3

A robust domain decomposition method for singularly perturbed parabolic semilinear reaction-diffusion problems

We consider the following singularly perturbed parabolic semilinear reaction-diffusion problem

$$\begin{cases} \mathscr{L}\mathrm{u}(x,t) = \mathrm{u}_t - \varepsilon \mathrm{u}_{xx} + f(x,t,\mathrm{u}) = 0, & \text{for } (x,t) \in \mathrm{Q} = \Omega \times (0,T],\ \Omega = (0,1), \\ \mathrm{u}(x,0) = \varphi(x), & \text{for } x \in \overline{\Omega}, \\ \mathrm{u}(0,t) = g_0(t), \mathrm{u}(1,t) = g_1(t), & \text{for } t \in (0,T], \end{cases} \tag{3.0.1}$$

where $\varepsilon \in (0,1]$ is the perturbation parameter. We assume that

$$f_{\mathrm{u}}(x,t,\mathrm{u}) \geq \alpha > 0 \quad \text{for all} \ \ (x,t,\mathrm{u}) \in \overline{\mathrm{Q}} \times \mathbb{R}. \tag{3.0.2}$$

Under suitable compatibility and regularity conditions on the data, problem (3.0.1) has a unique solution which exhibits boundary layers near $x = 0$ and $x = 1$; see [67, Chapt. 5, Theorem 6.4]. Problems of type (3.0.1) have found applications in diverse areas such as chemical kinetics, nematic liquid crystal cell, and many others [37, 68–70].

Following [71], the exact solution can be written as $\mathrm{u} = v + w$, where

$$||\partial_x^s \partial_t^r \mathrm{u}||_{\overline{\mathrm{Q}}} \leq C(1 + \varepsilon^{-s/2}(e^{-x\sqrt{\alpha/\varepsilon}} + e^{-(1-x)\sqrt{\alpha/\varepsilon}})) \ \text{for} \ \ 0 \leq s + 2r \leq 4, \tag{3.0.3}$$

and

$$||\partial_x^s v||_{\overline{\mathrm{Q}}} \leq C(1 + \varepsilon^{(2-s)/2}), \tag{3.0.4}$$

$$|\partial_x^s w(x,t)| \leq C\varepsilon^{-s/2}\left(\exp(-x\sqrt{\alpha/\varepsilon}) + \exp(-(1-x)\sqrt{\alpha/\varepsilon})\right), \tag{3.0.5}$$

for $(x,t) \in \overline{\mathrm{Q}}$, $s = 0, \ldots, 4$.

In this chapter, we aim to design and analyze a domain decomposition method of SWR type for problem (3.0.1). We first decompose the domain into three overlapping subdomains, then iteratively solve sub-problems posed on each subdomain. These sub-problems are formed using the central difference and the backward difference on a uniform mesh in space and time direction, respectively, and the same initial value of the governing equation, but with suitably designed boundary conditions along the interfacial boundaries. The sub-problems are solved iteratively until convergence is achieved. The analysis of uniform convergence is made in two steps, splitting the contribution to the global error from the iteration and the discretization errors. The approximations generated by the algorithm are proved to be almost second order accurate in space and first order accurate in time. More precisely, we show that only one iteration is necessary for the algorithm to reach the desired accuracy for smaller values of the perturbation parameter. At the end, some numerical results are given in support of the theory.

3.1 Domain decomposition method

To construct the algorithm, we discretize the continuous problem on three overlapping subdomains $\mathrm{Q}_p = \Omega_p \times (0,T],\ p = \ell, m, r$, where

$$\Omega_\ell = (0, 2\sigma_\varepsilon), \qquad \Omega_m = (\sigma_\varepsilon, 1-\sigma_\varepsilon), \qquad \Omega_r = (1-2\sigma_\varepsilon, 1),$$

and the transition parameter σ_ε is choosen as follows:

$$\sigma_\varepsilon = \min\left\{\frac{1}{4},\ 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N\right\}. \tag{3.1.1}$$

On each subdomain $\overline{\mathrm{Q}}_p = [a,b] \times [0,T]$, we introduce a mesh $\overline{\mathrm{Q}}_p^{N,M} = \overline{\Omega}_p^N \times \overline{\omega}^M$, where $\overline{\Omega}_p^N = \{x_i = i\Delta x, i = 0, 1\ldots, N,\ \Delta x = (b-a)/N\}$ and $\overline{\omega}^M = \{t_j = j\Delta t,\ j = 0, 1\ldots, M,\ \Delta t = T/M\}$ with $\mathrm{Q}_p^{N,M} = \overline{\mathrm{Q}}_p^{N,M} \cap \mathrm{Q}_p$, $\Omega_p^N = \overline{\Omega}_p^N \cap \Omega_p$ and $\omega^M = \overline{\omega}^M \cap (0,T]$. Letting $N = 2^n,\ n \geq 2$, on each subdomain $\mathrm{Q}_p^{N,M}$, $p = \ell, m, r$, we consider the following discretization

$$[\mathscr{L}_p^{N,M}\mathrm{U}_p]_{i,j} := [\delta_t \mathrm{U}_p]_{i,j} - \varepsilon[\delta_x^2 \mathrm{U}_p]_{i,j} + f(x_i, t_j, \mathrm{U}_{p;i,j}) = 0, \tag{3.1.2}$$

where

$$[\delta_x^2 \mathrm{U}_p]_{i,j} = \frac{1}{h_p^2}(\mathrm{U}_{p;i+1,j} - 2\mathrm{U}_{p;i,j} + \mathrm{U}_{p;i-1,j}),$$

and

$$[\delta_t \mathrm{U}_p]_{i,j} = \frac{1}{\Delta t}(\mathrm{U}_{p;i,j+1} - \mathrm{U}_{p;i,j}).$$

We now define the algorithmic procedure as follows.

Step 1. Initial Approximation: The algorithm starts with the following initial approximation

$$\mathrm{U}^{[0]}(x_i, t_j) = \begin{cases} 0, & 0 < x_i < 1,\ 0 < t_j \le T, \\ \mathrm{u}(x_i, 0), & \text{for } \ x_i \in \overline{\Omega}, \\ \mathrm{u}(0, t_j), & \text{for } \ t_j \in (0, T], \\ \mathrm{u}(1, t_j), & \text{for } \ t_j \in (0, T]. \end{cases} \tag{3.1.3}$$

Step 2. For each $k \ge 1$, the algorithm constructs k^{th} approximation $\mathrm{U}_p^{[k]},\ p = \ell, m, r$, by solving following problems

$$\begin{cases} [\mathscr{L}_\ell^{N,M} \mathrm{U}_\ell^{[k]}]_{i,j} = 0 & \text{for } (x_i, t_j) \in \mathrm{Q}_\ell^{N,M}, \\ \mathrm{U}_\ell^{[k]}(x_i, 0) = \varphi(x_i) & \text{for } x_i \in \overline{\Omega}_\ell^N, \\ \mathrm{U}_\ell^{[k]}(0, t_j) = g_0(t_j), \mathrm{U}_\ell^{[k]}(2\sigma_\varepsilon, t_j) = \mathcal{T}_{t_j} \mathrm{U}^{[k-1]}(2\sigma_\varepsilon, t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\mathscr{L}_r^{N,M} \mathrm{U}_r^{[k]}]_{i,j} = 0 & \text{for } (x_i, t_j) \in \mathrm{Q}_r^{N,M}, \\ \mathrm{U}_r^{[k]}(x_i, 0) = \varphi(x_i) & \text{for } x_i \in \overline{\Omega}_r^N, \\ \mathrm{U}_r^{[k]}(1 - 2\sigma_\varepsilon, t_j) = \mathcal{T}_{t_j} \mathrm{U}^{[k-1]}(1 - 2\sigma_\varepsilon, t_j), \mathrm{U}_r^{[k]}(1, t_j) = g_1(t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\mathscr{L}_m^{N,M} \mathrm{U}_m^{[k]}]_{i,j} = 0 & \text{for } (x_i, t_j) \in \mathrm{Q}_m^{N,M}, \\ \mathrm{U}_m^{[k]}(x_i, 0) = \varphi(x_i) & \text{for } x_i \in \overline{\Omega}_m^N, \\ \mathrm{U}_m^{[k]}(\sigma_\varepsilon, t_j) = \mathcal{T}_{t_j} \mathrm{U}_\ell^{[k]}(\sigma_\varepsilon, t_j), \mathrm{U}_m^{[k]}(1 - \sigma_\varepsilon, t_j) = \mathcal{T}_{t_j} \mathrm{U}_r^{[k]}(1 - \sigma_\varepsilon, t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

where symbol $\mathcal{T}_{t_j} \mathrm{U}^{[k]}$ denotes the piecewise linear interpolant at time level t_j on $\overline{\Omega}^N := (\overline{\Omega}_\ell^N \backslash \overline{\Omega}_m) \cup \overline{\Omega}_m^N \cup (\overline{\Omega}_r^N \backslash \overline{\Omega}_m)$.

Step 3. The solution to problem (3.0.1) can now be obtained by combining the solutions obtained in Step 2 in the following way

$$\mathrm{U}^{[k]}(x_i, t_j) = \begin{cases} \mathrm{U}_\ell^{[k]}(x_i, t_j),\ (x_i, t_j) \in \overline{\mathrm{Q}}_\ell^{N,M} \setminus \overline{\mathrm{Q}}_m, \\ \mathrm{U}_m^{[k]}(x_i, t_j),\ (x_i, t_j) \in \overline{\mathrm{Q}}_m^{N,M}, \\ \mathrm{U}_r^{[k]}(x_i, t_j),\ (x_i, t_j) \in \overline{\mathrm{Q}}_r^{N,M} \setminus \overline{\mathrm{Q}}_m. \end{cases} \tag{3.1.4}$$

Step 4. Termination: Stop the iterative process if condition

$$||\mathrm{U}^{[k]} - \mathrm{U}^{[k-1]}||_{\overline{\mathrm{Q}}^{N,M}} \leq tol \tag{3.1.5}$$

is true; otherwise return to Step 2 and continue the iterative process until the tolerance criterion is not satisfied.

3.2 Error Analysis

In this section, we seperately analyze the contribution to the global error from discretization error and iteration error. The analysis is based on the following auxiliary problems

$$\begin{cases} [\mathscr{L}_\ell^{N,M}\widetilde{\mathrm{U}}_\ell]_{i,j} = 0 & \text{for } (x_i,t_j) \in \mathrm{Q}_\ell^{N,M}, \\ \widetilde{\mathrm{U}}_\ell(x_i,0) = \mathrm{u}(x_i,0) & \text{for } x_i \in \overline{\Omega}_\ell^N, \\ \widetilde{\mathrm{U}}_\ell(0,t_j) = \mathrm{u}(0,t_j), \widetilde{\mathrm{U}}_\ell(2\sigma_\varepsilon,t_j) = \mathrm{u}(2\sigma_\varepsilon,t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\mathscr{L}_r^{N,M}\widetilde{\mathrm{U}}_r]_{i,j} = 0 & \text{for } (x_i,t_j) \in \mathrm{Q}_r^{N,M}, \\ \widetilde{\mathrm{U}}_r(x_i,0) = \mathrm{u}(x_i,0) & \text{for } x_i \in \overline{\Omega}_r^N, \\ \widetilde{\mathrm{U}}_r(1-2\sigma_\varepsilon,t_j) = \mathrm{u}(1-2\sigma_\varepsilon,t_j), \widetilde{\mathrm{U}}_r(1,t_j) = \mathrm{u}(1,t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\mathscr{L}_m^{N,M}\widetilde{\mathrm{U}}_m]_{i,j} = 0 & \text{for } (x_i,t_j) \in \mathrm{Q}_m^{N,M}, \\ \widetilde{\mathrm{U}}_m(x_i,0) = \mathrm{u}(x_i,0) & \text{for } x_i \in \overline{\Omega}_m^N, \\ \widetilde{\mathrm{U}}_m(\sigma_\varepsilon,t_j) = \mathrm{u}(\sigma_\varepsilon,t_j), \widetilde{\mathrm{U}}_m(1-\sigma_\varepsilon,t_j) = \mathrm{u}(1-\sigma_\varepsilon,t_j) & \text{for } t_j \in \omega^M, \end{cases}$$

where $\mathscr{L}_p^{N,M}$ is as defined in previous section, and u is the exact solution of (3.0.1).

Next we define

$$\widetilde{\mathrm{U}}(x_i,t_j) = \begin{cases} \widetilde{\mathrm{U}}_\ell(x_i,t_j), \ (x_i,t_j) \in \overline{\mathrm{Q}}_\ell^{N,M} \setminus \overline{\mathrm{Q}}_m, \\ \widetilde{\mathrm{U}}_m(x_i,t_j), \ (x_i,t_j) \in \overline{\mathrm{Q}}_m^{N,M}, \\ \widetilde{\mathrm{U}}_r(x_i,t_j), \ (x_i,t_j) \in \overline{\mathrm{Q}}_r^{N,M} \setminus \overline{\mathrm{Q}}_m. \end{cases} \tag{3.2.1}$$

We use the following notation in our analysis.

$$\begin{aligned} \xi_{\sigma_\varepsilon} &= \max\left\{\max_{t_j\in\omega^M} |(\widetilde{\mathrm{U}}_\ell - \widetilde{\mathrm{U}}_m)(\sigma_\varepsilon,t_j)|, \ \max_{t_j\in\omega^M} |(\widetilde{\mathrm{U}}_r - \widetilde{\mathrm{U}}_m)(1-\sigma_\varepsilon,t_j)|\right\}, \\ \xi_{2\sigma_\varepsilon} &= \max\left\{\max_{t_j\in\omega^M} |(\widetilde{\mathrm{U}}_\ell - \widetilde{\mathrm{U}}_m)(2\sigma_\varepsilon,t_j)|, \ \max_{t_j\in\omega^M} |(\widetilde{\mathrm{U}}_r - \widetilde{\mathrm{U}}_m)(1-2\sigma_\varepsilon,t_j)|\right\}, \\ \xi^{[k]} &= \max\{\max_{t_j\in\omega^M} |(\widetilde{\mathrm{U}}_\ell - \mathcal{T}_{t_j}\mathrm{U}^{[k-1]})(2\sigma_\varepsilon,t_j)|, \ \max_{t_j\in\omega^M} |(\widetilde{\mathrm{U}}_r - \mathcal{T}_{t_j}\mathrm{U}^{[k-1]})(1-2\sigma_\varepsilon,t_j)|\}, \\ \eta^{[k]} &= \max\{||\widetilde{\mathrm{U}}_\ell - \mathrm{U}^{[k]}||_{\overline{\mathrm{Q}}_\ell^{N,M}\setminus\overline{\mathrm{Q}}_m}, \ ||\widetilde{\mathrm{U}}_m - \mathrm{U}^{[k]}||_{\overline{\mathrm{Q}}_m^{N,M}}, \ ||\widetilde{\mathrm{U}}_r - \mathrm{U}^{[k]}||_{\overline{\mathrm{Q}}_r^{N,M}\setminus\overline{\mathrm{Q}}_m}\}. \end{aligned}$$

Lemma 3.2.1. *Let* u *be the exact solution of problem (3.0.1) and* $\widetilde{\mathrm{U}}_p,\ p = \ell, m, r,$ *be the solutions of the auxiliary problems defined in this section. Then*

$$||\mathrm{u} - \widetilde{\mathrm{U}}_p||_{\overline{\mathrm{Q}}_p^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N), \quad p = \ell, m, r. \tag{3.2.2}$$

Proof. For $(x_i, t_j) \in \mathrm{Q}_\ell^{N,M}$, the error function $\eta_\ell(x_i, t_j) = \mathrm{u}(x_i, t_j) - \widetilde{\mathrm{U}}_\ell(x_i, t_j)$ satisfies

$$[\delta_t \eta_\ell]_{i,j} - \varepsilon[\delta_x^2 \eta_\ell]_{i,j} + (f(x_i, t_j, \mathrm{u}_{i,j}) - f(x_i, t_j, \widetilde{\mathrm{U}}_{\ell;i,j})) = [(\delta_t - \partial_t)\mathrm{u}]_{i,j} + \varepsilon[(\partial_x^2 - \delta_x^2)\mathrm{u}]_{i,j} \tag{3.2.3}$$

with

$$\begin{cases} \eta_\ell(x_i, 0) = 0 & \text{for } x_i \in \overline{\Omega}_\ell^N, \\ \eta_\ell(0, t_j) = 0, \ \eta_\ell(2\sigma_\varepsilon, t_j) = 0 & \text{for } t_j \in \omega^M. \end{cases}$$

The error equation (3.2.3) can be written in the alternative form

$$[\mathscr{L}_\ell^{N,M}\eta_\ell]_{i,j} := [\delta_t \eta_\ell]_{i,j} - \varepsilon[\delta_x^2 \eta_\ell]_{i,j} + \left(\int_0^1 f_\mathrm{u}(x_i, t_j, \widetilde{\mathrm{U}}_{\ell;i,j} + s(\mathrm{u}_{i,j} - \widetilde{\mathrm{U}}_{\ell;i,j}))ds\right)\eta_{\ell;i,j}$$

$$= [(\delta_t - \partial_t)\mathrm{u}]_{i,j} + \varepsilon[(\partial_x^2 - \delta_x^2)\mathrm{u}]_{i,j},$$

or $$[\mathscr{L}_\ell^{N,M}\eta_\ell]_{i,j} := [\delta_t \eta_\ell]_{i,j} - \varepsilon[\delta_x^2 \eta_\ell]_{i,j} + [a_\ell \eta_\ell]_{i,j} = [(\delta_t - \partial_t)\mathrm{u}]_{i,j} + \varepsilon[(\partial_x^2 - \delta_x^2)\mathrm{u}]_{i,j},$$

where $a_{\ell;i,j} = \int_0^1 f_\mathrm{u}(x_i, t_j, \widetilde{\mathrm{U}}_{\ell;i,j} + s(\mathrm{u}_{i,j} - \widetilde{\mathrm{U}}_{\ell;i,j}))ds$. Now using Taylor expansions and (3.0.3) with $h_\ell \leq C\sqrt{\varepsilon}N^{-1}\ln N$, we get

$$\begin{aligned} \left|[\mathscr{L}_\ell^{N,M}\eta_\ell]_{i,j}\right| &\leq \frac{1}{2}(t_j - t_{j-1})\left\|\partial_t^2 \mathrm{u}(x_i, .)\right\|_{[t_{j-1}, t_j]} + \frac{\varepsilon}{12}h_\ell^2\left\|\partial_x^4 \mathrm{u}(., t_j)\right\|_{[x_{i-1}, x_{i+1}]} \\ &\leq C(\Delta t + N^{-2}\ln^2 N). \end{aligned}$$

So, applying the discrete maximum principle for $\mathscr{L}_\ell^{N,M}$ to the mesh functions $C(\Delta t + N^{-2}\ln^2 N) \pm \eta_\ell$, it holds

$$||\mathrm{u} - \widetilde{\mathrm{U}}_\ell||_{\overline{\mathrm{Q}}_\ell^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

In the same way, we get

$$||\mathrm{u} - \widetilde{\mathrm{U}}_r||_{\overline{\mathrm{Q}}_r^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Next, we obtain estimate for $||\mathrm{u} - \widetilde{\mathrm{U}}_m||_{\overline{\mathrm{Q}}_m^{N,M}}$. Defining $\eta_{m;i,j} = \mathrm{u}_{i,j} - \widetilde{\mathrm{U}}_{m;i,j}$, we have the following error equation

$$[\mathscr{L}_m^{N,M}\eta_m]_{i,j} := [\delta_t \eta_m]_{i,j} - \varepsilon[\delta_x^2 \eta_m]_{i,j} + [a_m \eta_m]_{i,j}$$

$$= [(\delta_t - \partial_t)\mathrm{u}]_{i,j} + \varepsilon[(\partial_x^2 - \delta_x^2)\mathrm{u}]_{i,j}, \tag{3.2.4}$$

where $a_{m;i,j} = \int_0^1 f_{\mathrm{u}}(x_i, t_j, \widetilde{\mathrm{U}}_{m;i,j} + s(\mathrm{u}_{i,j} - \widetilde{\mathrm{U}}_{m;i,j}))ds$. Since $\sigma_\varepsilon = \min\left\{\frac{1}{4},\ \ 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N\right\}$, we consider two cases: $2\sqrt{\frac{\varepsilon}{\alpha}}\ln N \geq 1/4$ and $2\sqrt{\frac{\varepsilon}{\alpha}}\ln N < 1/4$. First consider $2\sqrt{\frac{\varepsilon}{\alpha}}\ln N \geq 1/4$. Then $\varepsilon^{-1} \leq C\ln^2 N$ and $h_m = 1/(2N)$. So, using Taylor expansions and (3.0.3) it follows, from (3.2.4), that

$$\left|[\mathscr{L}_m^{N,M}\eta_m]_{i,j}\right| \leq C(\Delta t + N^{-2}\ln^2 N) \quad \text{for} \quad (x_i, t_j) \in \mathrm{Q}_m^{N,M}.$$

In the case when $2\sqrt{\frac{\varepsilon}{\alpha}}\ln N < 1/4$, the estimate on the first term of the right-hand side of equation (3.2.4) is obtained using Taylor expansion and (3.0.3). That is, for $(x_i, t_j) \in \mathrm{Q}_m^{N,M}$, $|[(\partial_t - \delta_t)\mathrm{u}]_{i,j}| \leq C\Delta t$. For the second term, use the decomposition of $\mathrm{u} = v + w$ and Taylor expansions to get

$$\begin{aligned}
\varepsilon\left|[(\partial_x^2 - \delta_x^2)\mathrm{u}]_{i,j}\right| &\leq \varepsilon\left|[(\partial_x^2 - \delta_x^2)v]_{i,j}\right| + \varepsilon\left|[(\partial_x^2 - \delta_x^2)w]_{i,j}\right| \\
&\leq C\varepsilon h_m^2\left\|\partial_x^4 v(.,t_j)\right\|_{[x_{i-1},x_{i+1}]} + C\varepsilon\left\|\partial_x^2 w(.,t_j)\right\|_{[x_{i-1},x_{i+1}]} \\
&\leq CN^{-2} + C\left\|\exp(-x\sqrt{\alpha/\varepsilon}) + \exp(-(1-x)\sqrt{\alpha/\varepsilon}\right\|_{[x_{i-1},x_{i+1}]} \\
&\leq CN^{-2} + C(e^{-\sigma_\varepsilon\sqrt{\alpha/\varepsilon}} + e^{-(1-(1-\sigma_\varepsilon))\sqrt{\alpha/\varepsilon}}) \\
&\leq CN^{-2} + 2e^{-\sigma_\varepsilon\sqrt{\alpha/\varepsilon}} \\
&= CN^{-2},
\end{aligned}$$

where we have used (3.0.4), (3.0.5) and $h_m \leq CN^{-1}$. Hence, for $(x_i, t_j) \in \mathrm{Q}_m^{N,M}$, we have

$$\left|[\mathscr{L}_m^{N,M}(\mathrm{u} - \widetilde{\mathrm{U}}_m)]_{i,j}\right| \leq C(\Delta t + N^{-2}\ln^2 N).$$

So, applying the discrete maximum principle for $\mathscr{L}_m^{N,M}$ to the mesh functions $C(\Delta t + N^{-2}\ln^2 N) \pm \eta_m$, it follows that

$$||\mathrm{u} - \widetilde{\mathrm{U}}_m||_{\overline{\mathrm{Q}}_m^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

□

Consider the following discrete problems

$$\begin{cases}
[\delta_t \mathrm{Z}_p]_{i,j} - \varepsilon[\delta_x^2 \mathrm{Z}_p]_{i,j} + [a_p \mathrm{Z}_p]_{i,j} = 0, & \text{for } (x_i, t_j) \in \mathrm{Q}_p^{N,M}, \\
\mathrm{Z}_p(a, t_j) = \mathrm{z}_0, \quad \mathrm{Z}_p(b, t_j) = \mathrm{z}_1, & \text{for } t_j \in \omega^M, \\
\mathrm{Z}_p(x_i, 0) = 0, & \text{for } x_i \in \overline{\Omega}_p^N,
\end{cases} \tag{3.2.5}$$

$$\begin{cases} [\delta_t \psi^{\pm}]_{i,j} - \varepsilon[\delta_x^2 \psi^{\pm}]_{i,j} + [\alpha\psi^{\pm}]_{i,j} = 0, \quad \text{for } (x_i, t_j) \in \mathrm{Q}_p^{N,M}, \\ \psi_p^-(a, t_j) = 1, \ \psi_p^-(b, t_j) = 0, \ \psi_p^+(a, t_j) = 0, \ \psi_p^+(b, t_j) = 1, \qquad \text{for } t_j \in \omega^M, \\ \psi_p^-(x_i, 0) = \frac{\varphi_1^N \varphi_2^i - \varphi_1^i \varphi_2^N}{\varphi_1^N - \varphi_2^N}, \quad \psi_p^+(x_i, 0) = \frac{\varphi_1^i - \varphi_2^i}{\varphi_1^N - \varphi_2^N}, \ \text{for } x_i \in \overline{\Omega}_p^N, \end{cases} \tag{3.2.6}$$

where $\varphi_i, \ i = 1, 2$, satisfy $\varphi_1 = \lambda_1 + \lambda_2, \quad \varphi_2 = \lambda_1 - \lambda_2$ with

$$\lambda_1 = 1 + \left(\frac{\sigma_\varepsilon}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^2, \ \lambda_2 = 2\left(\frac{\sigma_\varepsilon}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)\sqrt{1 + \left(\frac{\sigma_\varepsilon}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^2},$$

and $a_p(x_i, t_j) \geq \alpha > 0$, for $(x_i, t_j) \in \overline{\mathrm{Q}}_p^{N,M}, \ p = \ell, m, r$. It is easy to verify using the discrete maximum principle that $0 \leq \psi_p^{\pm}(x_i, t_j) \leq 1, \ (x_i, t_j) \in \overline{\mathrm{Q}}_p^{N,M}$.

Lemma 3.2.2. *Suppose that* $\mathrm{Z}_p(x_i, t_j)$ *and* $\psi_p^{\pm}(x_i, t_j)$ *are the solutions to the discrete problems (3.2.5) and (3.2.6) respectively. Then*

$$|\mathrm{Z}_p(x_i, t_j)| \leq |\mathrm{z}_0|\psi_p^-(x_i, t_j) + |\mathrm{z}_1|\psi_p^+(x_i, t_j), \quad (x_i, t_j) \in \overline{\mathrm{Q}}_p^{N,M}.$$

Proof. Suppose that W_p solves

$$\begin{cases} [\delta_t W_p]_{i,j} - \varepsilon[\delta_x^2 W_p]_{i,j} + [\alpha W_p]_{i,j} = 0, \ \ (x_i, t_j) \in \mathrm{Q}_p^{N,M}, \\ \mathrm{W}_p(a, t_j) = |\mathrm{z}_0|, \ \ \mathrm{W}_p(b, t_j) = |\mathrm{z}_1|, \ \text{for } t_j \in \omega^M, \\ \mathrm{W}_p(x_i, 0) = |\mathrm{z}_0|\frac{\varphi_1^N \varphi_2^i - \varphi_1^i \varphi_2^N}{\varphi_1^N - \varphi_2^N} + |\mathrm{z}_1|\frac{\varphi_1^i - \varphi_2^i}{\varphi_1^N - \varphi_2^N}, \ \text{for } x_i \in \overline{\Omega}_p^N. \end{cases} \tag{3.2.7}$$

Then W_p can be written as

$$\mathrm{W}_{p;i,j} = \psi_{p;i,j}^-|z_0| + \psi_{p;i,j}^+|z_1|, \quad x_i \in \overline{\mathrm{Q}}_p^{N,M}.$$

This can be verified by direct substitution. Using the discrete maximum principle, it follows that

$$|\mathrm{Z}_{p;i,j}| \leq \mathrm{W}_{p;i,j}, \quad x_i \in \overline{\mathrm{Q}}_p^{N,M}.$$

This completes the proof. □

In the next theorem we show that, when $\sigma_\varepsilon = 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N$, only one iteration is required for the algorithm to converge.

Theorem 3.2.3. *Let* u *be solution of problem (3.0.1) and* $\mathrm{U}^{[1]}$ *be the first iterate generated by the algorithm. If* $\sigma_\varepsilon = 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N$, *then*

$$||\mathrm{u} - \mathrm{U}^{[1]}||_{\overline{\mathrm{Q}}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Proof. Introducing the mesh function $\eta_\ell^{[1]}(x_i, t_j) = (\widetilde{\mathrm{U}}_\ell - \mathrm{U}_\ell^{[1]})(x_i, t_j)$, we write the error equation of $\eta_\ell^{[1]}$:

$$[\delta_t \eta_\ell^{[1]}]_{i,j} - \varepsilon[\delta_x^2 \eta_\ell^{[1]}]_{i,j} + [a_\ell^{[1]} \eta_\ell^{[1]}]_{i,j} = 0, \quad (x_i, t_j) \in \mathrm{Q}_\ell^{N,M} \tag{3.2.8}$$

with

$$\begin{cases} \eta_\ell^{[1]}(x_i, 0) = 0, & \text{for } x_i \in \overline{\Omega}_\ell^N, \\ \eta_\ell^{[1]}(0, t_j) = 0, \ |\eta_\ell^{[1]}(2\sigma_\varepsilon, t_j)| \le \xi^{[1]}, & \text{for } t_j \in \omega^M, \end{cases} \tag{3.2.9}$$

where $a_{\ell;i,j}^{[1]} = \int_0^1 f_{\mathrm{u}}(x_i, t_j, \mathrm{U}_{\ell;i,j}^{[1]} + s(\widetilde{\mathrm{U}}_{\ell;i,j} - \mathrm{U}_{\ell;i,j}^{[1]}))ds$. Now, using Lemma 3.2.2, it follows that, for $(x_i, t_j) \in \overline{\mathrm{Q}}_\ell^{N,M}$,

$$|\eta_\ell^{[1]}(x_i, t_j)| \le \xi^{[1]} \psi_\ell^+(x_i, t_j),$$

where ψ_ℓ^+ solves (3.2.6) and has the following form

$$\psi_\ell^+(x_i, t_j) = \frac{(\lambda_1 + \lambda_2)^i - (\lambda_1 - \lambda_2)^i}{(\lambda_1 + \lambda_2)^N - (\lambda_1 - \lambda_2)^N}.$$

Thus, for $(x_i, t_j) \in \overline{\mathrm{Q}}_\ell^{N,M} \backslash \overline{\mathrm{Q}}_m$, we have

$$\begin{aligned} |\eta_\ell^{[1]}(x_i, t_j)| &\le \xi^{[1]} \frac{(\lambda_1 + \lambda_2)^{N/2} - (\lambda_1 - \lambda_2)^{N/2}}{(\lambda_1 + \lambda_2)^N - (\lambda_1 - \lambda_2)^N} \\ &= \frac{\xi^{[1]}}{(\lambda_1 + \lambda_2)^{N/2} + (\lambda_1 - \lambda_2)^{N/2}} \le \frac{\xi^{[1]}}{(\lambda_1 + \lambda_2)^{N/2}}. \end{aligned}$$

Further, for $\sigma_\varepsilon = 2\sqrt{\varepsilon} \ln N / \sqrt{\alpha}$, it follows that

$$\left(1 + \frac{\sigma_\varepsilon}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^{-N} = \left(1 + 2\frac{\ln N}{N}\right)^{-N} \le 4N^{-2}, \quad N \ge 1 \ \text{ as } \lambda_2 \ge 2\left(\frac{\sigma_\varepsilon}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right),$$

where the arguments in [4, Lemma 5.1] are used to prove the last inequality. Thus, we have $|\eta_\ell^{[1]}(x_i, t_j)| \le 4\xi^{[1]} N^{-2}$ for $(x_i, t_j) \in \overline{\mathrm{Q}}_\ell^{N,M} \backslash \overline{\mathrm{Q}}_m$. Hence

$$||\widetilde{\mathrm{U}}_\ell - \mathrm{U}_\ell^{[1]}||_{\overline{\mathrm{Q}}_\ell^{N,M} \backslash \overline{\mathrm{Q}}_m} \le 4\xi^{[1]} N^{-2}. \tag{3.2.10}$$

Likewise

$$||\widetilde{\mathrm{U}}_r - \mathrm{U}_r^{[1]}||_{\overline{\mathrm{Q}}_r^{N,M} \backslash \overline{\mathrm{Q}}_m} \le 4\xi^{[1]} N^{-2}. \tag{3.2.11}$$

To estimate $||\widetilde{\mathrm{U}}_m - \mathrm{U}_m^{[1]}||_{\overline{\mathrm{Q}}_m^{N,M}}$, we consider the mesh function $\eta_m^{[1]}(x_i, t_j) = (\widetilde{\mathrm{U}}_m -$

$\mathrm{U}_m^{[1]})(x_i, t_j)$, which solves

$$[\delta_t \eta_m^{[1]}]_{i,j} - \varepsilon[\delta_x^2 \eta_m^{[1]}]_{i,j} + [a_m^{[1]} \eta_m^{[1]}]_{i,j} = 0, \quad (x_i, t_j) \in \mathrm{Q}_m^{N,M}, \tag{3.2.12}$$

where $a_{m;i,j}^{[1]} = \int_0^1 f_{\mathrm{u}}(x_i, t_j, \mathrm{U}_{m;i,j}^{[1]} + s(\widetilde{\mathrm{U}}_{m;i,j} - \mathrm{U}_{m;i,j}^{[1]}))ds$. Note that $\eta_m^{[1]}(x_i, 0) = 0$ for $x_i \in \overline{\Omega}_m^N$, and the following inequalities are fulfilled

$$|\eta_m^{[1]}(\sigma_\varepsilon, t_j)| = |(\widetilde{\mathrm{U}}_m - \mathcal{T}_{t_j}\mathrm{U}_\ell^{[1]})(\sigma_\varepsilon, t_j)| \leq |(\widetilde{\mathrm{U}}_m - \widetilde{\mathrm{U}}_\ell)(\sigma_\varepsilon, t_j)| + |(\widetilde{\mathrm{U}}_\ell - \mathrm{U}_\ell^{[1]})(\sigma_\varepsilon, t_j)|$$

$$\leq \xi_{\sigma_\varepsilon} + 4\xi^{[1]} N^{-2} \quad \text{for } t_j \in \omega^M,$$

and

$$\begin{aligned} |\eta_m^{[1]}(1-\sigma_\varepsilon, t_j)| &= |(\widetilde{\mathrm{U}}_m - \mathcal{T}_{t_j}\mathrm{U}_r^{[1]})(1-\sigma_\varepsilon, t_j)| \\ &\leq |(\widetilde{\mathrm{U}}_m - \widetilde{\mathrm{U}}_r)(1-\sigma_\varepsilon, t_j)| + |(\widetilde{\mathrm{U}}_r - \mathrm{U}_r^{[1]})(1-\sigma_\varepsilon, t_j)| \\ &\leq \xi_{\sigma_\varepsilon} + 4\xi^{[1]} N^{-2} \quad \text{for } t_j \in \omega^M, \end{aligned}$$

as $(\sigma_\varepsilon, t_j) \in \overline{\mathrm{Q}}_\ell^{N,M}$ and $(1-\sigma_\varepsilon, t_j) \in \overline{\mathrm{Q}}_r^{N,M}$. Thus, an application of Lemma 3.2.2 leads to the estimate

$$||\widetilde{\mathrm{U}}_m - \mathrm{U}_m^{[1]}||_{\overline{\mathrm{Q}}_m^{N,M}} \leq \xi_{\sigma_\varepsilon} + 4\xi^{[1]} N^{-2}. \tag{3.2.13}$$

Hence

$$\eta^{[1]} \leq \xi_{\sigma_\varepsilon} + 4\xi^{[1]} N^{-2}. \tag{3.2.14}$$

Note that $\xi^{[1]} \leq C$. Furthermore, since $(\sigma_\varepsilon, t_j) \in \overline{\mathrm{Q}}_\ell^{N,M}$ and $(1-\sigma_\varepsilon, t_j) \in \overline{\mathrm{Q}}_r^{N,M}$, it follows, from Lemma 3.2.1, that $\xi_{\sigma_\varepsilon} \leq C(\Delta t + N^{-2}\ln^2 N)$.
Using the triangle inequality, we write

$$||\mathrm{u} - \mathrm{U}^{[1]}||_{\overline{\mathrm{Q}}^{N,M}} \leq ||\mathrm{u} - \widetilde{\mathrm{U}}||_{\overline{\mathrm{Q}}^{N,M}} + ||\widetilde{\mathrm{U}} - \mathrm{U}^{[1]}||_{\overline{\mathrm{Q}}^{N,M}}.$$

Hence, using (3.2.14) and Lemma 3.2.1 we have the proof. □

In the next theorem, we establish uniform convergence of the iterates generated by the algorithm to the exact solution of problem (3.0.1) for $\sigma_\varepsilon = 1/4$.

Theorem 3.2.4. *Let* u *be the exact solution of (3.0.1) and* $\mathrm{U}^{[k]}$ *be the* k^{th} *iterate generated by the algorithm. If* $\sigma_\varepsilon = 1/4$, *then*

$$||\mathrm{u} - \mathrm{U}^{[k]}||_{\overline{\mathrm{Q}}^{N,M}} \leq C2^{-k} + C(\Delta t + N^{-2}\ln^2 N). \tag{3.2.15}$$

Proof. Consider the two mesh functions $\psi_\ell^{\pm}(x_i, t_j) = \frac{x_i}{2\sigma_\varepsilon}\xi^{[1]} \pm \eta_\ell^{[1]}(x_i, t_j)$, where $\eta_\ell^{[1]}$

satisfies (3.2.8)-(3.2.9). Then, it follows from (3.2.9) that $\psi_\ell^\pm$ satisfy the inequalities

$$\begin{cases} \psi_\ell^\pm(x_i,0) \geq 0, & \text{for } x_i \in \overline{\Omega}_\ell^N, \\ \psi_\ell^\pm(0,t_j) = 0, \psi_\ell^\pm(2\sigma_\varepsilon,t_j) \geq 0, & \text{for } t_j \in \omega^M. \end{cases}$$

The discrete maximum principle on $\overline{\mathrm{Q}}_\ell^{N,M}$ then yields

$$|\eta_\ell^{[1]}(x_i,t_j)| \leq \frac{x_i}{2\sigma_\varepsilon}\xi^{[1]} \quad \text{for } (x_i,t_j) \in \overline{\mathrm{Q}}_\ell^{N,M}.$$

This implies

$$||\widetilde{\mathrm{U}}_\ell - \mathrm{U}_\ell^{[1]}||_{\overline{\mathrm{Q}}_\ell^{N,M}\setminus\overline{\mathrm{Q}}_m} \leq \frac{\xi^{[1]}}{2}, \quad \text{as } x_i \leq \sigma_\varepsilon. \tag{3.2.16}$$

Analogously, for all $(x_i,t_j) \in \overline{\mathrm{Q}}_r^{N,M}\setminus\overline{\mathrm{Q}}_m$,

$$||\widetilde{\mathrm{U}}_r - \mathrm{U}_r^{[1]}||_{\overline{\mathrm{Q}}_r^{N,M}\setminus\overline{\mathrm{Q}}_m} \leq \frac{\xi^{[1]}}{2}. \tag{3.2.17}$$

We are left to find the estimate for $||\widetilde{\mathrm{U}}_m - \mathrm{U}_m^{[1]}||_{\overline{\mathrm{Q}}_m^{N,M}}$. With (3.2.12), we have $\eta_m^{[1]}(x_i,0) = 0$, for $x_i \in \overline{\Omega}_m^N$. Using the estimates (3.2.16)-(3.2.17) we get

$$\begin{aligned} |\eta_m^{[1]}(\sigma_\varepsilon,t_j)| = |(\widetilde{\mathrm{U}}_m - \mathcal{T}_{t_j}\mathrm{U}_\ell^{[1]})(\sigma_\varepsilon,t_j)| &\leq |(\widetilde{\mathrm{U}}_m - \widetilde{\mathrm{U}}_\ell)(\sigma_\varepsilon,t_j)| + |(\widetilde{\mathrm{U}}_\ell - \mathrm{U}_\ell^{[1]})(\sigma_\varepsilon,t_j)| \\ &\leq \xi_{\sigma_\varepsilon} + \frac{\xi^{[1]}}{2} \text{ for } t_j \in \omega^M, \end{aligned}$$

and

$$\begin{aligned} |\eta_m^{[1]}(1-\sigma_\varepsilon,t_j)| &= |(\widetilde{\mathrm{U}}_m - \mathcal{T}_{t_j}\mathrm{U}_r^{[1]})(1-\sigma_\varepsilon,t_j)| \\ &\leq |(\widetilde{\mathrm{U}}_m - \widetilde{\mathrm{U}}_r)(1-\sigma_\varepsilon,t_j)| + |(\widetilde{\mathrm{U}}_r - \mathrm{U}_r^{[1]})(1-\sigma_\varepsilon,t_j)| \\ &\leq \xi_{\sigma_\varepsilon} + \frac{\xi^{[1]}}{2} \text{ for } t_j \in \omega^M, \end{aligned}$$

as $(\sigma_\varepsilon,t_j) \in \overline{\mathrm{Q}}_\ell^{N,M}$ and $(1-\sigma_\varepsilon,t_j) \in \overline{\mathrm{Q}}_r^{N,M}$. Applying the discrete maximum principle on $\overline{\mathrm{Q}}_m^{N,M}$, we get

$$||\widetilde{\mathrm{U}}_m - \mathrm{U}_m^{[1]}||_{\overline{\mathrm{Q}}_m^{N,M}} \leq \xi_{\sigma_\varepsilon} + \frac{\xi^{[1]}}{2}. \tag{3.2.18}$$

Now, we have to find a bound on $\xi^{[2]}$ to estimate $\eta^{[2]}$. For $\sigma_\varepsilon = 1/4$, $(2\sigma_\varepsilon,t_j),(1-2\sigma_\varepsilon,t_j) \in \overline{\mathrm{Q}}_m^{N,M}$. Thus

$$\begin{aligned} |(\widetilde{\mathrm{U}}_\ell - \mathcal{T}_{t_j}\mathrm{U}^{[1]})(2\sigma_\varepsilon,t_j)| &\leq |(\widetilde{\mathrm{U}}_\ell - \widetilde{\mathrm{U}}_m)(2\sigma_\varepsilon,t_j)| + |(\widetilde{\mathrm{U}}_m - \mathrm{U}^{[1]})(2\sigma_\varepsilon,t_j)| \\ &\leq \xi_{2\sigma_\varepsilon} + \xi_{\sigma_\varepsilon} + \frac{\xi^{[1]}}{2} \text{ for } t_j \in \omega^M, \end{aligned}$$

$$\begin{aligned}|(\widetilde{\mathrm{U}}_r - \mathcal{T}_{t_j}\mathrm{U}^{[1]})(1-2\sigma_\varepsilon, t_j)| &= |(\widetilde{\mathrm{U}}_r - \widetilde{\mathrm{U}}_m)(1-2\sigma_\varepsilon, t_j)| + |(\widetilde{\mathrm{U}}_m - \mathrm{U}^{[1]})(1-2\sigma_\varepsilon, t_j)| \\ &\le \xi_{2\sigma_\varepsilon} + \xi_{\sigma_\varepsilon} + \frac{\xi^{[1]}}{2} \quad \text{for } t_j \in \omega^M.\end{aligned}$$

Therefore $\xi^{[2]} \le \xi_{2\sigma_\varepsilon} + \xi_{\sigma_\varepsilon} + \frac{\xi^{[1]}}{2}$ for $t_j \in \omega^M$, and so

$$\max\{\eta^{[1]}, \xi^{[2]}\} \le \lambda_\varepsilon + \frac{\xi^{[1]}}{2}, \quad \lambda_\varepsilon = \xi_{2\sigma_\varepsilon} + \xi_{\sigma_\varepsilon}.$$

Repetition of the previous arguments leads to

$$\max\{\eta^{[k]}, \xi^{[k+1]}\} \le \lambda_\varepsilon + \frac{\xi^{[k]}}{2}.$$

It is now easy to see that

$$\xi^{[k]} \le 2\lambda_\varepsilon + 2^{-(k-1)}\xi^{[1]}.$$

Hence

$$\eta^{[k]} \le 2\lambda_\varepsilon + 2^{-k}\xi^{[1]}. \tag{3.2.19}$$

Note that $\xi^{[1]} \le C$. Also, since $(2\sigma_\varepsilon, t_j), (1-2\sigma_\varepsilon, t_j) \in \overline{\mathrm{Q}}_m^{N,M}$, and $(\sigma_\varepsilon, t_j) \in \overline{\mathrm{Q}}_\ell^{N,M}$ and $(1-\sigma_\varepsilon, t_j) \in \overline{\mathrm{Q}}_r^{N,M}$, it follows, from Lemma 3.2.1, that $\lambda_\varepsilon \le C(\Delta t + N^{-2}\ln^2 N)$. Finally, combining (3.2.19) and Lemma 3.2.1, as in the previous theorem, we get the required result. □

3.3 Numerical Experiments

In this section, we provide the numerical results for two test problems to support the findings of the previous section. In our experiments, we choose $tol = N^{-2}\ln^2 N$. Further, we denote the final computed solution by $\mathrm{U}^{N,\Delta t}$.

Example 3.3.1. Consider the following singularly perturbed parabolic semilinear reaction-diffusion problem

$$\begin{cases} \mathrm{u}_t(x,t) - \varepsilon \mathrm{u}_{xx}(x,t) = \exp(-1) - \exp(-\mathrm{u}), & (x,t) \in \mathrm{Q} := \Omega \times (0,1], \\ \mathrm{u}(x,t) = 0, & (x,t) \in [0,1] \times \{0\}, \\ \mathrm{u}(0,t) = 0, \quad \mathrm{u}(1,t) = 0, & t \in (0,1]. \end{cases} \tag{3.3.1}$$

As the exact solution of problem (3.3.1) is unknown, we use the double mesh principle to estimate the maximum pointwise errors $\mathrm{E}_\varepsilon^{N,\Delta t} = ||\mathrm{U}^{N,\Delta t} - \mathrm{U}^{2N,\Delta t/4}||_{\overline{Q}^{N,M}}$, where $\mathrm{U}^{2N,\Delta t/4}$ denote the numerical approximation at grid point (x_i, t_j) on mesh having time step $\Delta t/4$ and $2N$ spatial mesh intervals in each subdomain.

Table 3.1: Errors $\mathrm{E}_{\varepsilon}^{N,\Delta t}$ and $\mathrm{E}^{N,\Delta t}$, and convergence rates $\rho^{N,\Delta t}$ for Example 3.3.1.

ε	$N=2^5$ $\Delta t=1/4$	$N=2^6$ $\Delta t=1/4^2$	$N=2^7$ $\Delta t=1/4^3$	$N=2^8$ $\Delta t=1/4^4$	$N=2^9$ $\Delta t=1/4^5$
10^{-1}	2.15E-02	6.38E-03	1.67E-03	4.24E-04	1.06E-04
10^{-2}	2.08E-02	5.74E-03	1.48E-03	3.72E-04	9.30E-05
10^{-3}	2.13E-02	5.98E-03	1.54E-03	3.88E-04	9.72E-05
10^{-4}	2.13E-02	6.02E-03	1.58E-03	4.10E-04	1.06E-04
10^{-5}	2.13E-02	6.02E-03	1.58E-03	4.10E-04	1.06E-04
10^{-6}	2.13E-02	6.02E-03	1.58E-03	4.10E-04	1.06E-04
10^{-7}	2.13E-02	6.02E-03	1.58E-03	4.10E-04	1.06E-04
10^{-8}	2.13E-02	6.02E-03	1.58E-03	4.10E-04	1.06E-04
$\mathrm{E}^{N,\Delta t}$	2.13E-02	6.02E-03	1.58E-03	4.10E-04	1.06E-04
$\rho^{N,\Delta t}$	1.82	1.93	1.95	1.95	

Table 3.2: Number of iterations required by the algorithm to converge for Example 3.3.1.

ε	$N=2^5$ $\Delta t=1/4$	$N=2^6$ $\Delta t=1/4^2$	$N=2^7$ $\Delta t=1/4^3$	$N=2^8$ $\Delta t=1/4^4$	$N=2^9$ $\Delta t=1/4^5$
10^{-1}	5	5	6	6	7
10^{-2}	3	4	4	5	6
10^{-3}	1	1	1	1	1
10^{-4}	1	1	1	1	1
10^{-5}	1	1	1	1	1
10^{-6}	1	1	1	1	1
10^{-7}	1	1	1	1	1
10^{-8}	1	1	1	1	1

We now calculate the uniform error for various values of N and Δt, by $\mathrm{E}^{N,\Delta t} = \max_{\varepsilon} \mathrm{E}_{\varepsilon}^{N,\Delta t}$, and corresponding rates of convergence are calculated by

$$\rho^{N,\Delta t} = \log_2\left(\mathrm{E}^{N,\Delta t}/\mathrm{E}^{2N,\Delta t/4}\right).$$

The errors $\mathrm{E}_{\varepsilon}^{N,\Delta t}$, $E^{N,\Delta t}$ and rates of uniform convergence $\rho^{N,\Delta t}$ that are computed for different values of ε, N, Δt are presented in Table 3.1. From it, one can observe the monotonically decreasing behavior of the maximum pointwise errors as N increases, Δt decreases and ε remains the same, which confirm that the method is convergent. Further, the last two rows of the table show that the method is parameter-uniform. Table 3.2 displays the iteration counts that are needed for the algorithm to converge. From it, one can see that only one iteration is necessary to achieve prescribed accuracy for the method when the perturbation parameter is small.

Example 3.3.2. Consider the following singularly perturbed parabolic semilinear reaction-diffusion problem

$$\begin{cases} \mathrm{u}_t(x,t) - \varepsilon \mathrm{u}_{xx}(x,t) + \exp(\mathrm{u}(x,t)) = \mathrm{f}(x,t) & (x,t) \in \mathrm{Q} := \Omega \times (0,1], \\ \mathrm{u}(x,t) = \phi(x) & 0 < x < 1, \\ \mathrm{u}(0,t) = g_0(t),\ \mathrm{u}(1,t) = g_1(t) & 0 \le t \le 1, \end{cases} \tag{3.3.2}$$

where f, ϕ, g_0 and g_1 are calculated from the exact solution

$$\mathrm{u}(x,t) = (1-e^{-t})\left(\frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1+e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x)\right).$$

Table 3.3: Errors $\mathrm{E}_\varepsilon^{N,\Delta t}$ and $\mathrm{E}^{N,\Delta t}$, and convergence rates $\rho^{N,\Delta t}$ for Example 3.3.2.

ε	$N=32$ $\Delta t = 1/4$	$N=64$ $\Delta t = 1/4^2$	$N=128$ $\Delta t = 1/4^3$	$N=256$ $\Delta t = 1/4^4$	$N=512$ $\Delta t = 1/4^5$
10^{-1}	6.79E-03	1.87E-03	4.78E-04	1.20E-04	3.01E-05
10^{-2}	1.97E-02	5.27E-03	1.34E-03	3.37E-04	8.44E-05
10^{-3}	4.11E-02	1.10E-02	2.80E-03	7.02E-04	1.76E-04
10^{-4}	4.90E-02	1.30E-02	3.31E-03	8.32E-04	2.08E-04
10^{-5}	5.08E-02	1.35E-02	3.43E-03	8.61E-04	2.15E-04
10^{-6}	5.10E-02	1.36E-02	3.45E-03	8.66E-04	2.17E-04
10^{-7}	5.10E-02	1.36E-02	3.45E-03	8.67E-04	2.17E-04
10^{-8}	5.10E-02	1.36E-02	3.45E-03	8.67E-04	2.17E-04
$E^{N,\Delta t}$	5.10E-02	1.36E-02	3.45E-03	8.67E-04	2.17E-04
$\rho^{N,\Delta t}$	1.91	1.98	1.99	2.00	

Table 3.4: Number of iterations required by the algorithm to converge for Example 3.3.2.

ε	$N=32$ $\Delta t = 1/4$	$N=64$ $\Delta t = 1/4^2$	$N=128$ $\Delta t = 1/4^3$	$N=256$ $\Delta t = 1/4^4$	$N=512$ $\Delta t = 1/4^5$
10^{-1}	3	4	4	4	5
10^{-2}	1	1	2	2	2
10^{-3}	1	1	1	1	1
10^{-4}	1	1	1	1	1
10^{-5}	1	1	1	1	1
10^{-6}	1	1	1	1	1
10^{-7}	1	1	1	1	1
10^{-8}	1	1	1	1	1

For problem (3.3.2), we determine the maximum pointwise errors by $\mathrm{E}_\varepsilon^{N,\Delta t} = ||\mathrm{u} - \mathrm{U}^{N,\Delta t}||_{\overline{\mathrm{Q}}^{N,M}}$, where u and $\mathrm{U}^{N,\Delta t}$ denotes the exact and numerical solutions, respectively. The errors $\mathrm{E}_\varepsilon^{N,\Delta t}$, $\mathrm{E}^{N,\Delta t}$, and rates of convergence are computed in a similar way as earlier. In Table 3.3 calculated errors $\mathrm{E}_\varepsilon^{N,\Delta t}$, $\mathrm{E}^{N,\Delta t}$, and rates of convergence $\rho^{N,\Delta t}$ are shown. Table 3.3 reveals that the method is parameter-uniform. Table 3.4 displays the number of iterations that are needed for the algorithm

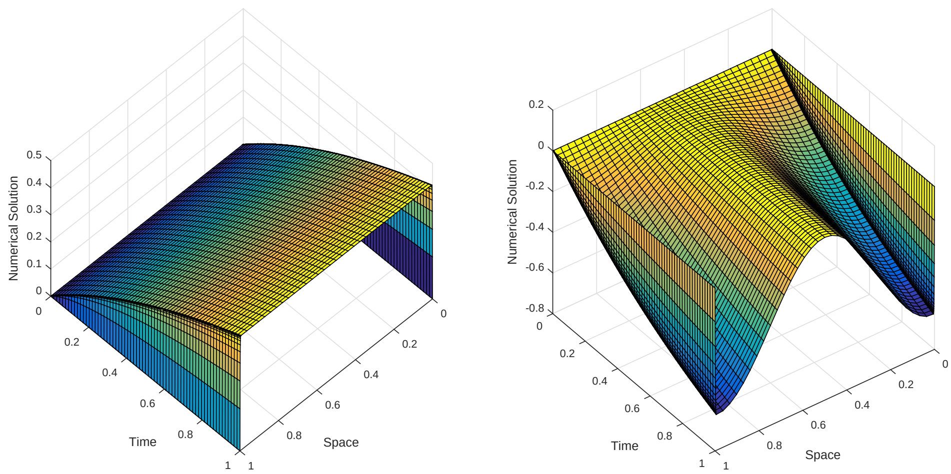

Figure 3.1: Numerical solutions of Example 3.3.1 and 3.3.2 for $\varepsilon = 10^{-6}$ with $N = 32, M = 64$ are depicted in the left and right figures, respectively.

to converge. From it, we observe that the convergence to the desired accuracy is achieved in only one iteration for smaller values of the perturbation parameter.

To visualize the boundary layer appearance, we have given the surface plots of the numerical solutions of problems (3.3.1) and (3.3.2) for $\varepsilon = 10^{-6}$ with $N = 32,\ M = 64$, in Fig. 3.1.

3.4 Conclusions

In this chapter, we have considered the numerical solution of singularly perturbed semilinear parabolic reaction-diffusion problems. For the numerical approximation of the problem, we designed a domain decomposition method of Schwarz waveform relaxation type. The construction of the method is based on decomposing problem domain into three overlapping subdomains and employing the backward Euler scheme in the time direction and the central difference scheme in the spatial direction. The method is shown to be capable of producing uniformly convergent results of first order in time and almost second order in space. Numerical experiments are carried out to demonstrate the effectiveness and robustness of the method.