]} \leq C$. Thus, we are left to bound φ.

Using Lemma 4.3.1, $\vartheta_{\sigma_1} + \vartheta_{\sigma_2} \leq C(\Delta t + N^{-2}\ln^2 N)$, where the fact that $(\sigma_1, t_j) \in \overline{Q}_{\ell\ell}^{N,M}$, $(1 - \sigma_1, t_j) \in \overline{Q}_{rr}^{N,M}$, $(\sigma_2, t_j) \in \overline{Q}_{\ell}^{N,M}$, $(1 - \sigma_2, t_j) \in \overline{Q}_{r}^{N,M}$, is also used. For bounding $\vartheta_{4\sigma_1}$, use a triangle inequality to get

$$\left|\left(\boldsymbol{u} - \mathcal{T}_j \widetilde{\boldsymbol{U}}_{\ell}\right)(4\sigma_1, t_j)\right| \leq \left|\left(\boldsymbol{u} - \mathcal{T}_j \boldsymbol{u}\right)(4\sigma_1, t_j)\right| + \left|\mathcal{T}_j\left(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_{\ell}\right)(4\sigma_1, t_j)\right|. \tag{4.3.11}$$

By stability of $\mathcal{T}_j$ and Lemma 4.3.1, we have

$$\left|\mathcal{T}_j\left(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_{\ell}\right)(4\sigma_1, t_j)\right| \leq \boldsymbol{C}(\Delta t + N^{-2}\ln^2 N), \quad t_j \in \omega^M.$$

For the interpolation error, using the solution decomposition $u_n = v_n + w_n$, we have

$$|(u_n - \mathcal{T}_j u_n)(4\sigma_1, t_j)| \leq |(v_n - \mathcal{T}_j v_n)(4\sigma_1, t_j)| + |(w_n - \mathcal{T}_j w_n)(4\sigma_1, t_j)|, \ t_j \in \omega^M, \ n = 1, 2. \tag{4.3.12}$$

The error bound corresponding to the regular part is obtained using standard interpolation error estimate and Lemma 4.1.1. We get

$$|(v_n - \mathcal{T}_j v_n)(4\sigma_1, t_j)| \leq Ch_\ell^2 ||\partial_x^2 v(., t_j)||_{[x_i, x_{i+1}]} \leq CN^{-2}, n = 1, 2, \ t_j \in \omega^M.$$

For the layer part the argument, using Lemma 4.1.2 and standard interpolation error estimates, proceeds in the following way

$$\begin{aligned} |(w_n - \mathcal{T}_j w_n)(4\sigma_1, t_j)| &\leq C|(\widehat{w}_{n,\varepsilon_1} - \mathcal{T}_j \widehat{w}_{n,\varepsilon_1})(4\sigma_1, t_j)| + |(\widehat{w}_{n,\varepsilon_2} - \mathcal{T}_j \widehat{w}_{n,\varepsilon_2})(4\sigma_1, t_j)| \\ &\leq ||\widehat{w}_{n,\varepsilon_1}(., t_j)||_{[x_i, x_{i+1}]} + Ch_\ell^2 ||\partial_x^2 \widehat{w}_{n,\varepsilon_2}(., t_j)||_{[x_i, x_{i+1}]} \\ &\leq C\|\mathcal{B}_{\varepsilon_1}\|_{[x_i, x_{i+1}]} + Ch_\ell^2 \varepsilon_2^{-1} \|\mathcal{B}_{\varepsilon_2}\|_{[x_i, x_{i+1}]} \\ &\leq CN^{-2} \ln^2 N, n = 1, 2, \ t_j \in \omega^M. \end{aligned}$$

Similarly

$$|(\boldsymbol{u} - \mathcal{T}_j \widetilde{\boldsymbol{U}}_r)(1 - 4\sigma_1, t_j)| \leq \boldsymbol{C}(\Delta t + N^{-2} \ln^2 N), \ t_j \in \omega^M.$$

Therefore

$$\vartheta_{4\sigma_1} \leq C(\Delta t + N^{-2} \ln^2 N).$$

For estimating $\vartheta_{4\sigma_2 - 3\sigma_1}$, we use a triangle inequality to get

$$\left| \left(\boldsymbol{u} - \mathcal{T}_j \widetilde{\boldsymbol{U}}_m \right) (4\sigma_2 - 3\sigma_1, t_j) \right| \leq |(\boldsymbol{u} - \mathcal{T}_j \boldsymbol{u}) (4\sigma_2 - 3\sigma_1, t_j)| + \left| \mathcal{T}_j \left(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_m \right) (4\sigma_2 - 3\sigma_1, t_j) \right|.$$

Using previous argument we get

$$\left| \mathcal{T}_j \left(\boldsymbol{u} - \widetilde{\boldsymbol{U}}_m \right) (4\sigma_2 - 3\sigma_1, t_j) \right| \leq \boldsymbol{C}(\Delta t + N^{-2} \ln^2 N), \quad t_j \in \omega^M.$$

To bound the interpolation error we use solution decomposition $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ of Lemma 4.1.1 to get

$$\begin{aligned} |(\boldsymbol{u} - \mathcal{T}_j \boldsymbol{u})(4\sigma_2 - 3\sigma_1, t_j)| &\leq |(\boldsymbol{v} - \mathcal{T}_j \boldsymbol{v})(4\sigma_2 - 3\sigma_1, t_j)| + |(\boldsymbol{w} - \mathcal{T}_j \boldsymbol{w})(4\sigma_2 - 3\sigma_1, t_j)| \\ &\leq \boldsymbol{C}h_m^2 ||\partial_x^2 \boldsymbol{v}(., t_j)||_{[x_i, x_{i+1}]} + \boldsymbol{C}||\boldsymbol{w}(., t_j)||_{[x_i, x_{i+1}]} \end{aligned}$$

$$\leq \boldsymbol{C}N^{-2},\ t_j \in \omega^M.$$

Similarly

$$|(\boldsymbol{u} - \mathcal{T}_j \widetilde{\boldsymbol{U}}_m)(1 - 4\sigma_2 + 3\sigma_1, t_j)| \leq \boldsymbol{C}(\Delta t + N^{-2}\ln^2 N),\ t_j \in \omega^M.$$

Hence

$$\vartheta_{4\sigma_2 - 3\sigma_1} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Combine the bounds for ϑ_{σ_1}, ϑ_{σ_2}, $\vartheta_{4\sigma_1}$, and $\vartheta_{4\sigma_2-3\sigma_1}$ to get $\varphi \leq C(\Delta t + N^{-2}\ln^2 N)$. This proves the lemma. □

Finally, we use (4.3.2) along with Lemmas 4.3.1 and 4.3.2, to arrive at the following main theorem of this chapter.

Theorem 4.3.3. *Suppose* $\boldsymbol{u}$ *is the solution of problem (4.0.1) and* $\boldsymbol{U}^{[k]}$ *is the* k^{th} *iterate of the algorithm defined in the previous section. Then*

$$||\boldsymbol{u} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq C(2^{-k} + \Delta t + N^{-2}\ln^2 N). \tag{4.3.13}$$

4.4 Numerical results

To verify our theoretical findings of Section 4.3 we consider three test examples and present numerical results in this section. The stopping rule of the algorithm for all three test examples is as follows

$$||\,\boldsymbol{U}^{[k+1]} - \boldsymbol{U}^{[k]}||_{\overline{Q}^{N,M}} \leq N^{-2}. \tag{4.4.1}$$

Example 4.4.1. Consider the following problem [32]

$$\begin{cases} \partial_t \boldsymbol{u} - \boldsymbol{E}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} & \textit{in } Q := (0,1) \times (0,1], \\ \boldsymbol{u}(x,0) = \boldsymbol{0} & \textit{in } [0,1], \\ \boldsymbol{u}(0,t) = \boldsymbol{0}, \boldsymbol{u}(1,t) = \boldsymbol{0} & \textit{in } (0,1], \end{cases} \tag{4.4.2}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 2(1+x)^2 & -\,(1+x^3) \\ -2\cos(\pi x/4) & 2.2e^{1-x} \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} \cos(\pi x/2) \\ x \end{pmatrix}.$$

The solution of this system is not known. So, we apply the double mesh principle (see [5]) to compute the errors and the convergence rates. We calculate $\boldsymbol{U}^{N,\Delta t}$ and $\boldsymbol{U}^{2N,\Delta t/2^\tau}$, for some constant τ. Note that for computing $\boldsymbol{U}^{2N,\Delta t/2^\tau}$ we use same

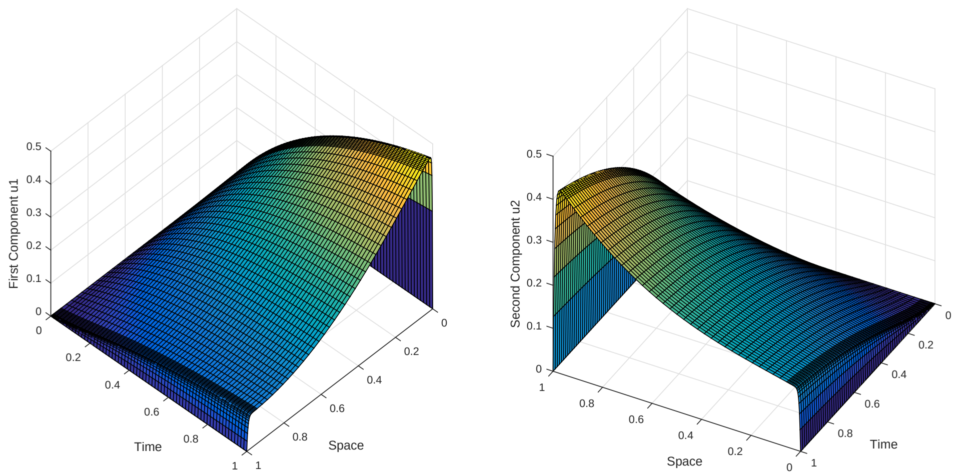

Figure 4.1: Numerical solution of Example 4.4.1 for $\varepsilon = 10^{-5},\ \varepsilon_2 = 10^{-4}$ with $N = 32,\ M = 64$ (left u_1, right u_2).

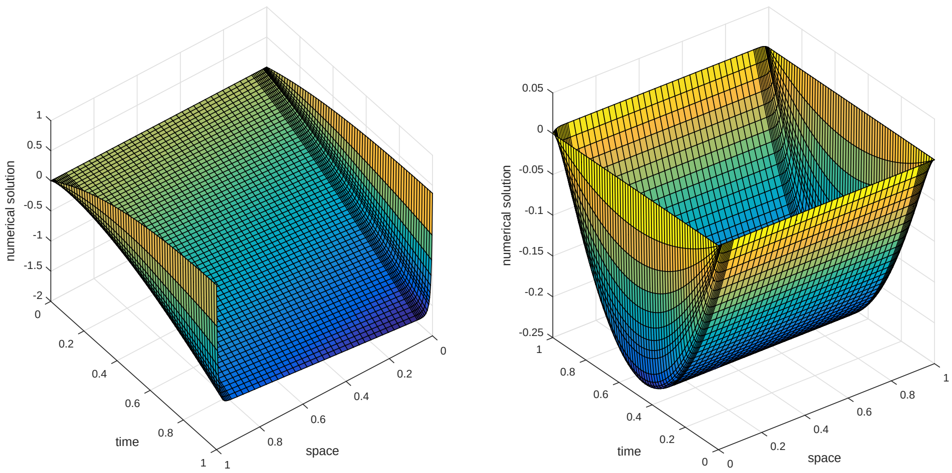

Figure 4.2: Numerical solution of Example 4.4.2 for $\varepsilon = 10^{-5},\ \varepsilon_2 = 10^{-4}$ with $N = 32,\ M = 64$ (left u_1, right u_2).

subdomain parameters σ_1 and σ_2 as with $\boldsymbol{U}^{N,\Delta t}$, but taking time stepping $\Delta t/2^{\tau}$ and mesh points in spatial direction to be $2N+1$. After computing $\boldsymbol{U}^{N,\Delta t}$ and $\boldsymbol{U}^{2N,\Delta t/2^{\tau}}$ the error is computed by

$$E^{N,\Delta t}_{\varepsilon_1,\varepsilon_2} = ||\,\boldsymbol{U}^{N,\Delta t} - \boldsymbol{U}^{2N,\Delta t/2^{\tau}}||_{\overline{Q}^{N,M}}.$$

We fix ε_1 by taking $\varepsilon_1 = 10^{-n}$, where n is a non-negative integer. We then calculate

$$E^{N,\Delta t}_{\varepsilon_1} = \max\{E^{N,\Delta t}_{\varepsilon_1,1}, E^{N,\Delta t}_{\varepsilon_1,10^{-1}}, \ldots, E^{N,\Delta t}_{\varepsilon_1,10^{-n}}\}.$$

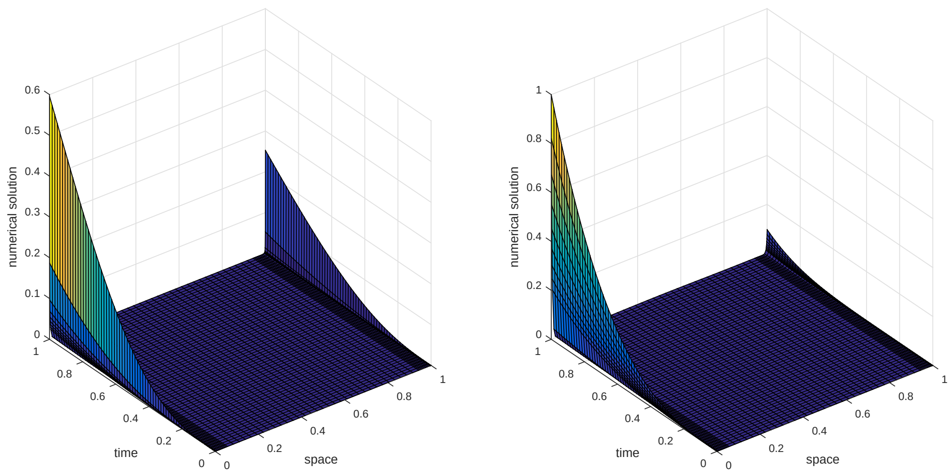

Figure 4.3: Numerical solution of Example 4.4.3 for $\varepsilon = 10^{-5},\ \varepsilon_2 = 10^{-4}$ with $N = 32,\ M = 64$ (left u_1, right u_2).

Table 4.1: The errors $E_{\varepsilon_1}^{N,\Delta t}$ and $E^{N,\Delta t}$, and the convergence rates $\varrho_{\varepsilon_1}^{N,\Delta t}$ and $\varrho^{N,\Delta t}$ for Example 4.4.1.

$\varepsilon_1 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 1$	2.19E-02	7.62E-03	2.16E-03	5.57E-04	1.40E-04
	1.52	1.82	1.95	1.99	
2	2.62E-02	8.52E-03	2.31E-03	5.91E-04	1.49E-04
	1.62	1.88	1.97	1.99	
3	2.83E-02	8.69E-03	2.33E-03	5.93E-04	1.49E-04
	1.70	1.90	1.97	1.99	
4	2.92E-02	8.98E-03	2.40E-03	6.12E-04	1.54E-04
	1.70	1.90	1.97	1.99	
5	3.07E-02	1.07E-02	3.19E-03	9.06E-04	2.47E-04
	1.52	1.75	1.81	1.88	
6	3.08E-02	1.07E-02	3.19E-03	9.06E-04	2.54E-04
	1.52	1.75	1.81	1.83	
7	3.08E-02	1.07E-02	3.19E-03	9.06E-04	2.54E-04
	1.53	1.75	1.81	1.83	
8	3.10E-02	1.09E-02	3.19E-03	9.06E-04	2.54E-04
	1.51	1.78	1.81	1.83	
$E^{N,\Delta t}$	3.10E-02	1.09E-02	3.19E-03	9.06E-04	2.54E-04
$\varrho^{N,\Delta t}$	1.51	1.78	1.81	1.83	

After that uniform error is calculated by $E^{N,\Delta t} = \max_{\varepsilon_1} E_{\varepsilon_1}^{N,\Delta t}$. We compute rates of convergence by

$$\varrho_{\varepsilon_1}^{N,\Delta t} = \log_2(E_{\varepsilon_1}^{N,\Delta t}/E_{\varepsilon_1}^{2N,\Delta t/2^\tau}), \qquad \varrho^{N,\Delta t} = \log_2(E^{N,\Delta t}/E^{2N,\Delta t/2^\tau}).$$

Table 4.2: Number of iterations taking $\varepsilon_1 = 10^{-8}$ for Example 4.4.1.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	3	4	5	6	6
1	2	3	4	5	6
2	2	3	5	6	7
3	3	4	5	6	7
4	3	4	5	5	6
5	3	3	3	3	3
6	2	2	2	2	2
7	1	1	1	1	1
8	1	1	1	1	1

Table 4.3: The errors $E_{\varepsilon_1}^{N,\Delta t}$ and $E^{N,\Delta t}$, and the convergence rates $\varrho_{\varepsilon_1}^{N,\Delta t}$ and $\varrho^{N,\Delta t}$ for Example 4.4.1.

$\varepsilon_1 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/2$	$N = 2^6$ $\Delta t = 1/2^2$	$N = 2^7$ $\Delta t = 1/2^3$	$N = 2^8$ $\Delta = 1/2^4$	$N = 2^9$ $\Delta t = 1/2^5$
$n = 1$	1.33E-02	8.53E-03	4.94E-03	2.71E-03	1.42E-03
	0.64	0.79	0.87	0.93	
2	1.66E-02	1.00E-02	5.56E-03	2.96E-03	1.53E-03
	0.73	0.85	0.91	0.95	
3	1.80E-02	1.04E-02	5.68E-03	2.99E-03	1.53E-03
	0.80	0.87	0.93	0.96	
4	1.90E-02	1.06E-02	5.74E-03	2.99E-03	1.53E-03
	0.84	0.89	0.94	0.97	
5	2.18E-02	1.21E-02	6.31E-03	3.19E-03	1.59E-03
	0.85	0.94	0.98	1.00	
6	2.18E-02	1.21E-02	6.31E-03	3.19E-03	1.59E-03
	0.85	0.94	0.98	1.00	
7	2.19E-02	1.21E-02	6.31E-03	3.19E-03	1.59E-03
	0.85	0.94	0.98	1.00	
8	2.24E-02	1.24E-02	6.31E-03	3.19E-03	1.59E-03
	0.86	0.97	0.99	1.00	
$E^{N,\Delta t}$	2.24E-02	1.24E-02	6.31E-03	3.19E-03	1.59E-03
$\varrho^{N,\Delta t}$	0.86	0.97	0.99	1.00	

For $\tau = 2$, Table 4.1 gives $E_{\varepsilon_1}^{N,\Delta t}$ and $\varrho_{\varepsilon_1}^{N,\Delta t}$ corresponding to our method, for various values of ε_1 and N, and Δt. It also gives the uniform errors $E^{N,\Delta t}$ and the uniform convergence rates $\varrho^{N,\Delta t}$. We observe that the results in Table 4.1 are well in accordance with Theorem 4.3.3. Table 4.2 gives, for fixed $\varepsilon_1 = 10^{-8}$ and different values of ε_2 and N, Δt, the number of iterations needed for stopping the algorithm. To see the influence of the error associated to the spatial discretization we divided the time step size by four, as displayed in Table 4.1. We also provide results in Table 4.3 by dividing the time step size by two. From this table the first order uniform convergence is observed, that is corresponding to the time discretization error. Next, we consider a test example whose solution is known.

Example 4.4.2. Consider the following problem [35]

$$\begin{cases} \partial_t \boldsymbol{u} - \boldsymbol{E}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} & in\ Q := \Omega \times (0,1], \\ \boldsymbol{u}(x,0) = \boldsymbol{0} & in\ \overline{\Omega}, \\ \boldsymbol{u}(0,t) = \boldsymbol{g}_0(t), \boldsymbol{u}(1,t) = \boldsymbol{g}_1(t) & in\ (0,1] \end{cases} \tag{4.4.3}$$

with

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where $f_1, f_2, \boldsymbol{g}_0$ and $\boldsymbol{g}_1$ are chosen in such a way that the solution of the problem $\boldsymbol{u} = (u_1, u_2)^T$ is

$$u_1(x,t) = t(\phi_1(x) + \phi_2(x) - 2) + (x+1)te^{-t},$$

$$u_2(x,t) = \varepsilon_1(1 - e^{-t})(\phi_1(x) - 1) + t(1-t)(\phi_2(x) - 1),$$

with

$$\phi_i(x) = \frac{e^{-x/\sqrt{\varepsilon_i}} + e^{-(1-x)/\sqrt{\varepsilon_i}}}{1 + e^{-1/\sqrt{\varepsilon_i}}}, \quad i = 1, 2.$$

The errors are computed using

$$E_{\varepsilon_1,\varepsilon_2}^{N,\Delta t} = ||\boldsymbol{u} - \boldsymbol{U}^{N,\Delta t}||_{\overline{Q}^{N,M}}.$$

After that the errors $E_{\varepsilon_1}^{N,\Delta t}$ and $E^{N,\Delta t}$ are calculated as described previously. Then the convergence rates $\varrho_{\varepsilon_1}^{N,\Delta t}$ and $\varrho^{N,\Delta t}$ are calculated by

$$\varrho_{\varepsilon_1}^{N,\Delta t} = \log_2(E_{\varepsilon_1}^{N,\Delta t}/E_{\varepsilon_1}^{2N,\Delta t/2^\tau}), \qquad \varrho^{N,\Delta t} = \log_2(E^{N,\Delta t}/E^{2N,\Delta t/2^\tau}),$$

for some constant τ. For $\tau = 2$, Table 4.4 gives $E_{\varepsilon_1}^{N,\Delta t}$ and $\varrho_{\varepsilon_1}^{N,\Delta t}$ corresponding to the method, for various values of ε_1 and N, and Δt. It also gives the uniform errors $E^{N,\Delta t}$ and the uniform convergence rates $\varrho^{N,\Delta t}$. Table 4.5 gives, for fixed $\varepsilon_1 = 10^{-8}$ and different values of ε_2 and N, Δt, the number of iterations needed for stopping the iterative process. Taking $\tau = 1$, results are provided in Table 4.6, where we observe the error corresponding to the time discretization. From these results we observe that the the error associated to the time discretization is dominating in the global error for this example. We next consider a test example in which the error associated to the spatial discretization is dominating in the global error.

Table 4.4: The errors $E_{\varepsilon_1}^{N,\Delta t}$ and $E^{N,\Delta t}$, and the convergence rates $\varrho_{\varepsilon_1}^{N,\Delta t}$ and $\varrho^{N,\Delta t}$ for Example 4.4.2.

$\varepsilon_1 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 1$	6.20E-02	1.77E-02	4.59E-03	1.16E-03	2.91E-04
	1.81	1.94	1.99	2.00	
2	8.32E-02	2.21E-02	5.72E-03	1.44E-03	3.62E-04
	1.91	1.95	1.99	2.00	
3	8.97E-02	2.38E-02	6.15E-03	1.55E-03	3.89E-04
	1.91	1.95	1.99	2.00	
4	9.20E-02	2.44E-02	6.31E-03	1.59E-03	3.99E-04
	1.91	1.95	1.99	2.00	
5	9.29E-02	2.46E-02	6.36E-03	1.60E-03	4.01E-04
	1.91	1.95	1.99	2.00	
6	9.31E-02	2.47E-02	6.39E-03	1.61E-03	4.04E-04
	1.91	1.95	1.99	2.00	
7	9.32E-02	2.48E-02	6.40E-03	1.61E-03	4.04E-04
	1.91	1.95	1.99	2.00	
8	9.33E-02	2.48E-02	6.41E-03	1.62E-03	4.05E-04
	1.91	1.95	1.99	2.00	
$E^{N,\Delta t}$	9.33E-02	2.48E-02	6.41E-03	1.62E-03	4.05E-04
$\varrho^{N,\Delta t}$	1.91	1.95	1.99	2.00	

Table 4.5: Number of iterations taking $\varepsilon_1 = 10^{-8}$ for Example 4.4.2.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	2	3	4	5	6
1	2	3	3	4	5
2	2	3	4	5	6
3	2	3	4	5	6
4	3	4	4	5	6
5	3	3	3	3	3
6	2	2	2	2	2
7	1	1	1	1	1
8	1	1	1	1	1

Example 4.4.3. Consider the following problem [35]

$$\begin{cases} \partial_t \boldsymbol{u} - \boldsymbol{E}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad in\ Q := \Omega \times (0,1], \\ \boldsymbol{u}(x,0) = \boldsymbol{0} \quad in\ \overline{\Omega}, \quad \boldsymbol{u}(0,t) = \boldsymbol{g}_0(t), \quad \boldsymbol{u}(1,t) = \boldsymbol{g}_1(t) \quad in\ (0,1], \end{cases} \tag{4.4.4}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 5 + e^{-1/(t\sin(\pi x))} & -(1 + (x^2 - x^4)t^2) \\ -(1 + (x^2 - x^4)t^2) & 5 + e^{-1/(t\sin(\pi x))} \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} 0 \\ t^2 e^{-1/(x - x^2)} \end{pmatrix},$$

Table 4.6: The errors $E_{\varepsilon_1}^{N,\Delta t}$ and $E^{N,\Delta t}$, and the convergence rates $\varrho_{\varepsilon_1}^{N,\Delta t}$ and $\varrho^{N,\Delta t}$ for Example 4.4.2.

$\varepsilon_1 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/2$	$N = 2^6$ $\Delta t = 1/2^2$	$N = 2^7$ $\Delta t = 1/2^3$	$N = 2^8$ $\Delta = 1/2^4$	$N = 2^9$ $\Delta t = 1/2^5$
$n = 1$	1.08E-01	6.21E-02	3.37E-02	1.77E-02	9.06E-03
	0.80	0.88	0.93	0.96	
2	1.61E-01	8.32E-02	4.23E-02	2.21E-02	1.13E-02
	0.95	0.98	0.94	0.97	
3	1.72E-01	8.97E-02	4.57E-02	2.38E-02	1.22E-02
	0.94	0.97	0.94	0.97	
4	1.76E-01	9.20E-02	4.70E-02	2.45E-02	1.25E-02
	0.94	0.97	0.94	0.97	
5	1.78E-01	9.29E-02	4.75E-02	2.47E-02	1.26E-02
	0.94	0.97	0.94	0.97	
6	1.78E-01	9.32E-02	4.76E-02	2.48E-02	1.27E-02
	0.94	0.97	0.94	0.97	
7	1.78E-01	9.33E-02	4.77E-02	2.48E-02	1.27E-02
	0.93	0.97	0.94	0.97	
8	1.78E-01	9.34E-02	4.77E-02	2.48E-02	1.27E-02
	0.93	0.97	0.94	0.97	
$E^{N,\Delta t}$	1.78E-01	9.34E-02	4.77E-02	2.48E-02	1.27E-02
$\varrho^{N,\Delta t}$	0.93	0.97	0.94	0.97	

Table 4.7: Number of iterations taking $\varepsilon_1 = 10^{-8}$ for Example 4.4.3.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 2$	5	6	7	8	9
3	5	6	6	7	7
4	4	5	4	5	5
5	2	2	2	2	2
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

$\boldsymbol{g}_0(t) = (\sin^3(t), t^3)^T$, and $\boldsymbol{g}_1(t) = ((1-e^{-t})^3, (1-\cos(t))^3)^T$.

The exact solution to this test example is not known, so we compute the errors and convergence rates using the double mesh principle similar to Example 4.4.1. The results are displayed in Tables 4.8-4.9; these results confirm that the error associated to the spatial discretization is dominating in the global error, in contrast to Examples 4.4.1 and 4.4.2. In summary, from the results presented for Examples 4.4.1-4.4.3, we observe that the method is uniformly convergent of first order in time direction and almost second order in spatial direction, which is in line with our theoretical findings. Further, the number of iterations needed for getting the desired accuracy

Table 4.8: The errors $E_{\varepsilon_1}^{N,\Delta t}$ and $E^{N,\Delta t}$, and the convergence rates $\varrho_{\varepsilon_1}^{N,\Delta t}$ and $\varrho^{N,\Delta t}$ for Example 4.4.3.

$\varepsilon_1 = 10^{-n}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 1/4$	$\Delta t = 1/4^2$	$\Delta t = 1/4^3$	$\Delta = 1/4^4$	$\Delta t = 1/4^5$
$n = 2$	1.58E-02	4.39E-03	1.12E-03	2.83E-04	7.08E-05
	1.85	1.97	1.99	2.00	
3	1.73E-02	4.77E-03	1.23E-03	3.0E-04	7.71E-05
	1.86	1.96	1.99	2.00	
4	1.91E-02	5.68E-03	1.58E-03	4.36E-04	1.20E-04
	1.75	1.84	1.86	1.86	
5	2.45E-02	1.06E-02	3.59E-03	1.20E-03	3.62E-04
	1.20	1.57	1.58	1.73	
6	2.48E-02	1.07E-02	3.61E-03	1.21E-03	3.78E-04
	1.21	1.57	1.58	1.67	
7	2.48E-02	1.07E-02	3.61E-03	1.21E-03	3.78E-04
	1.21	1.57	1.58	1.67	
8	2.48E-02	1.07E-02	3.61E-03	1.21E-03	3.78E-04
	1.21	1.57	1.58	1.67	
$E^{N,\Delta t}$	2.48E-02	1.07E-02	3.61E-03	1.21E-03	3.78E-04
$\varrho^{N,\Delta t}$	1.21	1.57	1.58	1.67	

Table 4.9: The errors $E_{\varepsilon_1}^{N,\Delta t}$ and $E^{N,\Delta t}$, and the convergence rates $\varrho_{\varepsilon_1}^{N,\Delta t}$ and $\varrho^{N,\Delta t}$ for Example 4.4.3.

$\varepsilon_1 = 10^{-n}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 1/2^5$	$\Delta t = 1/2^6$	$\Delta t = 1/2^7$	$\Delta = 1/2^8$	$\Delta t = 1/2^9$
$n = 2$	1.98E-03	8.57E-04	3.96E-04	1.91E-04	9.39E-05
	1.21	1.11	1.05	1.02	
3	3.32E-03	1.22E-03	4.91E-04	2.15E-04	1.00E-04
	1.44	1.31	1.19	1.10	
4	7.98E-03	2.60E-03	9.96E-04	3.86E-04	1.55E-04
	1.62	1.39	1.37	1.32	
5	1.53E-02	1.05E-02	4.16E-03	1.37E-03	3.74E-04
	0.55	1.33	1.60	1.87	
6	1.54E-02	1.05E-02	4.16E-03	1.44E-03	4.73E-04
	0.55	1.33	1.53	1.61	
7	1.54E-02	1.05E-02	4.16E-03	1.44E-03	4.73E-04
	0.55	1.33	1.53	1.61	
8	1.54E-02	1.05E-02	4.16E-03	1.44E-03	4.73E-04
	0.55	1.33	1.53	1.61	
$E^{N,\Delta t}$	1.54E-02	1.05E-02	4.16E-03	1.44E-03	4.73E-04
$\varrho^{N,\Delta t}$	0.55	1.33	1.53	1.61	

is small.

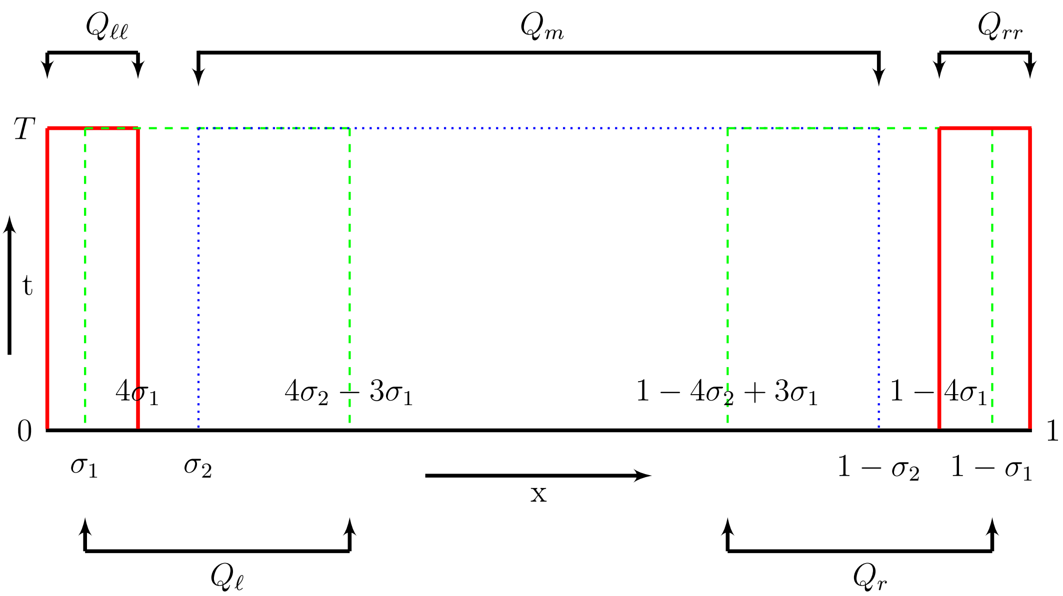

Figure 4.4: Decomposition of the computational domain.

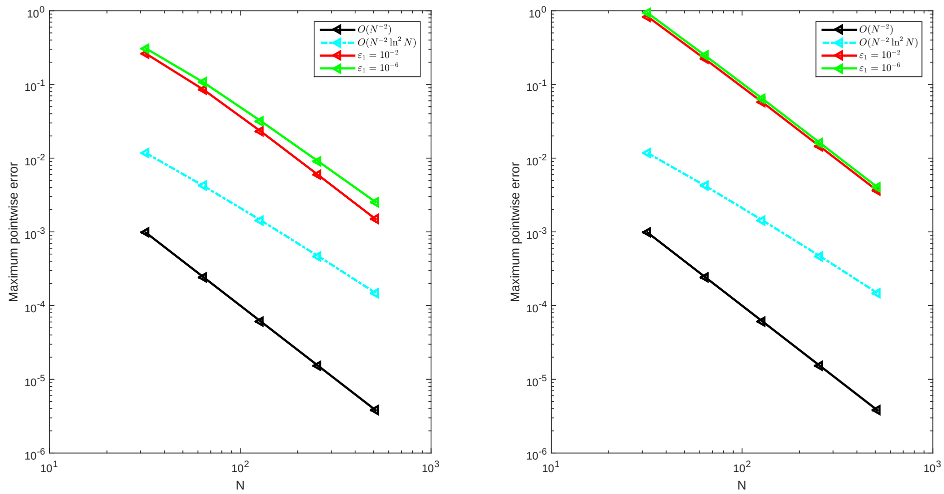

Figure 4.5: Loglog plot of the maximum pointwise errors for Examples 4.4.1 and 4.4.2 are depicted in the left and right subfigures, respectively.

In Figure 4.5, we show loglog plot of the maximum pointwise errors vs N for both the examples. The slopes of these plots also validate the theoretically obtained convergence result. The used CPU time in seconds for the proposed method for Examples 4.4.1 and 4.4.2 is given in Table 4.10. These results are computed using MATLAB software installed on a laptop equipped with an Intel(R) Core(TM) i3-3227U CPU with 1.90GHz speed and 8 GB RAM running on a 64 bit windows8 operating system.

Table 4.10: The used CPU time in seconds for Examples 4.4.1 and 4.4.2 with $\varepsilon_1 = 10^{-8},\ \varepsilon_2 = 10^{-7}$.

	$N = 2^5$ $\Delta t = 0.25$	$N = 2^6$ $\Delta t = 0.25/4$	$N = 2^7$ $\Delta t = 0.25/4^2$	$N = 2^8$ $\Delta t = 0.25/4^3$
Example 4.4.1	0.297093	0.462029	2.127035	41.279690
Example 4.4.2	0.430715	1.004809	6.124289	56.153489

4.5 Conclusions

In this chapter, we have designed and analyzed a domain decomposition method of Schwarz waveform relaxation type with overlap for solving singularly perturbed parabolic reaction-diffusion systems with distinct small parameters. The uniform convergence analysis is done using some auxiliary problems. The method is shown to be uniformly convergent of almost second order in space and first order in time. In addition, we perform some numerical experiments which support the theoretical error estimates. Moreover, numerically we observed that the convergence can be achieved within one iteration when the perturbation parameters are small and of same magnitude.